An introduction to linear ordinary differential equations with constant coefficients using the impulsive response method and factorization.

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Abstract

We present an approach to the impulsive response method for solving linear constant-coefficient ordinary differential equations of any order based on the factorization of the differential operator. The approach is elementary, we only assume a basic knowledge of calculus and linear algebra. In particular, we avoid the use of distribution theory, as well as of the other more advanced approaches: Laplace transform, linear systems, the general theory of linear equations with variable coefficients and variation of parameters. The approach presented here can be used in a first course on differential equations for science and engineering majors.

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1 Introduction

Linear constant-coefficient differential equations constitute an important chapter in the theory of ordinary differential equations, also in view of their many applications in various fields of science.

In introductory courses on differential equations, the treatment of second or higher order non-homogeneous equations is usually limited to illustrating the method of undetermined coefficients. Using this, one finds a particular solution when the forcing term is a polynomial, an exponential, a sine or a cosine, or a product of terms of this kind.

It is well known that the impulsive response method gives an explicit formula for a particular solution in the more general case in which the forcing term is an arbitrary continuous function. This method is generally regarded as too difficult to implement in a first course on differential equations. Students become aware of it only later, as an application of the theory of the Laplace transform [4] or of distribution theory [5].

An alternative approach which is sometimes used consists in developing the theory of linear systems first, considering then linear equations of order $n$ as a particular case of this theory. The problem with this approach is that one needs to “digest” the theory of linear systems, with all the issues related to the diagonalization of matrices and the Jordan form ([2], chapter 3).

Another approach is by the general theory of linear equations with variable coefficients, with the notion of Wronskian and the method of the variation of constants. This approach can be implemented also in the case of constant coefficients ([3], chapter 2). However in introductory courses, the variation of constants method is often limited to first-order equations. Moreover, this method may be very long to implement in specific calculations, even for rather simple equations. Finally, within this approach, the occurrence of the particular solution as a convolution integral is rather indirect, and appears only at the end of the theory (see, for example, [3], exercise 4 p. 89).

The purpose of these notes is to give an elementary presentation of the impulsive response method using only very basic results from linear algebra and calculus in one or many variables.

We write the $n$-th order equation in the form

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = f(x),$$  \hspace{1cm} (1.1)

where we use the notation $y(x)$ in place of $y(t)$ or $y^{(k)}$ denotes as usual the derivative of order $k$ of $y$, $a_1, a_2, \ldots, a_n$ are complex constants, and the forcing term $f$ is a complex-valued continuous function in an interval $I$. When $f \neq 0$ the equation is called non-homogeneous. When $f = 0$ we get the associated homogeneous equation

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = 0.$$  \hspace{1cm} (1.2)

The basic tool of our investigation is the so called impulsive response. This is the function defined as follows. Let $p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n$ ($\lambda \in \mathbb{C}$) be the characteristic polynomial, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the roots of $p(\lambda)$ (not necessarily distinct, each counted with its multiplicity). We define the impulsive response $g = g_{\lambda_1, \ldots, \lambda_n}$
recursively by the following formulas: for \( n = 1 \) we set \( g_{\lambda_1}(x) = e^{\lambda_1 x} \), for \( n \geq 2 \) we set

\[
g_{\lambda_1 \cdots \lambda_n}(x) = e^{\lambda_n x} \int_0^x e^{-\lambda_n t} g_{\lambda_1 \cdots \lambda_{n-1}}(t) \, dt \quad (x \in \mathbb{R}). \tag{1.3}
\]

It turns out that \( g \) solves the homogeneous equation (1.2) with the initial conditions

\[
y(0) = y'(0) = \cdots = y^{(n-2)}(0) = 0, \quad y^{(n-1)}(0) = 1.
\]

Moreover, the impulsive response allows one to solve the non-homogeneous equation with an arbitrary continuous forcing term and with arbitrary initial conditions. Indeed, if \( g \) denotes the impulsive response of order \( n \) and if \( 0 \in I \), we shall see that the general solution of (1.1) in the interval \( I \) can be written as

\[
y(x) = y_p(x) + y_h(x),
\]

where the function \( y_p \) is given by the convolution integral

\[
y_p(x) = \int_0^x g(x - t)f(t) \, dt, \tag{1.4}
\]

and solves (1.1) with trivial initial conditions at the point \( x = 0 \) (i.e., \( y_p^{(k)}(0) = 0 \) for \( k = 0, 1, \ldots, n - 1 \)), whereas the function

\[
y_h(x) = \sum_{k=0}^{n-1} c_k g^{(k)}(x) \tag{1.5}
\]
gives the general solution of the associated homogeneous equation (1.2) as the coefficients \( c_k \) vary in \( \mathbb{C} \). In other words, the \( n \) functions \( g, g', g'', \ldots, g^{(n-1)} \) are linearly independent solutions of this equation and form a basis of the vector space of its solutions.

We will begin with the case of first-order equations, in which (1.4) and (1.5) are easily proved, proceed with the basic case of second-order equations, and treat finally the case of arbitrary \( n \). For simplicity, we start our discussion for \( n = 1, 2 \) with the case of real coefficients (i.e., \( a_1, \ldots, a_n \in \mathbb{R} \) in (1.1) and (1.2)), and real-valued forcing terms. The same techniques and proofs apply to the case of complex coefficients and complex-valued forcing terms. The complex notation will be useful to treat the case \( n > 2 \) by induction on \( n \). For \( n \geq 2 \) we will not assume, a priori, existence, uniqueness, and extendability of the solutions of a linear initial value problem (homogeneous or not). We shall rather prove these facts directly, in the case of constant coefficients, by obtaining explicit formulas for the solutions.

The proof of (1.4) that we give is by induction on \( n \) and is based on the factorization of the differential operator acting on \( y \) in (1.1) into first-order factors, along with the formula for solving first-order linear equations. It requires, moreover, the interchange of the order of integration in a double integral, that is, the Fubini theorem. The proof is constructive in that it directly produces the particular solution \( y_p \) as a convolution integral between the impulsive response \( g \) and the forcing term \( f \). In particular if we take \( f = 0 \), we get that the unique solution of the homogeneous initial value problem with
all vanishing initial data is the zero function. This implies immediately the uniqueness of the initial value problem (homogeneous or not) with arbitrary initial data.

We remark that in the usual approach to linear equations with constant coefficients, the proof of uniqueness relies on an estimate for the rate of growth of any solution $y$ of the homogeneous equation, together with its derivatives $y', y'', \ldots, y^{(n-1)}$, in terms of the coefficients $a_1, a_2, \ldots, a_n$ in (1.2) (see, e.g., [3], chapter 2, Theorems 13 and 14).

As regards (1.5), one can give a similar proof by induction on $n$ using factorization. We shall rather prove (1.5) by computing the linear relationship between the coefficients $c_k$ and the initial data $b_k = y^{(k)}(0) = y_p^{(k)}(0)$ for any given $n$. If $x_0$ is any point of $I$, we can just replace $\int_0^x$ with $\int_{x_0}^x$ in (1.4), and $g^{(k)}(x)$ with $g^{(k)}(x - x_0)$ in (1.5). The function $y_p$ satisfies then $y_p^{(k)}(x_0) = 0$ for $0 \leq k \leq n - 1$, and the relationship between $c_k$ and $b_k = y^{(k)}(x_0) = y_p^{(k)}(x_0)$ remains the same as before.

We will then look for explicit formulas for the impulsive response $g$. For $n = 1$ $g$ is an exponential, for $n = 2$ it is easily computed. For generic $n$ one can use the recursive formula (1.3) to compute $g$ at any prescribed order. A simple formula is obtained for $n = 3$, or, for generic $n$, in the case when the roots of $p(\lambda)$ are all equal or all different. In the general case we prove by induction on $k$ that if $\lambda_1, \ldots, \lambda_k$ are the distinct roots of $p(\lambda)$, of multiplicities $m_1, \ldots, m_k$, then there exist polynomials $G_1, \ldots, G_k$, of degrees $m_1 - 1, \ldots, m_k - 1$, such that

$$g(x) = \sum_{j=1}^{k} G_j(x)e^{\lambda_j x}. \quad (1.6)$$

An explicit formula for the polynomials $G_j$ is obtained. We will ultimately give the formula for the polynomials $G_j$ based on the partial fraction expansion of $1/p(\lambda)$. This formula can be proved by induction, but it is most easily proved using the Laplace transform or distribution theory.

Using (1.6) in (1.5) gives the well known formula for the general solution of (1.2) in terms of the complex exponentials $e^{\lambda_j x}$. The case of real coefficients follows easily from this result. Using (1.6) in (1.4) gives a simple proof of the method of undetermined coefficients in the case when $f(x) = P(x)e^{\lambda_0 x}$, where $P$ is a complex polynomial and $\lambda_0 \in \mathbb{C}$.

Several examples with worked-out solutions are given throughout the paper. We also propose some exercises to complement the main text and to get acquainted with the impulsive response method.

This paper is an outgrowth of [1], where the method of factorization was illustrated for second-order equations, but the general case only outlined. Here we treat equations of arbitrary order $n$. For completeness, we include the case $n = 2$ from [1], by suitably rearranging the material to better fit the new presentation.

2 The case $n = 1$

Consider the linear first-order differential equation

$$y' + ay = f(x), \quad (2.1)$$
where \( y' = \frac{dy}{dx}, \) \( a \) is a real constant, and the forcing term \( f \) is a real-valued continuous function in an interval \( I \subset \mathbb{R} \). It is well known that the general solution of (2.1) is given by
\[
y(x) = e^{-ax} \int e^{ax} f(x) \, dx, \tag{2.2}
\]
where \( \int e^{ax} f(x) \, dx \) denotes the set of all primitives of the function \( e^{ax} f(x) \) in the interval \( I \) (i.e., its indefinite integral). Suppose that \( 0 \in I \), and consider the integral function \( \int_0^x e^{at} f(t) \, dt \). By the Fundamental Theorem of Calculus, this is the primitive of \( e^{ax} f(x) \) that vanishes at 0. The theorem of the additive constant for primitives implies that
\[
\int e^{ax} f(x) \, dx = \int_0^x e^{at} f(t) \, dt + k \quad (k \in \mathbb{R}),
\]
and we can rewrite (2.2) in the form
\[
y(x) = e^{-ax} \int_0^x e^{at} f(t) \, dt + ke^{-ax} \\
= \int_0^x e^{-a(x-t)} f(t) \, dt + ke^{-ax} \\
= \int_0^x g(x-t) f(t) \, dt + kg(x), \tag{2.3}
\]
where \( g(x) = e^{-ax} \). The function \( g \) is called the impulsive response of the differential equation \( y' + ay = 0 \). It is the unique solution of the initial value problem
\[
\begin{cases}
y' + ay = 0 \\
y(0) = 1
\end{cases}
\]
Formula (2.3) illustrates a well known result in the theory of linear differential equations. Namely, the general solution of (2.1) is the sum of the general solution of the associated homogeneous equation \( y' + ay = 0 \) and of any particular solution of (2.1). In (2.3) the function
\[
y_p(x) = \int_0^x g(x-t) f(t) \, dt \tag{2.4}
\]
is the particular solution of (2.1) that vanishes at \( x = 0 \).

If \( x_0 \) is any point of \( I \), it is easy to verify that
\[
y(x) = \int_{x_0}^x g(x-t) f(t) \, dt + y_0 g(x-x_0) \quad (x \in I)
\]
is the unique solution of (2.1) in the interval \( I \) that satisfies \( y(x_0) = y_0 \) \( (y_0 \in \mathbb{R}) \).

As we shall see, formula (2.4) gives a particular solution of the non-homogeneous equation also in the case of higher-order linear constant-coefficient differential equations, by suitably defining the impulsive response \( g \).

**Remark 1: the convolution.** To better understand the structure of the particular solution (2.4), let us recall that if \( h_1 \) and \( h_2 \) are suitable functions (for example if \( h_1 \) and
h_2 \text{ are piecewise continuous on } \mathbb{R} \text{ with } h_1 \text{ bounded and } h_2 \text{ absolutely integrable over } \mathbb{R}, \\
\text{or if } h_1 \text{ and } h_2 \text{ are two signals, that is, they are piecewise continuous on } \mathbb{R} \text{ and vanish for } x < 0, \text{ then their convolution } h_1 \ast h_2 \text{ is defined as}
\[ h_1 \ast h_2 (x) = \int_{-\infty}^{+\infty} h_1(x-t)h_2(t) \, dt. \]

The change of variables \( x - t = s \) shows that convolution is commutative:
\[ h_1 \ast h_2 (x) = h_2 \ast h_1 (x) = \int_{-\infty}^{+\infty} h_1(s)h_2(x-s) \, ds. \]

Moreover, one can show that convolution is associative: \( (h_1 \ast h_2) \ast h_3 = h_1 \ast (h_2 \ast h_3) \).

If \( h_1 \) and \( h_2 \) are two signals, it is easy to verify that \( h_1 \ast h_2 \) is also a signal and that for any \( x \in \mathbb{R} \) one has
\[ h_1 \ast h_2 (x) = \int_{0}^{x} h_1(x-t)h_2(t) \, dt. \]

In a similar way, if \( h_1 \) and \( h_2 \) vanish for \( x > 0 \), the same holds for \( h_1 \ast h_2 \) and we have
\[ h_1 \ast h_2 (x) = \int_{-x}^{0} h_1(x-t)h_2(t) \, dt. \]

If \( \theta \) denotes the Heaviside step function, given by
\[ \theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases} \]
and if we let
\[ \tilde{\theta}(x) = -\theta(-x) = \begin{cases} 0 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0, \end{cases} \]
then for any two functions \( g \) and \( f \) we have
\[ \int_{0}^{x} g(x-t)f(t) \, dt = \begin{cases} \theta g \ast \theta f (x) & \text{if } x \geq 0 \\ -\tilde{\theta} g \ast \tilde{\theta} f (x) & \text{if } x \leq 0. \end{cases} \] (2.5)

In other words, the integral \( \int_{0}^{x} g(x-t)f(t) \, dt \) is the convolution of \( \theta g \) and \( \theta f \) for \( x \geq 0 \), it is the opposite of the convolution of \( \tilde{\theta} g \) and \( \tilde{\theta} f \) for \( x \leq 0 \).

3 The case \( n = 2 \)

Consider the second-order non-homogeneous linear differential equation
\[ y'' + ay' + by = f(x), \] (3.1)
where \( y'' = \frac{d^2y}{dx^2} \), \( a, b \) are real constants, and the forcing term \( f \) is a real-valued continuous function in an interval \( I \subset \mathbb{R} \), i.e., \( f \in C^0(I) \). For \( f = 0 \) we get the associated homogeneous equation
\[ y'' + ay' + by = 0. \] (3.2)
We will write (3.1) and (3.2) in operator form as $Ly = f(x)$ and $Ly = 0$, where $L$ is the \textit{linear second-order differential operator with constant coefficients} defined by

$$Ly = y'' + ay' + by,$$

for any function $y$ at least twice differentiable. Denoting by $\frac{d}{dx}$ the differentiation operator, we have

$$L = \left(\frac{d}{dx}\right)^2 + a \frac{d}{dx} + b. \quad (3.3)$$

$L$ defines a map $C^2(\mathbb{R}) \to C^0(\mathbb{R})$ that to each real-valued function $y$ at least twice differentiable over $\mathbb{R}$ with continuous second derivative associates the continuous function $Ly$. The fundamental property of $L$ is its linearity, that is,

$$L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2, \quad (3.4)$$

$\forall c_1, c_2 \in \mathbb{R}, \forall y_1, y_2 \in C^2(\mathbb{R})$. This formula implies some important facts. First, if $y_1$ and $y_2$ are any two solutions of the homogeneous equation on $\mathbb{R}$, then any linear combination $c_1y_1 + c_2y_2$ is also a solution of this equation on $\mathbb{R}$. In other words, the set

$$V = \{ y \in C^2(\mathbb{R}) : Ly = 0 \}$$

is a real vector space. We will prove that $\dim V = 2$. Note that if $y \in V$, then $y \in C^\infty(\mathbb{R})$, as follows from (3.2) by isolating $y''$ and differentiating any number of times.

Secondly, if $y_1$ and $y_2$ are two solutions of (3.1) on a given interval $I \subset I$, then their difference $y_1 - y_2$ solves (3.2). It follows that if we know a particular solution $y_p$ of the non-homogeneous equation on an interval $I'$, then any other solution of (3.1) on $I'$ is given by $y_p + y_h$, where $y_h$ is a solution of the associated homogeneous equation. We shall see that the solutions of (3.1) are defined on the whole of the interval $I$ on which $f$ is continuous (and are of course of class $C^2$ there).

The fact that $L$ has constant coefficients (i.e., $a$ and $b$ in (3.3) are constants) allows one to find explicit formulas for the solutions of (3.1) and (3.2). To this end, it is useful to consider complex-valued functions, $y: \mathbb{R} \to \mathbb{C}$. If $y = y_1 + iy_2$ (with $y_1, y_2: \mathbb{R} \to \mathbb{R}$) is such a function, the derivative $y'$ may be defined by linearity as $y' = y_1' + iy_2'$. It follows that $L(y_1 + iy_2) = Ly_1 + iLy_2$. In a similar way one defines the integral of $y$:

$$\int y(x) \, dx = \int y_1(x) \, dx + i \int y_2(x) \, dx, \quad \int \limits_{c}^{d} y(x) \, dx = \int \limits_{c}^{d} y_1(x) \, dx + i \int \limits_{c}^{d} y_2(x) \, dx.$$

The theorem of the additive constant for primitives and the Fundamental Theorem of Calculus extend to complex-valued functions.

It is then easy to verify, using either Euler’s formula or the series expansion of the exponential function and differentiation term by term, that

$$\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}, \quad \forall \lambda \in \mathbb{C}.$$  

It follows that $Le^{\lambda x} = (\lambda^2 + a\lambda + b)e^{\lambda x}$, and the complex exponential $e^{\lambda x}$ is a solution of (3.2) if and only if $\lambda$ is a root of the \textit{characteristic polynomial} $p(\lambda) = \lambda^2 + a\lambda + b$. Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the roots of $p(\lambda)$, so that

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2).$$
The operator $L$ factors in a similar way as a product (composition) of first-order differential operators:

$$L = \left(\frac{d}{dx} - \lambda_1\right) \left(\frac{d}{dx} - \lambda_2\right). \quad (3.5)$$

Indeed we have

$$\left(\frac{d}{dx} - \lambda_1\right) \left(\frac{d}{dx} - \lambda_2\right) y = \left(\frac{d}{dx} - \lambda_1\right) (y' - \lambda_2 y) = y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y,$$

which coincides with $Ly$ since $\lambda_1 + \lambda_2 = -a$, and $\lambda_1\lambda_2 = b$. Note that in (3.5) the order with which the two factors are composed is unimportant. In other words, the two operators $\left(\frac{d}{dx} - \lambda_1\right)$ and $\left(\frac{d}{dx} - \lambda_2\right)$ commute:

$$\left(\frac{d}{dx} - \lambda_1\right) \left(\frac{d}{dx} - \lambda_2\right) = \left(\frac{d}{dx} - \lambda_2\right) \left(\frac{d}{dx} - \lambda_1\right).$$

The idea is now to use (3.5) to reduce the problem to first-order differential equations. It is useful to consider linear differential equations with complex coefficients, whose solutions will be, in general, complex-valued. For example the first-order homogeneous equation $y' - \lambda y = 0$ with $\lambda \in \mathbb{C}$ has the general solution

$$y(x) = k e^{\lambda x} \quad (k \in \mathbb{C}).$$

(Indeed if $y' = \lambda y$, then $\frac{d}{dx} \left(y(x)e^{-\lambda x}\right) = 0$, whence $y(x)e^{-\lambda x} = k$.) The first-order non-homogeneous equation

$$y' - \lambda y = \left(\frac{d}{dx} - \lambda\right) y = f(x) \quad (\lambda \in \mathbb{C}),$$

with complex-valued forcing term $f : I \subset \mathbb{R} \to \mathbb{C}$ continuous in $I \ni 0$, has the general solution

$$y(x) = e^{\lambda x} \int e^{-\lambda t} f(t) dt = e^{\lambda x} \int_0^x e^{-\lambda t} f(t) dt + k e^{\lambda x} = \int_0^x g_\lambda(x-t) f(t) dt + k g_\lambda(x) \quad (x \in I, \ k = y(0) \in \mathbb{C}). \quad (3.6)$$

Here $g_\lambda(x) = e^{\lambda x}$ is the impulsive response of the differential operator $\left(\frac{d}{dx} - \lambda\right)$. It is the (unique) solution of $y' - \lambda y = 0, \ y(0) = 1$. Formula (3.6) can be proved as (2.3) in the real case. In particular, the solution of the first-order problem

$$\begin{cases} y' - \lambda y = f(x) \\ y(0) = 0 \end{cases} \quad (\lambda \in \mathbb{C})$$

is unique and is given by

$$y(x) = \int_0^x e^{\lambda(x-t)} f(t) dt. \quad (3.7)$$

The following result gives a particular solution of (3.1) as a convolution integral.
Theorem 3.1. Let \( f \in C^0(I) \), and suppose that \( 0 \in I \). Then the initial value problem

\[
\begin{align*}
\begin{cases}
y'' + ay' + by &= f(x) \\
y(0) &= 0, \\
y'(0) &= 0
\end{cases}
\end{align*}
\tag{3.8}
\]

has a unique solution, defined on the whole of \( I \), and given by the formula

\[
y(x) = \int_0^x g(x-t)f(t) \, dt \quad (x \in I),
\tag{3.9}
\]

where \( g \) is the function defined by

\[
g(x) = \int_0^x e^{\lambda_2(x-t)}e^{\lambda_1 t} \, dt \quad (x \in \mathbb{R}).
\tag{3.10}
\]

In particular if we take \( f = 0 \), we get that the only solution of the homogeneous problem

\[
\begin{align*}
\begin{cases}
y'' + ay' + by &= 0 \\
y(0) &= 0, \\
y'(0) &= 0
\end{cases}
\end{align*}
\tag{3.11}
\]

is the zero function \( y = 0 \).

**Proof.** We rewrite the differential equation (3.1) in the form

\[
\left( \frac{d}{dx} - \lambda_1 \right) \left( \frac{d}{dx} - \lambda_2 \right) y = f(x).
\]

Letting

\[
h = \left( \frac{d}{dx} - \lambda_2 \right) y = y' - \lambda_2 y,
\]

we see that \( y \) solves the problem (3.8) if and only if

\[
h \quad \text{solves} \quad \begin{cases} h' - \lambda_1 h = f(x) \\
h(0) = 0 \end{cases} \quad \text{and} \quad y \quad \text{solves} \quad \begin{cases} y' - \lambda_2 y = h(x) \\
y(0) = 0 \end{cases}.
\]

From (3.7) we have

\[
h(x) = \int_0^x e^{\lambda_1(x-t)}f(t) \, dt,
\]

\[
y(x) = \int_0^x e^{\lambda_2(x-t)}h(t) \, dt.
\]

Substituting \( h \) from the first formula into the second, we obtain \( y(x) \) as a repeated integral (for any \( x \in I \)):

\[
y(x) = \int_0^x e^{\lambda_2(x-t)} \left( \int_0^t e^{\lambda_1(t-s)} f(s) \, ds \right) \, dt.
\tag{3.12}
\]

To fix ideas, let us suppose that \( x > 0 \). Then in the integral with respect to \( t \) we have \( 0 \leq t \leq x \), whereas in the integral with respect to \( s \) we have \( 0 \leq s \leq t \). We can then rewrite \( y(x) \) as a double integral:

\[
y(x) = e^{\lambda_2 x} \int_{T_x} e^{(\lambda_1 - \lambda_2) t} e^{-\lambda_1 s} f(s) \, ds \, dt,
\]

where \( T_x \) is the set of points \( (t, s) \) such that \( 0 \leq t \leq x \) and \( 0 \leq s \leq t \).
where $T_x$ is the triangle in the $(s, t)$ plane defined by $0 \leq s \leq t \leq x$, with vertices at the points $(0, 0), (0, x), (x, x)$. In (3.12) we first integrate with respect to $s$ and then with respect to $t$. Since the triangle $T_x$ is convex both horizontally and vertically, and since the integrand function

$$F(s, t) = e^{(\lambda_1 - \lambda_2) t} e^{-\lambda_1 s} f(s)$$

is continuous in $T_x$, we can interchange the order of integration and integrate with respect to $t$ first. Given $s$ (between 0 and $x$) the variable $t$ in $T_x$ varies between $s$ and $x$, see the picture below.

![Graph showing triangle $T_x$ and integration over the triangle](image)

We thus obtain

$$y(x) = \int_0^x \left( \int_s^x e^{\lambda_2 (x-t)} e^{\lambda_1 (t-s)} \, dt \right) f(s) \, ds.$$  

By substituting $t$ with $t + s$ in the integral with respect to $t$ we finally get

$$y(x) = \int_0^x \left( \int_0^{x-s} e^{\lambda_2 (x-s-t)} e^{\lambda_1 t} \, dt \right) f(s) \, ds \quad (3.13)$$

$$= \int_0^x g(x-s) f(s) \, ds,$$

which is (3.9). For $x < 0$ we can reason in a similar way and we get the same result. \qed

The integral in formula (3.10) can be computed exactly as in the real field. We obtain the following expression of the function $g$:

1) if $\lambda_1 \neq \lambda_2 \ (\Leftrightarrow \ \Delta = a^2 - 4b \neq 0)$ then

$$g(x) = \frac{1}{\lambda_1 - \lambda_2} \left( e^{\lambda_1 x} - e^{\lambda_2 x} \right); \quad (3.14)$$

2) if $\lambda_1 = \lambda_2 \ (\Leftrightarrow \ \Delta = 0)$ then

$$g(x) = x e^{\lambda_1 x}. \quad (3.15)$$
Note that \( g \) is always a real function. Letting \( \alpha = -a/2 \) and 
\[
\beta = \begin{cases} 
\sqrt{-\Delta}/2 & \text{if } \Delta < 0 \\
\sqrt{\Delta}/2 & \text{if } \Delta > 0,
\end{cases}
\]
so that \( \lambda_{1,2} = \begin{cases} 
\alpha \pm i\beta & \text{if } \Delta < 0 \\
\alpha \pm \beta & \text{if } \Delta > 0,
\end{cases} \)
we have 
\[
g(x) = \begin{cases} 
\frac{1}{2\beta} (e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}) = \frac{1}{\beta} e^{\alpha x} \sin(\beta x) & \text{if } \Delta < 0 \\
\frac{1}{2\beta} (e^{(\alpha+\beta)x} - e^{(\alpha-\beta)x}) = \frac{1}{\beta} e^{\alpha x} \text{sh}(\beta x) & \text{if } \Delta > 0.
\end{cases} \tag{3.16}
\]
Also notice that \( g \in C^\infty(\mathbb{R}) \).

It is easy to check that \( g \) solves the following homogeneous initial value problem:
\[
\begin{align*}
\begin{cases} 
y'' + ay' + by = 0 \\
y(0) = 0, \quad y'(0) = 1
\end{cases}
\end{align*} \tag{3.17}
\]
Conversely, if \( y \) solves (3.17) then \( y = g \) given by (3.10). This is proved using (3.5)-(3.6) and reasoning as in the proof of Theorem 3.1. The function \( g \) is called the \textit{impulsive response} of the differential operator \( L \). The name comes from the initial conditions in (3.17), namely \( g(0) = 0, \ g'(0) = 1 \).

**Remark 2: the impulsive response as a convolution.** Denoting \( g \) by \( g_{\lambda_1,\lambda_2} \) and recalling (2.5), we can rewrite (3.10) in the suggestive form 
\[
g_{\lambda_1,\lambda_2}(x) = \int_0^x g_{\lambda_2}(x-t)g_{\lambda_1}(t) \, dt = \begin{cases} 
\theta g_{\lambda_2} \ast \theta g_{\lambda_1}(x) & \text{if } x \geq 0 \\
-\theta g_{\lambda_2} \ast \theta g_{\lambda_1}(x) & \text{if } x \leq 0,
\end{cases} \tag{3.18}
\]
showing that \( g \) is symmetric in \( \lambda_1 \leftrightarrow \lambda_2 \). This formula relates the (complex) first-order impulsive responses to the impulsive response of order 2, and will be generalized, in section 5, to linear equations of order \( n \) (cf. (5.2)). We also observe that the proof of Theorem 3.1 can be simplified, at least at a formal level, by rewriting (3.12) in convolution form (for example for \( x \geq 0 \)) and then using the associativity of this:
\[
y(x) = \theta g_{\lambda_2} \ast (\theta g_{\lambda_1} \ast \theta f)(x) = (\theta g_{\lambda_2} \ast \theta g_{\lambda_1}) \ast \theta f(x). \tag{3.19}
\]
This is just the equality (3.12)=(3.13) for \( x \geq 0 \). The interchange of the order of integration in the double integral considered above is then equivalent to the associative property of convolution for signals. Now (3.19) and (3.18) imply (3.9) for \( x \geq 0 \).

It is interesting to verify directly that the function \( y \) given by (3.9) solves (3.1). First let us prove the following formula for the derivative \( y' \):
\[
y'(x) = \frac{d}{dx} \int_0^x g(x-t)f(t) \, dt = g(0)f(x) + \int_0^x g'(x-t)f(t) \, dt \quad (x \in I). \tag{3.20}
\]
Indeed, given \( h \) such that \( x + h \in I \), we have
\[
y(x+h) - y(x) = \frac{1}{h} \left( \int_0^{x+h} g(x+h-t)f(t) \, dt - \int_0^x g(x-t)f(t) \, dt \right). \tag{3.21}
\]
As \( g \in C^2(\mathbb{R}) \), we can apply Taylor’s formula with the Lagrange remainder
\[
g(x_0 + h) = g(x_0) + g'(x_0)h + \frac{1}{2} g''(\xi)h^2
\]
at the point \( x_0 = x - t \), where \( \xi \) is some point between \( x_0 \) and \( x_0 + h \). Substituting this in (3.21) and using
\[
\int_0^{x+h} g(x-t)f(t)\,dt = \int_0^x g(x-t)f(t)\,dt + \int_x^{x+h} g(x-t)f(t)\,dt,
\]
gives
\[
\frac{1}{h} (y(x + h) - y(x)) = \frac{1}{h} \int_x^{x+h} g(x-t)f(t)\,dt + \int_0^{x+h} g'(x-t)f(t)\,dt
\]
\[
+ \frac{1}{2} h \int_0^{x+h} g''(\xi)f(t)\,dt.
\]
for some \( \xi \) between \( x - t \) and \( x - t + h \). When \( h \) tends to zero, the first term in the right-hand side of (3.22) tends to \( g(0)f(x) \), by the Fundamental Theorem of Calculus. The second term in (3.22) tends to \( \int_0^x g'(x-t)f(t)\,dt \), by the continuity of the integral function. Finally, the third term tends to zero, since the integral that occurs in it is a bounded function of \( h \) in a neighborhood of \( h = 0 \). (This is easy to prove.) We thus obtain formula (3.20). Recalling that \( g(0) = 0 \), we finally get
\[
y'(x) = \int_0^x g'(x-t)f(t)\,dt \quad (x \in I).
\]
In the same way we compute the second derivative:
\[
y''(x) = \left( \frac{d}{dx} \right)^2 \int_0^x g(x-t)f(t)\,dt
\]
\[
= \frac{d}{dx} \int_0^x g'(x-t)f(t)\,dt
\]
\[
= g'(0)f(x) + \int_0^x g''(x-t)f(t)\,dt
\]
\[
= f(x) + \int_0^x g''(x-t)f(t)\,dt,
\]
where we used \( g'(0) = 1 \). It follows that
\[
y''(x) + ay'(x) + by(x) = f(x) + \int_0^x (g'' + ag' + bg)(x-t) f(t)\,dt
\]
\[
= f(x), \quad \forall x \in I,
\]
g being a solution of the homogeneous equation. Therefore the function \( y \) given by (3.9) solves (3.1) in the interval \( I \). The initial conditions \( y(0) = 0 = y'(0) \) are immediately verified.

We now come to the solution of the initial value problem with arbitrary initial data at the point \( x = 0 \).
**Theorem 3.2.** Let \( f \in C^0(I) \), \( 0 \in I \), and let \( y_0, y'_0 \) be two arbitrary real numbers. Then the initial value problem

\[
\begin{aligned}
    y'' + ay' + by &= f(x) \\
    y(0) &= y_0, \quad y'(0) = y'_0
\end{aligned}
\]  

(3.23)

has a unique solution, defined on the whole of \( I \), and given by

\[
y(x) = \int_0^x g(x-t) f(t) \, dt + (y'_0 + ay_0) g(x) + y_0 g'(x) \quad (x \in I).
\]  

(3.24)

In particular (taking \( f = 0 \)), the solution of the homogeneous problem

\[
\begin{aligned}
    y'' + ay' + by &= 0 \\
    y(0) &= y_0, \quad y'(0) = y'_0
\end{aligned}
\]  

(3.25)

is unique, of class \( C^\infty \) on the whole of \( \mathbb{R} \), and is given by

\[
y_h(x) = (y'_0 + ay_0) g(x) + y_0 g'(x) \quad (x \in \mathbb{R}).
\]  

(3.26)

**Proof.** The uniqueness of the solutions of the problem (3.23) follows from the fact that if \( y_1 \) and \( y_2 \) both solve (3.23), then their difference \( \tilde{y} = y_1 - y_2 \) solves the problem (3.11), whence \( \tilde{y} = 0 \) by Theorem 3.1. Now notice that the function \( g' \) satisfies the homogeneous equation (like \( g \)). Indeed, since \( L \) has constant coefficients, we have

\[
Lg' = L \frac{d}{dx} g = \left[ \left( \frac{d}{dx} \right)^2 + a \frac{d}{dx} + b \right] \frac{d}{dx} g = \frac{d}{dx} Lg = 0.
\]

By the linearity of \( L \) and by Theorem 3.1 it follows that the function \( y \) given by (3.24) satisfies \( (Ly)(x) = f(x), \forall x \in I \). It is immediate that \( y(0) = y_0 \). Finally, since

\[
y'(x) = \int_0^x g'(x-t) f(t) \, dt + (y'_0 + ay_0) g'(x) + y_0 g''(x),
\]

we have

\[
y'(0) = y'_0 + ay_0 + y_0 g''(0)
\]

\[
= y'_0 + ay_0 + y_0 (-a g'(0) - b g(0)) = y'_0.
\]

It is also possible to give a constructive proof, analogous to that of Theorem 3.1. Indeed, by proceeding as in the proof of this theorem and using (3.6), we find that \( y \) solves the problem (3.23) if and only if \( y \) is given by

\[
y(x) = \int_0^x g(x-s) f(s) \, ds + (y'_0 - \lambda_2 y_0) g(x) + y_0 e^{\lambda_2 x}.
\]

This formula agrees with (3.24) in view of the equality \( e^{\lambda_2 x} = g'(x) - \lambda_1 g(x) \), which follows from (3.17) and (3.5) by observing that the function \( h(x) = g'(x) - \lambda_1 g(x) \) solves \( \left( \frac{d}{dx} - \lambda_2 \right) h = 0, h(0) = 1 \).
Theorem 3.3. Let $f \in C^0(I)$, $x_0 \in I$, $y_0, y'_0 \in \mathbb{R}$. The solution of the initial value problem

$$\begin{align*}
\left\{ \begin{array}{l}
y'' + ay' + by = f(x) \\
y(x_0) = y_0, \quad y'(x_0) = y'_0
\end{array} \right. 
\tag{3.27}
\end{align*}$$

is unique, it is defined on the whole of $I$, and is given by

$$y(x) = \int_{x_0}^{x} g(x-t)f(t) \, dt + (y'_0 + ay_0) g(x-x_0) + y_0 g'(x-x_0) \quad (x \in I). \tag{3.28}$$

In particular (taking $f = 0$), the solution of the homogeneous problem

$$\begin{align*}
\left\{ \begin{array}{l}
y'' + ay' + by = 0 \\
y(x_0) = y_0, \quad y'(x_0) = y'_0,
\end{array} \right. 
\tag{3.29}
\end{align*}$$

with $x_0 \in \mathbb{R}$ arbitrary, is unique, of class $C^\infty$ on the whole of $\mathbb{R}$, and is given by

$$y_h(x) = (y'_0 + ay_0) g(x-x_0) + y_0 g'(x-x_0) \quad (x \in \mathbb{R}). \tag{3.30}$$

**Proof.** Let $y$ be a solution of the problem (3.27), and let $\tau_{x_0}$ denote the translation map defined by $\tau_{x_0}(x) = x + x_0$. Set

$$\tilde{y}(x) = y \circ \tau_{x_0}(x) = y(x + x_0).$$

Since the differential operator $L$ has constant coefficients, it is invariant under translations, that is,

$$L(y \circ \tau_{x_0}) = (Ly) \circ \tau_{x_0}.$$

It follows that

$$(L\tilde{y})(x) = (Ly)(x + x_0) = f(x + x_0).$$

Since moreover $\tilde{y}(0) = y(x_0) = y_0$, $\tilde{y}'(0) = y'(x_0) = y'_0$, we see that $y$ solves the problem (3.27) in the interval $I$ if and only if $\tilde{y}$ solves the initial value problem

$$\begin{align*}
\left\{ \begin{array}{l}
\tilde{y}'' + a\tilde{y}' + b\tilde{y} = \tilde{f}(x) \\
\tilde{y}(0) = y_0, \quad \tilde{y}'(0) = y'_0
\end{array} \right.
\end{align*}$$

in the translated interval $I - x_0 = \{t - x_0 : t \in I\}$ and with the translated forcing term $\tilde{f}(x) = f(x + x_0)$. By Theorem 3.2 we have that $\tilde{y}$ is unique and is given by

$$\tilde{y}(x) = \int_{0}^{x} g(x-s)\tilde{f}(s) \, ds + (y'_0 + ay_0) g(x) + y_0 g'(x).$$

Therefore $y$ is also unique and is given by

$$y(x) = \tilde{y}(x-x_0) = \int_{0}^{x-x_0} g(x-x_0 - s)f(s + x_0) \, ds + (y'_0 + ay_0) g(x-x_0) + y_0 g'(x-x_0).$$

Formula (3.28) follows immediately from this by making the change of variable $s + x_0 = t$ in the integral with respect to $s$. \qed
Substitution of (3.15) and (3.16) in (3.30) yields the following formula for the solution of the homogeneous problem (3.29):

\[
y_h(x) = \begin{cases} 
  y_0 e^{\alpha(x-x_0)} \cos \beta(x-x_0) + \frac{1}{\beta} (y_0' - \alpha y_0) e^{\alpha(x-x_0)} \sin \beta(x-x_0) & \text{if } \Delta < 0 \\
  y_0 e^{\alpha(x-x_0)} \cosh \beta(x-x_0) + \frac{1}{\beta} (y_0' - \alpha y_0) e^{\alpha(x-x_0)} \sinh \beta(x-x_0) & \text{if } \Delta > 0 \\
  e^{\alpha(x-x_0)} [y_0 + (y_0' - \alpha y_0)(x-x_0)] & \text{if } \Delta = 0,
\end{cases}
\]

(3.31)

where for \( \Delta = 0 \) we let \( \alpha = \lambda_1 \) (so that, in all cases, \( \alpha = -a/2 \)). Note the similarity between the solutions with \( \Delta \neq 0 \), where the trigonometric functions for \( \Delta < 0 \) are replaced by the hyperbolic ones for \( \Delta > 0 \). Also note that for \( \beta \to 0 \), both solutions with \( \Delta \neq 0 \) go over to the solution with \( \Delta = 0 \).

**Remark 3: existence, uniqueness and extendability of the solutions.** It is well known that a linear initial value problem with variable coefficients (homogeneous or not) has a unique solution defined on the entire interval \( I \) in which the coefficients and the forcing term are continuous (on the whole of \( \mathbb{R} \) in the homogeneous constant-coefficient case). (See, e.g., [3], Theorems 1 and 3 pp. 104-105, for the homogeneous variable-coefficient case.) Now theorems 3.1, 3.2 and 3.3 give an elementary proof of this fact (together with an explicit formula for the solution) for a linear second-order problem with constant coefficients.

**Corollary 3.4.** Let \( f \in C^0(I) \) and let \( x_0 \in I \) be fixed. Every solution \( y \) of \( Ly = f(x) \) in the interval \( I \) can be written as \( y = y_p + y_h \), where

\[
y_p(x) = \int_{x_0}^x g(x-t) f(t) \, dt
\]

(3.32)
solves (3.1) with the initial conditions \( y_p(x_0) = y_p'(x_0) = 0 \), and \( y_h \) is the solution of the homogeneous equation (3.2) such that \( y_h(x_0) = y(x_0) \), \( y_h'(x_0) = y'(x_0) \).

**Proof.** Let \( y \) be any solution of (3.1) in \( I \), and let \( y_0 = y(x_0), \ y_0' = y'(x_0) \). Then \( y \) solves the problem (3.27), so by uniqueness \( y \) is given by (3.28). Thus \( y = y_p + y_h \), where \( y_p \), given by (3.32), solves (3.1) with trivial initial conditions at \( x_0 \), and \( y_h \), given by (3.30)-(3.31), solves (3.2) with the same initial conditions as \( y \) at \( x_0 \).

**Corollary 3.5.** The set \( V \) of real solutions of the homogeneous equation (3.2) on \( \mathbb{R} \) is a real vector space of dimension 2, and the two functions \( g, g' \) form a basis of \( V \).

**Proof.** Let \( y \) be a real solution of (3.2) on \( \mathbb{R} \). Let \( y_0 = y(0), \ y_0' = y'(0) \), and define \( y_h \) by formula (3.26). Then \( y \) and \( y_h \) both satisfy (3.25), whence \( y = y_h \) by Theorem 3.2. Note that \( y \in C^\infty(\mathbb{R}) \). We conclude by (3.26) that every element of the vector space \( V \) can be written as a linear combination of \( g \) and \( g' \) in a unique way. (The coefficients in this combination are uniquely determined by the initial data at the point \( x = 0 \).) In particular, this holds for the trivial solution \( y(x) = 0 \ \forall x \), i.e., \( cy(x) + dg'(x) = 0 \) for \( c, d \in \mathbb{R} \) and \( \forall x \), implies \( c = d = 0 \). Thus \( g \) and \( g' \) are linearly independent and form a basis of \( V \).

Another basis of \( V \) can be obtained from the following result.
Theorem 3.6. Every complex solution of the homogeneous equation $Ly = 0$ can be written in the form $y = c_1 y_1 + c_2 y_2$ ($c_1, c_2 \in \mathbb{C}$), where $y_1$ and $y_2$ are given by

$$(y_1(x), y_2(x)) = \begin{cases} (e^{\lambda_1 x}, e^{\lambda_2 x}) & \text{if } \lambda_1 \neq \lambda_2 \\ (e^{\lambda_1 x}, x e^{\lambda_1 x}) & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Conversely, any function of this form is a solution of $Ly = 0$.

Proof 1. This proof does not use the impulsive response but only factorization. It can be used to give an independent proof of existence and uniqueness for the solutions of the homogeneous initial value problem (3.29). (The constants $c_1, c_2$ can be uniquely determined in terms of $y_0, y_0'$.)

By (3.5), we rewrite the homogeneous differential equation (3.2) in the form

$$Ly = \left( \frac{d}{dx} - \lambda_1 \right) \left( \frac{d}{dx} - \lambda_2 \right) y = 0.$$ 

We now argue as in the proof of Theorem 3.1. Letting $h = \left( \frac{d}{dx} - \lambda_2 \right) y = y' - \lambda_2 y$, we see that $y$ solves $Ly = 0$ if and only if

$$\begin{cases} h' - \lambda_1 h = 0 \\ y' - \lambda_2 y = h(x) \end{cases} \iff \begin{cases} h(x) = k e^{\lambda_1 x} \quad (k \in \mathbb{C}) \\ y' - \lambda_2 y = k e^{\lambda_1 x}. \end{cases}$$

By formula (2.2) (which gives the general solution of (2.1) also for $a \in \mathbb{C}$, cf. (3.6)), we see that $y$ solves $Ly = 0$ if and only if $y$ is given by

$$y(x) = e^{\lambda_2 x} \int e^{-\lambda_2 x} k e^{\lambda_1 x} dx$$

$$= k e^{\lambda_2 x} \int e^{(\lambda_1 - \lambda_2) x} dx$$

$$= \begin{cases} k e^{\lambda_2 x} (x + k') & \text{if } \lambda_1 = \lambda_2 \\ k e^{\lambda_2 x} \frac{e^{(\lambda_1 - \lambda_2) x}}{\lambda_1 - \lambda_2} + k' & \text{if } \lambda_1 \neq \lambda_2 \end{cases}$$

$$= \begin{cases} c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} & \text{if } \lambda_1 = \lambda_2 \\ c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_2 x} & \text{if } \lambda_1 \neq \lambda_2, \end{cases}$$

where $k' \in \mathbb{C}$, and we have set $c_1 = kk'$, $c_2 = k$ for $\lambda_1 = \lambda_2$, and $c_1 = k \frac{k}{\lambda_1 - \lambda_2}$, $c_2 = kk'$ for $\lambda_1 \neq \lambda_2$. This proves the first part of the Theorem. The converse part also follows from this. Indeed for $\lambda_1 \neq \lambda_2$ we already know that $y_1$ and $y_2$ solve $Ly = 0$ so by (complex) linearity, any linear combination $c_1 y_1 + c_2 y_2$ with $c_1, c_2 \in \mathbb{C}$ is a solution of $Ly = 0$. For $\lambda_1 = \lambda_2$ we know that $y_1(x) = e^{\lambda_1 x}$ is a solution of $Ly = 0$. Taking $k = 1$, $k' = 0$ in the previous formulae, we see that $y_2(x) = x e^{\lambda_1 x}$ is also a solution of $Ly = 0$, and the result follows by linearity.

Proof 2. This proof uses the impulsive response and is well suited for generalization to higher-order equations.

We observe that any complex solution of $Ly = 0$ on $\mathbb{R}$ can be written as a complex linear combination of $g$ and $g'$ in a unique way. Indeed if $y = \text{Re } y + i \text{Im } y$ solves $Ly = 0$, then $L(\text{Re } y) = 0 = L(\text{Im } y)$, since $L$ has real coefficients. By Corollary 3.5,
Re \( y \) and Im \( y \) are unique real linear combinations of \( g \) and \( g' \), so \( y \) is a unique complex linear combination of \( g \) and \( g' \). In particular, the set \( V_C \) of complex solutions of the homogeneous equation \( Ly = 0 \) on \( \mathbb{R} \) is a complex vector space of dimension 2, with a basis given by \( \{g, g'\} \). Now by (3.14) and (3.15) we see that any linear combination of \( g \) and \( g' \) can be rewritten as a linear combination of \( y_1 \) and \( y_2 \). Indeed if \( y = cg + dg' \) with \( c, d \in \mathbb{C} \), then \( y = c_1 y_1 + c_2 y_2 \), where

\[
\begin{aligned}
&c_1 = \frac{c + d\lambda_1}{\lambda_1 - \lambda_2}, \\
&c_2 = \frac{c + d\lambda_2}{\lambda_2 - \lambda_1} \quad \text{for} \quad \lambda_1 \neq \lambda_2, \\
&c_1 = d, \quad c_2 = c + d\lambda_1 \quad \text{for} \quad \lambda_1 = \lambda_2.
\end{aligned}
\]

The converse part follows by verifying that \( y_1 \) and \( y_2 \) solve \( Ly = 0 \). The case \( \lambda_1 \neq \lambda_2 \) is clear. If \( \lambda_1 = \lambda_2 \) then \( L = \left( \frac{d}{dx} - \lambda_1 \right)^2 \) by (3.5), and not only \( Ly_1 = 0 \), but also

\[
Ly_2(x) = \left( \frac{d}{dx} - \lambda_1 \right) \left( \left( \frac{d}{dx} - \lambda_1 \right) (xe^{\lambda_1 x}) \right) = \left( \frac{d}{dx} - \lambda_1 \right) e^{\lambda_1 x} = 0.
\]

Note that in this case, \( y_2 \) is just the impulsive response \( g \).

**Corollary 3.7.** If \( \lambda_1 \) and \( \lambda_2 \) are real, then the functions \( y_1 \) and \( y_2 \) form a basis of \( V \). If \( \lambda_{1,2} = \alpha \pm i\beta \) with \( \beta \neq 0 \), then a basis of \( V \) is given by the functions

\[
\tilde{y}_1(x) = \Re y_1(x) = e^{\alpha x}\cos \beta x, \quad \tilde{y}_2(x) = \Im y_1(x) = e^{\alpha x}\sin \beta x.
\]

**Proof.** By Theorem 3.6 the functions \( y_1 \) and \( y_2 \) form a basis of \( V_C \). Indeed they belong to \( V_C \) and every element of \( V_C \) is a linear combination of them. Since \( \dim V_C = 2 \), \( y_1 \) and \( y_2 \) must be linearly independent. If \( \lambda_1 \) and \( \lambda_2 \) are real, then \( y_1 \) and \( y_2 \) are real-valued functions and form a basis of \( V \) as well. Indeed they are linearly independent in \( V_C \), thus also in \( V \), and \( \dim V = 2 \). On the other hand, if \( \lambda_{1,2} = \alpha \pm i\beta \) with \( \beta \neq 0 \), then using \( \tilde{y}_1 = \frac{1}{2}(y_1 + y_2), \tilde{y}_2 = \frac{1}{2i}(y_1 - y_2) \), we see that the functions \( \tilde{y}_1 \) and \( \tilde{y}_2 \) solve \( Ly = 0 \) and are linearly independent in \( V_C \) (and thus also in \( V \)). Indeed if \( c\tilde{y}_1 + d\tilde{y}_2 = 0 \) for \( c, d \in \mathbb{C} \), then we get \( \frac{1}{2}(c - id)y_1 + \frac{1}{2}(c + id)y_2 = 0 \), whence \( c \pm id = 0 \) by the linear independence of \( y_1 \) and \( y_2 \), and finally \( c = d = 0 \). Since \( \tilde{y}_1 \) and \( \tilde{y}_2 \) are linearly independent in \( V \) and \( \dim V = 2 \), \( \tilde{y}_1 \) and \( \tilde{y}_2 \) form a basis of \( V \).

**Remark 4: complex coefficients.** Theorems 3.1, 3.2, 3.3, 3.6 and Corollary 3.4 remain valid if \( L \) is a linear second-order differential operator with constant complex coefficients, i.e., if \( a, b \in \mathbb{C} \) in (3.3), and \( f \) is a complex-valued function. In this case \( \lambda_1 \) and \( \lambda_2 \) in (3.5) are arbitrary complex numbers. The impulsive response \( g \) of \( L \) is defined again by (3.10)-(3.18), it is given explicitly by formulas (3.14)-(3.15), and solves (3.17). The solution of the initial value problem (3.27), with \( a, b, y_0, y_0' \in \mathbb{C} \) and \( f : I \subset \mathbb{R} \to \mathbb{C} \) continuous in \( I \ni x_0 \), is given again by (3.28). Finally, the set \( V \) of solutions of (3.2) on \( \mathbb{R} \) is a complex vector space of dimension 2, with a basis given by \( \{g, g'\} \) or by \( \{y_1, y_2\} \). In the next section the results obtained here for \( n = 2 \) will be generalized to linear equations of order \( n \), by working directly with complex coefficients and using induction on \( n \).

**Remark 5: the approach by distributions.** The meaning of formulas (3.9)-(2.5), or (3.24)-(2.5), is best appreciated in the context of distribution theory. Given a linear
constant-coefficient differential equation, written in the symbolic form \( Ly = f(x) \), we can study the equation \( LT = S \) for \( T \) and \( S \) in a suitable convolution algebra, for example the algebra \( \mathcal{D}'_+ \) of distributions with support in \([0, +\infty)\), which generalizes the algebra of signals. One can show that the elementary solution in \( \mathcal{D}'_+ \), that is, the solution of \( LT = \delta \), or equivalently, the inverse of \( L\delta \) (\( \delta \) the Dirac distribution), is given precisely by \( \theta g \), where \( g \) is the impulsive response of the differential operator \( L \). If \( y \) solves \( Ly = f(x) \) with trivial initial conditions at \( x = 0 \), then \( \theta y = \theta \delta + (y_0 + ay_0)\delta + y_0\delta' \), whence

\[
L(\theta y) = \theta f + (y_0' + ay_0)\delta + y_0\delta',
\]

whence

\[
\theta y = \theta g \ast \theta f + (y_0' + ay_0)\theta g + y_0\theta g'.
\]

This is just (3.24) for \( x \geq 0 \). This approach requires, however, a basic knowledge of distribution theory, which goes far beyond our goals and will not be used here. (See [5], chapters II and III, for a nice presentation.)

We now look at some examples.

**Example 1.** Solve the initial value problem

\[
\begin{align*}
y'' - 2y' + y &= \frac{e^x}{x+2} \\
y(0) &= 0, \quad y'(0) = 0.
\end{align*}
\]

**Solution.** The characteristic polynomial is \( p(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \), so that \( \lambda_1 = \lambda_2 = 1 \), and the impulsive response is

\[ g(x) = xe^x. \]

The forcing term \( f(x) = \frac{e^x}{x+2} \) is continuous for \( x > -2 \) and for \( x < -2 \). Since the initial conditions are posed at \( x = 0 \), we can work in the interval \( I = (-2, +\infty) \). By Theorem 3.1 we get the following solution of the proposed problem in \( I \):

\[
y(x) = \int_0^x (x-t)e^{x-t}(e^t)^{x-t}dt = e^x \int_0^x \frac{x-t}{t+2}dt \\
= e^x \left( \int_0^x \frac{x+2}{t+2} - 1 \right) dt = e^x \left( (x+2) \log(t+2) - t \right)_0^x \\
= e^x \left( (x+2) \log(x+2) - x - (x+2) \log 2 \right) \\
= e^x(x+2) \log \left( \frac{x+2}{2} \right) - xe^x.
\]

**Example 2.** Solve the initial value problem

\[
\begin{align*}
y'' + y &= \frac{1}{\cos x} \\
y(0) &= 0, \quad y'(0) = 0.
\end{align*}
\]

**Solution.** We have \( p(\lambda) = \lambda^2 + 1 \), whence \( \lambda_1 = \lambda_2 = i \), and the impulsive response is

\[ g(x) = \sin x. \]
The initial data are given at \( x = 0 \) and we can work in the interval \( I = (\pi/2, \pi/2) \), where the forcing term \( f(x) = \frac{1}{\cos x} \) is continuous. By formula (3.9) we get

\[
y(x) = \int_0^x \sin(x - t) \frac{1}{\cos t} \, dt = \int_0^x (\sin x \cos t - \cos x \sin t) \frac{1}{\cos t} \, dt
\]

\[
= \sin x \int_0^x dt - \cos x \int_0^x \frac{\sin t}{\cos t} \, dt = x \sin x + \cos x \log(\cos x).
\]

**Example 3.** Solve the initial value problem

\[
\begin{align*}
y'' - y' &= \frac{1}{\cosh x} \\
y(0) &= 0, \quad y'(0) = 0.
\end{align*}
\]

**Solution.** We have \( p(\lambda) = \lambda^2 - \lambda = \lambda(\lambda - 1) \), thus \( \lambda_1 = 1, \quad \lambda_2 = 0 \), and the impulsive response is \( g(x) = e^x - 1 \).

The forcing term \( f(x) = \frac{1}{\cosh x} \) is continuous on the whole of \( \mathbb{R} \). Formula (3.9) gives

\[
y(x) = \int_0^x \left( e^{x-t} - 1 \right) \frac{1}{\cosh t} \, dt
\]

The two integrals are easily computed:

\[
\int \frac{1}{\cosh t} \, dt = 2 \int \frac{1}{e^t + e^{-t}} \, dt = 2 \int \frac{e^t}{e^{2t} + 1} \, dt = 2 \arctan(e^t) + C,
\]

\[
\int \frac{e^{-t}}{\cosh t} \, dt = 2 \int \frac{1}{e^{2t} + 1} \, dt = 2 \int \frac{1 + e^{2t} - e^{-2t}}{1 + e^{2t}} \, dt = 2t - \log(1 + e^{2t}) + C^'.
\]

We finally get

\[
y(x) = e^x \left[ 2x - \log(1 + e^{2x}) \right]_0^x - 2 \left[ \arctan(e^t) \right]_0^x
\]

\[
= e^x \left[ 2x - \log(1 + e^{2x}) \right] + e^x \log 2 - 2 \arctan(e^x) + \frac{\pi}{2}.
\]

**Exercises**

1. Solve the following initial value problems:

   \[
   \begin{align*}
   \text{(a)} & \quad \begin{cases} y'' + 2y' + y = \frac{e^{x^2}}{x^2 + 1} \\
   y(0) = 0, \quad y'(0) = 0.
   \end{cases} \\
   \text{(b)} & \quad \begin{cases} y'' - 2y' + 2y = \frac{e^x}{1 + \cos x} \\
   y(0) = 0, \quad y'(0) = 0.
   \end{cases} \\
   \text{(c)} & \quad \begin{cases} y'' - 3y' + 2y = \frac{e^{x^2}}{(e^x + 1)^2} \\
   y(0) = 0, \quad y'(0) = 0.
   \end{cases} \\
   \text{(d)} & \quad \begin{cases} y'' + 4y = \frac{1}{\sin 2x} \\
   y(\frac{\pi}{4}) = 0, \quad y'(\frac{\pi}{4}) = 0.
   \end{cases} \\
   \text{(e)} & \quad \begin{cases} y'' - y = \frac{1}{\sinh x} \\
   y(1) = 0, \quad y'(1) = 0.
   \end{cases} \\
   \text{(f)} & \quad \begin{cases} y'' - 2y' = \frac{e^x}{\cosh 2x} \\
   y(0) = 0, \quad y'(0) = 0.
   \end{cases}
   \end{align*}
\]
4 The general case

Consider the linear constant-coefficient non-homogeneous differential equation of order \( n \)
\[
y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = f(x), \tag{4.1}
\]
where \( y^{(k)} = \frac{d^k y}{dx^k} = \left(\frac{d}{dx}\right)^k y \), \( a_1, a_2, \ldots, a_n \in \mathbb{C} \), and the forcing term \( f : I \to \mathbb{C} \) is a complex-valued continuous function in an interval \( I \subset \mathbb{R} \), i.e., \( f \in C^0(I) \). When \( f = 0 \) we obtain the associated homogeneous equation
\[
y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = 0. \tag{4.2}
\]
We will write (4.1) and (4.2) in operator form as \( L y = f(x) \) and \( L y = 0 \), where \( L \) is the linear differential operator of order \( n \) with constant coefficients defined by
\[
L y = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y
\]
for any function \( y \) at least \( n \) times differentiable. Denoting by \( \frac{d}{dx} \) the differentiation operator, we have
\[
L = \left(\frac{d}{dx}\right)^n + a_1 \left(\frac{d}{dx}\right)^{n-1} + \cdots + a_{n-1} \frac{d}{dx} + a_n. \tag{4.3}
\]
\( L \) defines a map \( C^n(\mathbb{R}) \to C^0(\mathbb{R}) \) that to each complex-valued function \( y \) at least \( n \)-times differentiable over \( \mathbb{R} \) with continuous \( n \)th derivative associates the continuous function \( L y \). The fundamental property of \( L \) is its linearity, i.e., (3.4) holds, \( \forall c_1, c_2 \in \mathbb{C}, \forall y_1, y_2 \in C^n(\mathbb{R}) \). Again this implies that if \( y_1 \) and \( y_2 \) are any two solutions of (4.2) on \( \mathbb{R} \), then any linear combination of them is again a solution of (4.2) on \( \mathbb{R} \). It follows that the set
\[
V = \{ y \in C^n(\mathbb{R}) : L y = 0 \}
\]
is a complex vector space. We will prove that \( \dim V = n \), and will see how to get a basis of \( V \). Note that if \( y \in V \), then \( y \in C^\infty(\mathbb{R}) \), as follows from (4.2) by isolating \( y^{(n)} \) and differentiating any number of times.

The linearity of \( L \) also implies, as in the case of \( n = 1, 2 \), that the general solution of (4.1) on some interval \( I' \subset I \) is the sum of the general solution of (4.2) on \( I' \) plus any particular solution of (4.1) on \( I' \). We will prove that the solutions of the initial value problem for the differential equation (4.1) at any point \( x_0 \in I \) and with any initial data are defined on the entire interval \( I \) on which \( f \) is continuous (and are of course of class \( C^n \) there).

The fact that \( L \) has constant coefficients allows one to find explicit formulas for the solutions of (4.1)-(4.2). We first observe that the complex exponential \( e^{\lambda x} \), \( \lambda \in \mathbb{C} \), is a solution of (4.2) if and only if \( \lambda \) is a root of the characteristic polynomial
\[
p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n. \tag{4.4}
\]
Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C} \) be the roots of \( p(\lambda) \), not necessarily all distinct, each counted with its multiplicity. The polynomial \( p(\lambda) \) factors as
\[
p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).
\]
The differential operator \( L \) can be factored in a similar way as

\[
L = \left( \frac{d}{dx} - \lambda_1 \right) \left( \frac{d}{dx} - \lambda_2 \right) \cdots \left( \frac{d}{dx} - \lambda_n \right),
\]

where the order of composition of the factors is unimportant as they all commute.

The idea is now to use (4.5) to reduce the problem to first-order differential equations. As for \( n = 2 \), we shall not assume, a priori, existence, uniqueness, and extendability of the solutions of a linear initial value problem. We will prove these facts directly, in the case of constant coefficients, by obtaining explicit formulas for the solutions through (4.5).

The following result gives a particular solution of (4.1) as a convolution integral, and generalizes Theorem 3.1.

**Theorem 4.1.** Let \( f \in C^0(I) \), and suppose that \( 0 \in I \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be \( n \) complex numbers (not necessarily all distinct), and let \( L \) be the differential operator (4.5). Then the initial value problem

\[
\begin{cases}
Ly = f(x) \\
y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0
\end{cases}
\]

has a unique solution, defined on the whole of \( I \), and given by the formula

\[
y(x) = \int_0^x g(x-t)f(t) \, dt \quad (x \in I),
\]

where \( g = g_{\lambda_1 \cdots \lambda_n} \) is the function defined recursively as follows: for \( n = 1 \) we set \( g_{\lambda}(x) = e^{\lambda x} \) (\( \lambda \in \mathbb{C} \)), for \( n \geq 2 \) we set

\[
g_{\lambda_1 \cdots \lambda_n}(x) = \int_0^x g_{\lambda_n}(x-t) g_{\lambda_1 \cdots \lambda_n-1}(t) \, dt \quad (x \in \mathbb{R}).
\]

The function \( g_{\lambda_1 \cdots \lambda_n} \) is of class \( C^\infty \) on the whole of \( \mathbb{R} \). In particular if we take \( f = 0 \), we get that the unique solution of the homogeneous problem

\[
\begin{cases}
Ly = 0 \\
y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0
\end{cases}
\]

is the zero function \( y = 0 \).

**Proof.** We proceed by induction on \( n \). First we prove that \( g_{\lambda_1 \cdots \lambda_n} \in C^\infty(\mathbb{R}) \). Indeed this holds for \( n = 1 \). Suppose it holds for \( n - 1 \). Then it holds also for \( n \), since by (4.8)

\[
g_{\lambda_1 \cdots \lambda_n}(x) = e^{\lambda_n x} \int_0^x e^{-\lambda_n t} g_{\lambda_1 \cdots \lambda_n-1}(t) \, dt,
\]

so \( g_{\lambda_1 \cdots \lambda_n} \) is the product of two functions of class \( C^\infty \) on \( \mathbb{R} \). (The integral function of a \( C^\infty \) function is \( C^\infty \).)

We now prove (4.7). The theorem holds for \( n = 1, 2 \). Assuming it holds for \( n - 1 \), let us prove it for \( n \). Consider then the problem (4.6) with \( L \) given by (4.5):

\[
\begin{cases}
\left( \frac{d}{dx} - \lambda_1 \right) \left( \frac{d}{dx} - \lambda_2 \right) \cdots \left( \frac{d}{dx} - \lambda_n \right) y = f(x) \\
y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0
\end{cases}
\]
Letting \( h = \left( \frac{d}{dx} - \lambda_n \right) y \), we see that \( y \) solves the problem (4.6) if and only if \( h \) solves
\[
\begin{align*}
\begin{cases}
\left( \frac{d}{dx} - \lambda_1 \right) \left( \frac{d}{dx} - \lambda_2 \right) \ldots \left( \frac{d}{dx} - \lambda_{n-1} \right) h &= f(x) \\
h(0) = h'(0) = \ldots = h^{(n-2)}(0) = 0,
\end{cases}
\end{align*}
\]
and \( y \) solves
\[
\begin{align*}
\begin{cases}
y' - \lambda_n y &= h(x) \\
y(0) &= 0.
\end{cases}
\end{align*}
\]
(The initial conditions for \( h = y' - \lambda_n y \) follow by computing \( h', h'', \ldots, h^{(n-2)} \) and setting \( x = 0 \).) By the inductive hypothesis and by (3.7), we have
\[
h(x) = \int_0^x g_{\lambda_1 \ldots \lambda_{n-1}}(x-t)f(t)
dt,
\]
\[
y(x) = \int_0^x e^{\lambda_n(x-t)}h(t)
dt.
\]
Substituting \( h \) from the first formula into the second, we obtain \( \forall x \in I \):
\[
y(x) = \int_0^x g_{\lambda_1}(x-t) \left( \int_0^t g_{\lambda_1 \ldots \lambda_{n-1}}(t-s)f(s)\ ds \right) dt
\]
\[
= \int_0^x \left( \int_s^x g_{\lambda_1}(x-t)g_{\lambda_1 \ldots \lambda_{n-1}}(t-s)\ dt \right) f(s)\ ds
\]
\[
= \int_0^x \left( \int_0^{x-s} g_{\lambda_1}(x-s-t)g_{\lambda_1 \ldots \lambda_{n-1}}(t)\ dt \right) f(s)\ ds
\]
\[
= \int_0^x g_{\lambda_1 \ldots \lambda_n}(x-s)f(s)\ ds.
\]
We have interchanged the order of integration in the second line, substituted \( t \) with \( t + s \) in the third, and used (4.8) in the last.

By induction one can also verify that the function \( g_{\lambda_1 \ldots \lambda_n} \) solves the following homogeneous initial value problem:
\[
\begin{align*}
\begin{cases}
Ly &= 0 \\
y(0) = y'(0) = \ldots = y^{(n-2)}(0) = 0, \quad y^{(n-1)}(0) = 1.
\end{cases}
\end{align*}
\]
(4.10)

Conversely, if \( y \) solves (4.10) then \( y \) is given by (4.8). In other words, the solution of the initial value problem (4.10) is unique and is calculable for \( n \geq 2 \) by the recursive formula (4.8). This is proved by induction, reasoning as in the proof of Theorem 4.1.

For example take \( n = 3 \) and suppose \( y \) solves the problem
\[
\begin{align*}
\begin{cases}
\left( \frac{d}{dx} - \lambda_1 \right) \left( \frac{d}{dx} - \lambda_2 \right) \left( \frac{d}{dx} - \lambda_3 \right) y &= 0 \\
y(0) = y'(0) = 0, \quad y''(0) = 1.
\end{cases}
\end{align*}
\]
Letting \( h = \left( \frac{d}{dx} - \lambda_3 \right) y \), the function \( h \) solves
\[
\begin{align*}
\begin{cases}
\left( \frac{d}{dx} - \lambda_1 \right) \left( \frac{d}{dx} - \lambda_2 \right) h &= 0 \\
h(0) = 0, \quad h'(0) = 1,
\end{cases}
\end{align*}
\]

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whence \( h = g_{\lambda_1\lambda_2} \). Therefore \( y \) solves
\[
\begin{cases}
y' - \lambda_3 y = g_{\lambda_1\lambda_2}(x) \\
y(0) = 0,
\end{cases}
\]
and from (3.7) we get
\[
y(x) = \int_0^x g_{\lambda_3}(x-t) g_{\lambda_1\lambda_2}(t) \, dt = g_{\lambda_1\lambda_2\lambda_3}(x).
\]

The function \( g = g_{\lambda_1\ldots\lambda_n} \) is called the impulsive response of the differential operator \( L \). We will see later how to compute it explicitly using (4.8). Note that as a function of \( \lambda_1, \ldots, \lambda_n \), \( g_{\lambda_1\ldots\lambda_n} \) is symmetric in the interchange \( \lambda_i \leftrightarrow \lambda_j \), \( \forall i, j = 1, \ldots, n \). This follows from the fact that \( g_{\lambda_1\ldots\lambda_n} \) solves the problem (4.10) with \( L \) symmetric in \( \lambda_i \leftrightarrow \lambda_j \), \( \forall i, j \).

**Remark 6.** It is interesting to verify directly that the function \( y \) given by (4.7) solves the non-homogeneous equation \( Ly = f(x) \). Indeed, using repeatedly the formula
\[
\frac{d}{dx} \int_0^x F(x,t) \, dt = F(x,x) + \int_0^x \frac{\partial F}{\partial x}(x,t) \, dt,
\]
we get from (4.7):
\[
\begin{align*}
y'(x) &= g(0)f(x) + \int_0^x g'(x-t)f(t) \, dt = \int_0^x g'(x-t)f(t) \, dt \\
y''(x) &= g'(0)f(x) + \int_0^x g''(x-t)f(t) \, dt = \int_0^x g''(x-t)f(t) \, dt \\
&
\vdots \\
y^{(n-1)}(x) &= g^{(n-2)}(0)f(x) + \int_0^x g^{(n-1)}(x-t)f(t) \, dt = \int_0^x g^{(n-1)}(x-t)f(t) \, dt \\
y^{(n)}(x) &= g^{(n-1)}(0)f(x) + \int_0^x g^{(n)}(x-t)f(t) \, dt = f(x) + \int_0^x g^{(n)}(x-t)f(t) \, dt,
\end{align*}
\]
where we used (4.10). It follows that
\[
(Ly)(x) = f(x) + \int_0^x (Lg)(x-t)f(t) \, dt = f(x), \quad \forall x \in I,
\]
being \( Lg = 0 \). The initial conditions \( y^{(k)}(0) = 0 \) for \( 0 \leq k \leq n-1 \) are immediately verified from these formulas.

For the initial value problem with arbitrary (complex) initial data at \( x = 0 \), we have the following result, that generalizes Theorem 3.2.

**Theorem 4.2.** Let \( f \in C^0(I), 0 \in I \), and let \( b_0, b_1, \ldots, b_{n-1} \) be \( n \) complex numbers. Let \( L \) be the differential operator in (4.3) or (4.5), and let \( g \) be the impulsive response of \( L \). Then the initial value problem
\[
\begin{cases}
Ly = f(x) \\
y(0) = b_0, \ y'(0) = b_1, \ldots, \ y^{(n-1)}(0) = b_{n-1}
\end{cases}
\]

(4.11)
has a unique solution, defined on the whole of \(I\), and given for any \(x \in I\) by

\[
y(x) = \int_0^x g(x-t) f(t) \, dt + c_0 g(x) + c_1 g'(x) + \cdots + c_{n-1} g^{(n-1)}(x),
\]

(4.12)

where

\[
\begin{aligned}
c_0 &= b_{n-1} + a_1 b_{n-2} + \cdots + a_{n-2} b_1 + a_{n-1} b_0 \\
c_1 &= b_{n-2} + a_1 b_{n-3} + \cdots + a_{n-3} b_1 + a_{n-2} b_0 \\
& \vdots \\
c_{n-3} &= b_2 + a_1 b_1 + a_2 b_0 \\
c_{n-2} &= b_1 + a_1 b_0 \\
c_{n-1} &= b_0.
\end{aligned}
\]

(4.13)

In particular (taking \(f = 0\)), the solution of the homogeneous problem

\[
\begin{aligned}
Ly &= 0 \\
y(0) &= b_0, \quad y'(0) = b_1, \ldots, \quad y^{(n-1)}(0) = b_{n-1}
\end{aligned}
\]

is unique, of class \(C^\infty\) on the whole of \(\mathbb{R}\), and is given by

\[
y_h(x) = \sum_{k=0}^{n-1} c_k g^{(k)}(x) \quad (x \in \mathbb{R}).
\]

(4.14)

**Proof.** The uniqueness follows from the fact that if \(y_1, y_2\) both solve the problem (4.11), then their difference \(\tilde{y} = y_1 - y_2\) solves the problem (4.9), whence \(\tilde{y} = 0\) by Theorem 4.1.

To show that \(y(x)\) given by (4.12) satisfies \(Ly(x) = f(x)\), just observe that any derivative \(g^{(k)}\) of \(g\) satisfies the homogeneous equation, since

\[
Lg^{(k)} = L\left( \frac{d}{dx} \right)^k g = \left( \frac{d}{dx} \right)^k Lg = 0.
\]

By Theorem 4.1 and the linearity of \(L\) it follows that

\[
(Ly)(x) = f(x) + L \left( \sum_{k=0}^{n-1} c_k g^{(k)} \right)(x) = f(x) + \sum_{k=0}^{n-1} c_k (Lg^{(k)})(x) = f(x).
\]

There remain to prove the relations (4.13) between the coefficients \(c_k\) and \(b_k = g^{(k)}(0)\) \((0 \leq k \leq n - 1)\). By computing \(y', y'', \ldots, y^{(n-1)}\) from (4.12) and imposing the initial conditions (4.11), we obtain the linear system

\[
\begin{aligned}
c_{n-1} &= b_0 \\
c_{n-2} + c_{n-1} g^{(n)}(0) &= b_1 \\
c_{n-3} + c_{n-2} g^{(n)}(0) + c_{n-1} g^{(n+1)}(0) &= b_2 \\
c_{n-4} + c_{n-3} g^{(n)}(0) + c_{n-2} g^{(n+1)}(0) + c_{n-1} g^{(n+2)}(0) &= b_3 \\
& \vdots \\
c_0 + c_1 g^{(n)}(0) + c_2 g^{(n+1)}(0) + \cdots + c_{n-1} g^{(2n-2)}(0) &= b_{n-1},
\end{aligned}
\]

(4.15)
where we used \( g^{(k)}(0) = 0 \) for \( 0 \leq k \leq n - 2 \), \( g^{(n-1)}(0) = 1 \). The derivatives \( g^{(k)}(0) \) with \( k \geq n \) can be computed from the differential equation \( Lg = 0 \) by isolating \( g^{(n)}(x) \), differentiating and letting \( x = 0 \). One gets

\[
\begin{align*}
g^{(n)}(0) &= -a_1 g^{(n-1)}(0) = -a_1, \\
g^{(n+1)}(0) &= -a_1 g^{(n)}(0) - a_2 g^{(n-1)}(0) = a_1^2 - a_2, \\
g^{(n+2)}(0) &= -a_1 g^{(n+1)}(0) - a_2 g^{(n)}(0) - a_3 g^{(n-1)}(0) = -a_1^3 + 2a_1a_2 - a_3,
\end{align*}
\]

and so on. These relations, substituted in the system (4.15), allow one to solve it recursively and we get exactly the relations (4.13).

We can also proceed more systematically as follows. We rewrite (4.13) in matrix form as

\[
c = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = A \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix},
\]

where

\[
A = \begin{pmatrix}
a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_2 & a_1 & 1 \\
a_{n-2} & a_{n-3} & \cdots & \cdots & a_2 & a_1 & 1 & 0 \\
a_{n-3} & a_{n-4} & \cdots & \cdots & a_2 & a_1 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
a_2 & a_1 & 1 & 0 & \cdots & \cdots & 0 & 0 \\
a_1 & 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
\end{pmatrix},
\]

and the system (4.15) as \( b = Bc \), where

\[
B = \begin{pmatrix}
0 & 0 & 0 & \cdots & \cdots & 0 & 1 \\
0 & 0 & \cdots & \cdots & 0 & 1 & g^{(n)}(0) \\
0 & 0 & \cdots & \cdots & 1 & g^{(n)}(0) & g^{(n+1)}(0) \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 1 & g^{(n)}(0) & \cdots & \cdots & g^{(2n-3)}(0) \\
1 & g^{(n)}(0) & g^{(n+1)}(0) & \cdots & g^{(2n-3)}(0) & g^{(2n-2)}(0) \\
\end{pmatrix}.
\]

Using the equations

\[
Lg = 0, \quad Lg' = 0, \quad Lg'' = 0, \quad \ldots, \quad Lg^{(n-2)} = 0
\]

at \( x = 0 \), it is easily checked that the matrices \( A \) and \( B \) are the inverse of each other. Finally, let us mention that the theorem can also be proved by induction on \( n \), with a constructive proof analogous to that of Theorem 4.1.

The proof of the following result is similar to that of Theorem 3.3.
Theorem 4.3. Let \( f \in C^0(I) \), \( x_0 \in I \), \( b_0, b_1, \ldots, b_{n-1} \in \mathbb{C} \), and let \( L \) and \( g \) be as in Theorem 4.2. The solution of the initial value problem

\[
\begin{aligned}
Ly &= f(x) \\
y(x_0) &= b_0, \ g'(x_0) = b_1, \ldots, \ y^{(n-1)}(x_0) = b_{n-1}
\end{aligned}
\]  

(4.16)

is unique, it is defined on the whole of \( I \), and is given \( \forall x \in I \) by

\[
y(x) = \int_{x_0}^{x} g(x-t)f(t)\,dt + \sum_{k=0}^{n-1} c_k g^{(k)}(x-x_0),
\]

(4.17)

where the coefficients \( c_k \) are given by (4.13). In particular (taking \( f = 0 \)), the solution of the homogeneous problem

\[
\begin{aligned}
Ly &= 0 \\
y(x_0) &= b_0, \ g'(x_0) = b_1, \ldots, \ y^{(n-1)}(x_0) = b_{n-1},
\end{aligned}
\]

(4.16)

with \( x_0 \in \mathbb{R} \) arbitrary, is unique, of class \( C^\infty \) on the whole of \( \mathbb{R} \), and is given by

\[
y_h(x) = \sum_{k=0}^{n-1} c_k g^{(k)}(x-x_0) \quad (x \in \mathbb{R}).
\]

(4.18)

Now let \( y \) be any solution of (4.1) in the interval \( I \ni x_0 \), and let \( b_k = y^{(k)}(x_0) \) \((0 \leq k \leq n-1)\). Then \( y \) solves the problem (4.16), so by uniqueness \( y \) is given by (4.17). We thus obtain the following analogue of Corollary 3.4.

Corollary 4.4. Let \( f \in C^0(I) \) and let \( x_0 \in I \) be fixed. Every solution \( y \) of \( Ly = f(x) \) in the interval \( I \) can be written as \( y = y_p + y_h \), where \( y_p(x) = \int_{x_0}^{x} g(x-t)f(t)\,dt \) solves \( Ly = f(x) \) with the initial conditions \( y_p^{(k)}(x_0) = 0 \) for \( 0 \leq k \leq n-1 \), and \( y_h \) solves \( Ly_h = 0 \) with the same initial conditions as \( y \) at \( x_0 \). The function \( y_h \) is given by (4.18), where the \( c_k \) are given by (4.13) with \( b_k = y^{(k)}(x_0) \) \((0 \leq k \leq n-1)\).

The following result generalizes Corollary 3.5.

Corollary 4.5. The set \( V \) of solutions of the homogeneous equation \( Ly = 0 \) on \( \mathbb{R} \) is a complex vector space of dimension \( n \), with a basis given by \( \{g, g', g'', \ldots, g^{(n-1)}\} \). If \( L \) has real coefficients \( \text{(i.e.,} \ a_1, \ldots, a_n \text{ in (4.3) are all real numbers)} \), then the set \( V_\mathbb{R} \) of real solutions of \( Ly = 0 \) on \( \mathbb{R} \) is a real vector space of dimension \( n \), with a basis given by \( \{g, g', g'', \ldots, g^{(n-1)}\} \).

Proof. From (4.14) and the uniqueness of the homogeneous initial value problem, it follows that every solution of \( Ly = 0 \) on \( \mathbb{R} \) can be written as a linear combination of \( g, g', g'', \ldots, g^{(n-1)} \) in a unique way. (The coefficients \( c_k \) in this combination are uniquely determined by the initial data \( b_k \) at \( x = 0 \) through (4.13).) In particular, this holds for the trivial solution \( y(x) = 0 \) \( \forall x \), i.e., \( c_0 g(x) + \cdots + c_{n-1} g^{(n-1)}(x) = 0 \) for \( c_0, \ldots, c_{n-1} \in \mathbb{C} \) and \( \forall x \in \mathbb{R} \), implies \( c_0 = \cdots = c_{n-1} = 0 \). Thus the \( n \) functions \( g, g', g'', \ldots, g^{(n-1)} \) are linearly independent on \( \mathbb{R} \) and form a basis of \( V \).

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Exercises

1. Verify that the matrices $A$ and $B$ in Theorem 4.2 are the inverse of each other.

2. Prove Theorem 4.3 by imitating the proof of Theorem 3.3.

5 Explicit formulas for the impulsive response

To summarize our results so far, we have seen that the impulsive response $g$, i.e., the solution of the homogeneous problem (4.10), allows one to solve the non-homogeneous equation (4.1) with an arbitrary forcing term and with arbitrary initial conditions.

We now consider the problem of explicitly computing $g$ for an equation of order $n$. By iterating (4.8) we obtain the following formula for the impulsive response of order $n$ as an $(n-1)$-times repeated integral of exponential functions:

$$g_{\lambda_1 \ldots \lambda_n}(x) = \int_0^x e^{\lambda_n(x-t_{n-1})} \left( \int_0^{t_{n-1}} e^{\lambda_{n-1}(t_{n-1}-t_{n-2})} \ldots \right. \left. \left( \int_0^{t_3} e^{\lambda_3(t_3-t_2)} \left( \int_0^{t_2} e^{\lambda_2(t_2-t_1)} e^{\lambda_1 t_1} dt_1 \right) dt_2 \right) \ldots dt_{n-2} \right) dt_{n-1}. \quad (5.1)$$

For $x \geq 0$ this is just the convolution of the $n$ functions $\theta g_{\lambda_j}$ ($1 \leq j \leq n$), whereas for $x \leq 0$ it is the opposite of the convolution of the functions $\theta g_{\lambda_j}$ (see (2.5)):

$$g_{\lambda_1 \ldots \lambda_n}(x) = \begin{cases} \theta g_{\lambda_n} * \theta g_{\lambda_{n-1}} * \ldots * \theta g_{\lambda_2} * \theta g_{\lambda_1}(x) & \text{if } x \geq 0 \\ -\theta g_{\lambda_n} * \theta g_{\lambda_{n-1}} * \ldots * \theta g_{\lambda_2} * \theta g_{\lambda_1}(x) & \text{if } x \leq 0. \end{cases} \quad (5.2)$$

We can also write the repeated integral in (5.1) as an $(n-1)$-dimensional integral

$$g_{\lambda_1 \ldots \lambda_n}(x) = e^{\lambda_n x} \int_{T_x} e^{t_{n-1}(\lambda_{n-1}-\lambda_n)} \ldots e^{t_2(\lambda_2-\lambda_3)} e^{t_1(\lambda_1-\lambda_2)} dt_1 dt_2 \ldots dt_{n-1},$$

where $T_x$ is the $(n-1)$-dimensional simplex of $\mathbb{R}^{n-1}$ defined (for, e.g., $x > 0$) by

$$T_x = \{(t_1, t_2, \ldots, t_{n-1}) \in \mathbb{R}^{n-1} : \ 0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n-1} \leq x\}.$$

For example for $n = 3$, (5.1) becomes

$$g_{\lambda_1 \lambda_2 \lambda_3}(x) = \int_0^x e^{\lambda_3(x-t_2)} \left( \int_0^{t_2} e^{\lambda_2(t_2-t_1)} e^{\lambda_1 t_1} dt_1 \right) dt_2,$$

and we easily obtain the following result.
Corollary 5.1. The impulsive response $g$ of the differential operator

$$L = \left( \frac{d}{dx} \right)^3 + a_1 \left( \frac{d}{dx} \right)^2 + a_2 \frac{d}{dx} + a_3$$

is given as follows. Let $\lambda_1, \lambda_2, \lambda_3$ be the roots of the characteristic polynomial

$$p(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3.$$

1) If $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (all distinct roots), then

$$g(x) = \frac{e^{\lambda_1 x}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{e^{\lambda_2 x}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{e^{\lambda_3 x}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}.$$

2) If $\lambda_1 = \lambda_2 \neq \lambda_3$, then

$$g(x) = \frac{1}{(\lambda_1 - \lambda_3)^2} \left[ e^{\lambda_3 x} - e^{\lambda_1 x} + (\lambda_1 - \lambda_3) x e^{\lambda_3 x} \right].$$

3) If $\lambda_1 = \lambda_2 = \lambda_3$, then

$$g(x) = \frac{1}{2} x^2 e^{\lambda_1 x}.$$

Note that if $a_1, a_2, a_3$ are all real, then $g$ is real-valued (as it must be). Indeed in case 1), if $\lambda_1 \in \mathbb{R}$ and $\lambda_2, \lambda_3 = \alpha \pm i\beta$ with $\beta \neq 0$, one computes

$$g(x) = \frac{1}{(\lambda_1 - \alpha)^2 + \beta^2} \left[ e^{\lambda_1 x} + \frac{1}{\beta} (\alpha - \lambda_1) e^{\alpha x} \sin \beta x - e^{\alpha x} \cos \beta x \right].$$

In all other cases $\lambda_1, \lambda_2, \lambda_3$ are real if $a_1, a_2, a_3 \in \mathbb{R}$.

When the roots of $p(\lambda)$ are all distinct or all equal, there is a fairly simple expression for $g$.

Proposition 5.2. For generic $n$, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of the characteristic polynomial $p(\lambda)$ in (4.4).

1) If $\lambda_i \neq \lambda_j$ for $i \neq j$ (all distinct roots), then the impulsive response is

$$g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x},$$

where

$$c_j = \frac{1}{\prod_{i \neq j} (\lambda_j - \lambda_i)} = \frac{1}{p'(\lambda_j)} \quad (1 \leq j \leq n).$$

2) If $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, then

$$g(x) = \frac{1}{(n - 1)!} x^{n-1} e^{\lambda_1 x}. \quad (5.3)$$

Proof. This is readily proved by induction on $n$ using (4.8). For 1) it is useful to remind that the function $g = g_{\lambda_1 \ldots \lambda_n}$ is symmetric in the interchange $\lambda_i \leftrightarrow \lambda_j, \forall i, j = 1, \ldots, n$. \qed
In the general case we can simplify formula (5.2) as follows. Let \( \lambda_1, \lambda_2, \ldots, \lambda_k \) be the distinct roots of \( p(\lambda) \), of multiplicities \( m_1, m_2, \ldots, m_k \), with \( m_1 + m_2 + \cdots + m_k = n \). Since convolution is commutative and associative, we can group in (5.2) the convolutions relative to each root \( \lambda_j \) \((1 \leq j \leq k)\). We can then rewrite the impulsive response of order \( n \) (for, say, \( x \geq 0 \)) as

\[
g = (\theta g_{\lambda_k} \cdots \theta g_{\lambda_1}) (\theta g_{\lambda_{k-1}} \cdots \theta g_{\lambda_1}) \cdots (\theta g_{\lambda_2} \cdots \theta g_{\lambda_1}).
\]

The term \((\theta g_{\lambda_j} \cdots \theta g_{\lambda_1})\) in this formula contains \( m_j \) factors and the convolution is repeated \( m_j - 1 \) times. This term gives (for \( x \geq 0 \)) the impulsive response \( g_{\lambda_j} \) of the differential operator \((\frac{d}{dx} - \lambda_j)^{m_j}\). By (5.2) and (5.3) we have

\[
g_{\lambda_j}(x) = \begin{cases} 
\theta g_{\lambda_j} \cdots \theta g_{\lambda_1}(x) & \text{if } x \geq 0 \\
-\theta g_{\lambda_j} \cdots \theta g_{\lambda_1}(x) & \text{if } x \leq 0
\end{cases}
\]

\[
g(x) = \frac{1}{(m_j - 1)!} x^{m_j-1} e^{\lambda_j x}, \quad \forall x \in \mathbb{R}.
\]

Defining for short

\[
g_{\lambda,m}(x) := \frac{1}{(m - 1)!} x^{m-1} e^{\lambda x} \quad (\lambda \in \mathbb{C}, \ m \in \mathbb{N}^+),
\]

we obtain

\[
g(x) = \begin{cases} 
\theta g_{\lambda_k, m_k} \cdots \theta g_{\lambda_{k-1}, m_{k-1}} \cdots \theta g_{\lambda_1, m_1}(x) & \text{if } x \geq 0 \\
-\theta g_{\lambda_k, m_k} \cdots \theta g_{\lambda_{k-1}, m_{k-1}} \cdots \theta g_{\lambda_1, m_1}(x) & \text{if } x \leq 0
\end{cases}
\]

(5.5)

\[
g(x) = \int_0^x g_{\lambda_k, m_k}(x-t_{k-1}) \left( \int_0^{t_{k-1}} g_{\lambda_{k-1}, m_{k-1}}(t_{k-1}-t_{k-2}) \cdots \\
\cdots \left( \int_0^{t_3} g_{\lambda_3, m_3}(t_3-t_2) \left( \int_0^{t_2} g_{\lambda_2, m_2}(t_2-t_1)g_{\lambda_1, m_1}(t_1) dt_1 \right) dt_2 \right) \cdots dt_{k-2} \right) dt_{k-1}.
\]

This formula is useful for computing \( g \) in the case of multiple roots. In fact, using (5.5), we can prove the following result.

**Theorem 5.3.** There exist polynomials \( G_1, G_2, \ldots, G_k \), of degrees \( m_1 - 1, m_2 - 1, \ldots, m_k - 1 \), respectively, such that

\[
g(x) = G_1(x)e^{\lambda_1 x} + G_2(x)e^{\lambda_2 x} + \cdots + G_k(x)e^{\lambda_k x}.
\]

More precisely, the impulsive response \( g \) for \( k \) distinct roots is given by (5.6), where \( G_1 \) is the polynomial of degree \( m_1 - 1 \) given by (for \( k \geq 2 \))

\[
G_1(x) = \sum_{r_1=0}^{m_1-1} \sum_{r_2=0}^{m_2-1} \sum_{r_3=0}^{m_3-1} \cdots \sum_{r_k=0}^{m_k-1} \frac{(-1)^{m_1-1-r_1}}{r_1!} \frac{(-1)^{m_2-1-r_2}}{r_2!} \frac{(-1)^{m_3-1-r_3}}{r_3!} \cdots \frac{(-1)^{m_k-1-r_k_1}}{r_k!}
\]

\[
\times \left( \frac{m_1+m_2-2-r_1}{m_2-1} \right) \left( \frac{m_3-1+r_1-r_2}{m_3-1} \right) \left( \frac{m_4-1+r_2-r_3}{m_4-1} \right) \cdots \left( \frac{m_k-1+r_{k-2}-r_{k-1}}{m_k-1} \right)
\]

\[
\times \left( \lambda_1 - \lambda_2 \right)^{m_1+m_2-1-r_1} \left( \lambda_1 - \lambda_3 \right)^{m_3+r_1-r_2} \left( \lambda_1 - \lambda_4 \right)^{m_4+r_2-r_3} \cdots \left( \lambda_1 - \lambda_k \right)^{m_k+r_{k-2}-r_{k-1}}.
\]

(5.7)

To get \( G_j, \ j \geq 2 \), just exchange \( \lambda_1 \leftrightarrow \lambda_j \) and \( m_1 \leftrightarrow m_j \) in this formula.

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Proof. The proof of the first statement is by induction on the number \( k \) of distinct roots of \( p(\lambda) \). The result holds for \( k = 1 \), with \( G_1(x) = \frac{x^{m_1-1}}{(m_1-1)!} \) by formula (5.3). For \( k = 2 \) we have from (5.5)

\[
g(x) = \int_0^x g_{\lambda_2,m_2}(x-t)g_{\lambda_1,m_1}(t) \, dt = \frac{e^{\lambda_2x}}{(m_2-1)!(m_1-1)!} \int_0^x (x-t)^{m_2-1}t^{m_1-1}e^{(\lambda_1-\lambda_2)t} \, dt. \quad (5.8)
\]

We need the following lemma, whose proof is postponed at the end.

**Lemma 5.4.** Let \( p, q \in \mathbb{N} \) and \( b, c \in \mathbb{C}, \ b \neq c \). Then, \( \forall x \in \mathbb{R}, \) we have

\[
\frac{1}{p!q!} e^{bx} \int_0^x (x-t)^p t^q e^{(c-b)t} \, dt = P(x, p, q, b-c) e^{bx} + P(x, q, p-c-b) e^{cx}, \quad (5.9)
\]

where \( P(x, p, q, a), \ a \in \mathbb{C} \setminus \{0\}, \) is the polynomial of degree \( p \) in \( x \) given by

\[
P(x, p, q, a) = \sum_{r=0}^p \frac{(-1)^{p-r}}{r!} \left( p + q - r \right) \frac{x^r}{a^{p+q+1-r}}. \quad (5.10)
\]

Suppose this is proved. From (5.8)-(5.10) we immediately get the following formula for the impulsive response \( g \) in the case of two distinct roots \( \lambda_1, \lambda_2 \) of multiplicities \( m_1, m_2 \):

\[
g(x) = G_1(x)e^{\lambda_1 x} + G_2(x)e^{\lambda_2 x}, \quad (5.11)
\]

\[
G_1(x) = \sum_{r=0}^{m_1-1} \frac{(-1)^{m_1-1-r}}{r!} \left( m_1 + m_2 - r - 2 \right) \frac{x^r}{(\lambda_1 - \lambda_2)^{m_1+2-r-1} m_1}, \quad (5.12)
\]

\[
G_2(x) = \sum_{r=0}^{m_2-1} \frac{(-1)^{m_2-1-r}}{r!} \left( m_1 + m_2 - r - 2 \right) \frac{x^r}{(\lambda_2 - \lambda_1)^{m_2+2-r-1} m_2}. \quad (5.13)
\]

Note that \( G_1, G_2 \) have degrees \( m_1 - 1, m_2 - 1 \), respectively, and that \( G_2 \) is obtained from \( G_1 \) by interchanging \( \lambda_1 \leftrightarrow \lambda_2 \) and \( m_1 \leftrightarrow m_2 \), so that \( g \) is symmetric under this interchange (clear from (5.5)). Note also that (5.12) agrees with (5.7) for \( k = 2 \).

The step from \( k - 1 \) to \( k \) in the inductive proof is now similar. Suppose the result holds for \( k - 1 \) roots, and let us prove it holds for \( k \) roots. The inductive hypothesis and (5.5) imply that for \( x \geq 0 \)

\[
\theta g_{\lambda_{k-1},m_{k-1}} \ast \cdots \ast \theta g_{\lambda_1,m_1}(x) = \tilde{G}_1(x)e^{\lambda_1 x} + \cdots + \tilde{G}_{k-1}(x)e^{\lambda_{k-1} x},
\]

for some polynomials \( \tilde{G}_1, \ldots, \tilde{G}_{k-1} \), of degrees \( m_1 - 1, \ldots, m_{k-1} - 1 \), respectively. Adding the root \( \lambda_k \), of multiplicity \( m_k \), and using (5.5), gives the impulsive response

\[
g(x) = \theta g_{\lambda_k,m_k} \ast (\theta g_{\lambda_{k-1},m_{k-1}} \ast \cdots \ast \theta g_{\lambda_1,m_1})(x)
= \frac{e^{\lambda_k x}}{(m_k - 1)!} \int_0^x (x-t)^{m_k-1} \left[ \tilde{G}_1(t)e^{(\lambda_1-\lambda_k)t} + \cdots + \tilde{G}_{k-1}(t)e^{(\lambda_{k-1}-\lambda_k)t} \right] \, dt. \quad (5.14)
\]
The final formula for $g(x)$ holds for any $x \in \mathbb{R}$. By expanding each polynomial $\tilde{G}_j$ in (5.14) in powers of $t$, we obtain integrals of the form (5.9), with $p = m_k - 1$, $0 \leq q \leq m_j - 1$, $b = \lambda_k$, and $c = \lambda_j$ ($1 \leq j \leq k - 1$). Lemma 5.4 implies that $g$ is of the form (5.6), with $G_1, \ldots, G_{k-1}$ of degrees $m_1 - 1, \ldots, m_{k-1} - 1$, respectively, and $G_k$ of degree $\leq m_k - 1$. However, since $g$ must be symmetric under $\lambda_i \leftrightarrow \lambda_j$ and $m_i \leftrightarrow m_j$, $\forall i, j = 1, \ldots, k$ (this follows from (5.5)), the polynomial $G_k$ must actually be of degree $m_k - 1$.

In order to prove formula (5.7), we rewrite (5.9) in convolution form, by observing that the left-hand side of (5.9) is just

$$\int_0^x g_{b,p+1}(x-t)g_{c,q+1}(t)\,dt = \begin{cases} \theta g_{b,p+1} \ast \theta g_{c,q+1}(x) & \text{if } x \geq 0 \\ -\theta g_{b,p+1} \ast \theta g_{c,q+1}(x) & \text{if } x \leq 0 \end{cases}$$

($g_{\lambda,m}$ given by (5.4)). Recalling (5.10), we obtain for $x \geq 0$

$$\theta g_{b,p+1} \ast \theta g_{c,q+1}(x) = \sum_{r=0}^p (-1)^{p-r} \frac{\binom{p+q-r}{q}}{(b-c)^{p+q+1-r}} g_{b,r+1}(x) \quad (\text{for } k = 3).$$

(5.15) The same expression is obtained for $-\theta g_{b,p+1} \ast \theta g_{c,q+1}(x)$ with $x \leq 0$. Note that (5.15) holds only for $b \neq c$. For $b = c$ we have the simpler formula (6.4) below.

Using (5.15) in (5.5) and iterating, it is possible to compute the polynomials $G_j$ for $k$ distinct roots. For example for $k = 3$, by computing $g = (\theta g_{\lambda_1,m_1} \ast \theta g_{\lambda_2,m_2}) \ast \theta g_{\lambda_3,m_3}$ for $x \geq 0$, we get $g = G_1e^{\lambda_1} + G_2e^{\lambda_2} + G_3e^{\lambda_3}$, where

$$G_1(x) = \sum_{r=0}^{m_1-1} \sum_{s=0}^{r} \frac{(-1)^{m_1-1-s}}{s!} \frac{(m_1+m_2-2-r)}{(m_2-1)^{m_1+m_2-r-1}} \frac{(m_3-1+r-s)}{(m_3-1)^{m_3+r-s}} x^s;$$

$$G_2(x) = \sum_{r=0}^{m_2-1} \sum_{s=0}^{r} \frac{(-1)^{m_2-1-s}}{s!} \frac{(m_1+m_2-2-r)}{(m_1-1)^{m_1+m_2-r-1}} \frac{(m_3-1+r-s)}{(m_3-1)^{m_3+r-s}} x^s;$$

$$G_3(x) = \sum_{r=0}^{m_3-1} \sum_{s=0}^{r} \frac{(-1)^{m_1}}{s!} \frac{(m_1+m_2-2-r)}{(m_1-1)^{m_1+m_2-r-1}} \frac{(m_3-1+r-s)}{(m_3-1)^{m_3+r-s}} x^s + \sum_{r=0}^{m_2-1} \sum_{s=0}^{r} \frac{(-1)^{m_2}}{s!} \frac{(m_1+m_2-2-r)}{(m_1-1)^{m_1+m_2-r-1}} \frac{(m_3-1+r-s)}{(m_3-1)^{m_3+r-s}} x^s.$$
In general, using (5.15) in (5.5) \( k - 1 \) times, we obtain precisely formula (5.7) for the polynomial \( G_1 \). Note that the coefficient of \( x^s \) in \( G_1(x) = \sum_{s=0}^{m_1-1} c_s x^s \) is

\[
c_s = \sum_{r_1=s}^{m_1-1} \sum_{r_2=s}^{r_1-1} \cdots \sum_{r_{k-2}=s}^{r_{k-3}-1} \frac{(-1)^{m_1-1-s}}{s!} \times \frac{(m_1+m_2-2-r_1)(m_3-1-r_2) \cdots (m_k-1+r_{k-3}-r_{k-2})}{m_2-1 \cdots m_k-1} \times (\lambda_1-\lambda_2)^{m_1+2-r_1} \cdots (\lambda_1-\lambda_{k-1})^{m_k-1+r_{k-3}-r_{k-2}}.
\]

The last statement of the theorem follows from the symmetry of \( g \) under \( \lambda_i \leftrightarrow \lambda_j \), \( m_i \leftrightarrow m_j \), \( \forall i, j \).

**Proof of Lemma 5.4.** We shall prove that if \( p, q \in \mathbb{N} \) and \( a \in \mathbb{C} \setminus \{0\} \), then

\[
\frac{1}{p!q!} \int_0^x (x-t)^p t^q e^{at} dt = P(x, p, q, -a) + P(x, q, p, a)e^{ax}.
\]

Lemma 5.4 follows immediately from this by letting \( a = c - b \). Set

\[
F(x, a, p, q) = \int_0^x (x-t)^p t^q e^{at} dt.
\]

To compute this observe that the powers of \( t \) in the integral can be traded by corresponding powers of the differential operator \( \frac{\partial}{\partial a} \) differentiating with respect to the parameter \( a \). Indeed by iterating the formula \( \frac{\partial}{\partial a} e^{at} = t e^{at} \), one easily gets the formulas

\[
(\frac{\partial}{\partial a})^q e^{at} = t^q e^{at}, \quad (x - \frac{\partial}{\partial a})^p e^{at} = (x-t)^p e^{at},
\]

\[
(x - \frac{\partial}{\partial a})^p(\frac{\partial}{\partial a})^q e^{at} = (x-t)^p t^q e^{at}.
\]

Using this in \( F \) and taking the derivatives with respect to \( a \) outside the integral, we obtain

\[
F(x, a, p, q) = (x - \frac{\partial}{\partial a})^p(\frac{\partial}{\partial a})^q \int_0^x e^{at} dt
= (x - \frac{\partial}{\partial a})^p(\frac{\partial}{\partial a})^q \left( \frac{e^{ax} - 1}{a} \right).
\]

This formula holds for \( p = 0 \) or \( q = 0 \) as well, the zero power being the identity operator. To compute \( (\frac{\partial}{\partial a})^q \left( \frac{e^{ax} - 1}{a} \right) \) use Leibniz rule

\[
(\frac{\partial}{\partial a})^q(f g) = \sum_{r=0}^q \binom{q}{r} \left[ (\frac{\partial}{\partial a})^r f \right] \left[ (\frac{\partial}{\partial a})^{q-r} g \right],
\]

and the formulas

\[
(\frac{\partial}{\partial a})^r(e^{ax} - 1) = \begin{cases} e^{ax} - 1 & \text{if } r = 0 \\ x^r e^{ax} & \text{if } r \geq 1, \end{cases}
\]

\[
(\frac{\partial}{\partial a})^{q-r} a^{-1} = (-1)^{q-r}(q-r)! a^{-1-(q-r)},
\]

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to get
\[
\left( \frac{\partial}{\partial a} \right)^q \left( \frac{e^ax - 1}{a} \right) = (e^ax - 1)(-1)^q \sum_{r=0}^{q} \frac{q^r}{r!} x^r e^ax (-1)^q x^r (q-r)! \frac{x^r}{a^{q+1-r}}.
\]

Using this in (5.17) gives
\[
F(x, a, p, q) = (x - \frac{\partial}{\partial a})^p \left\{ (-1)^{q+1} \frac{q!}{aq+1} + e^ax \sum_{r=0}^{q} (-1)^q \frac{q!}{r!} \frac{x^r}{a^{q+1-r}} \right\}
= (-1)^{q+1} q! (x - \frac{\partial}{\partial a})^p a^{-(q+1)} + \sum_{r=0}^{q} (-1)^q \frac{q!}{r!} x^r (x - \frac{\partial}{\partial a})^p \left( e^ax a^{-(q+1-r)} \right).
\]

(5.18)

To compute the term \((x - \frac{\partial}{\partial a})^p a^{-(q+1)}\) in this formula, use the binomial expansion
\[
(x - \frac{\partial}{\partial a})^p = \sum_{r=0}^{p} \binom{p}{r} x^r (-\frac{\partial}{\partial a})^{p-r},
\]
and the general formula for the derivatives of inverse powers
\[
(-\frac{\partial}{\partial a})^p a^{-m} = \frac{(p+m-1)!}{(m-1)!} a^{-m-p},
\]

(5.19)

to get
\[
(x - \frac{\partial}{\partial a})^p a^{-(q+1)} = \sum_{r=0}^{p} \binom{p}{r} x^r (-\frac{\partial}{\partial a})^{p-r} a^{-(q+1)} = \sum_{r=0}^{p} \binom{p}{r} \frac{(p+q-r)!}{q!} a^{-(q+1-r)} x^r.
\]

(5.20)

To compute the term \((x - \frac{\partial}{\partial a})^p (e^ax a^{-(q+1-r)})\) in (5.18), observe that
\[
(x - \frac{\partial}{\partial a})^m e^ax = 0, \quad \forall m \geq 1.
\]

It follows that for any function \(g\):
\[
(x - \frac{\partial}{\partial a})(e^ax g) = [(x - \frac{\partial}{\partial a}) e^ax] g - e^ax \frac{\partial}{\partial a} g = -e^ax \frac{\partial}{\partial a} g,
(x - \frac{\partial}{\partial a})^2 (e^ax g) = (x - \frac{\partial}{\partial a})(-e^ax \frac{\partial}{\partial a} g) = e^ax \left( \frac{\partial}{\partial a} \right)^2 g
\]
\[
\vdots
\]
\[
(x - \frac{\partial}{\partial a})^p (e^ax g) = e^ax (-\frac{\partial}{\partial a})^p g.
\]
Using this with \( g(a) = a^{-(q+1-r)} \) and \((5.19)\), we obtain

\[
(x - \frac{\partial}{\partial a})^p \left( e^{ax} a^{-(q+1-r)} \right) = e^{ax} \left( -\frac{\partial}{\partial a} \right)^p a^{-(q+1-r)}
\]

\[
= e^{ax} \frac{(p+q-r)!}{(q-r)!} a^{r-p-q-1}.
\]

Substituting this and \((5.20)\) in \((5.18)\) we finally get

\[
F(x, a, p, q) = (-1)^q \sum_{r=0}^{p} \binom{p}{r} (p + q - r)! a^{r-p-q-1}
\]

\[
+ \sum_{r=0}^{q} (-1)^q \frac{q!}{r!} x^r e^{ax} \frac{(p+q-r)!}{(q-r)!} a^{r-p-q-1}.
\]

This yields precisely \((5.16)\) with \( P \) given by \((5.10)\), as easily seen.

Another way to compute the polynomials \( G_j \) is given by the following result.

**Theorem 5.5.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_k \) be the distinct roots of the characteristic polynomial \((4.4)\), of multiplicities \( m_1, m_2, \ldots, m_k \), and let

\[
\prod_{j=1}^{k} \frac{1}{(\lambda - \lambda_j)^{m_j}} = \sum_{j=1}^{k} \left( \frac{c_{j,1}}{\lambda - \lambda_j} + \frac{c_{j,2}}{(\lambda - \lambda_j)^2} + \cdots + \frac{c_{j,m_j}}{(\lambda - \lambda_j)^{m_j}} \right)
\]

\((5.21)\)

be the partial fraction expansion of \( \frac{1}{p(\lambda)} \) over \( \mathbb{C} \). Then the impulsive response of the differential operator \((4.3)\) is given by \((5.6)\), where

\[
G_j(x) = c_{j,1} x + c_{j,2} x^2 + \frac{c_{j,3} x^3}{2!} + \cdots + c_{j,m_j} \frac{x^{m_j-1}}{(m_j - 1)!} \quad (1 \leq j \leq k).
\]

**Proof.** This theorem can be proved by induction (on \( n \) using \((4.8)\), or on \( k \) using \((5.5)\) and Lemma 5.4). However, a much simpler proof can be given using the Laplace transform or distribution theory. Indeed, one can show that the Laplace transform \( \mathcal{L} \) of the impulsive response \( g \) is precisely the reciprocal of the characteristic polynomial:

\[
(\mathcal{L}g)(\lambda) = \frac{1}{p(\lambda)}.
\]

Decomposing \( 1/p(\lambda) \) according to \((5.21)\) and taking the inverse Laplace transform gives the result (see [4], eq. (21) p. 81). See also [5], pp. 141-142, for another proof using distribution theory in the convolution algebra \( D'_+ \).

We now use the impulsive response to compute the general solution of the homogeneous equation \( Ly = 0 \) in terms of the complex exponentials \( e^{\lambda_j x} \). The following result generalizes Theorem 3.6.

**Theorem 5.6.** Every solution of \( Ly = 0 \) can be written in the form

\[
y_h(x) = P_1(x)e^{\lambda_1 x} + P_2(x)e^{\lambda_2 x} + \cdots + P_k(x)e^{\lambda_k x},
\]

\((5.22)\)

where \( P_j \) \((1 \leq j \leq k)\) is a complex polynomial of degree \( \leq m_j - 1 \). Conversely, any function of this form is a solution of \( Ly = 0 \).
Proof. Let $y_h$ be a solution of $Ly = 0$. By Corollary 4.4, $y_h$ can be written as a linear combination of $g, g', \ldots, g^{(n-1)}$, i.e., it is given by (4.14) for some complex numbers $c_0, c_1, \ldots, c_{n-1}$. Substituting (5.6) in (4.14) we get an expression of the form (5.22), where the degrees of the polynomials $P_j$ can be less than or equal to $m_j - 1$ (depending on the coefficients $c_0, c_1, \ldots, c_{n-1}$).

For the converse, observe that since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct roots of $p(\lambda)$, of multiplicities $m_1, m_2, \ldots, m_k$, $p(\lambda)$ can be factored according to the formula

$$p(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\cdots(\lambda - \lambda_k)^{m_k}.$$ 

Similarly, the differential operator $L$ factors as

$$L = \left(\frac{d}{dx} - \lambda_1\right)^{m_1}\left(\frac{d}{dx} - \lambda_2\right)^{m_2}\cdots\left(\frac{d}{dx} - \lambda_k\right)^{m_k},$$

(5.23)

where the order of composition of the various factors is irrelevant. We now observe that if $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}^+$, then

$$\left(\frac{d}{dx} - \lambda\right)^m y = 0 \iff y(x) = P(x)e^{\lambda x},$$

(5.24)

where $P$ is a complex polynomial of degree $\leq m - 1$. Indeed given a function $y$ and letting $h(x) = y(x)e^{-\lambda x}$, we compute

$$h'(x) = y'(x)e^{-\lambda x} - \lambda y(x)e^{-\lambda x} = \left[\left(\frac{d}{dx} - \lambda\right)y\right](x)e^{-\lambda x},$$

$$h''(x) = \left(y''(x) - 2\lambda y'(x) + \lambda^2 y(x)\right)e^{-\lambda x} = \left[\left(\frac{d}{dx} - \lambda\right)^2 y\right](x)e^{-\lambda x},$$

$$\vdots$$

$$h^{(m)}(x) = \left[\left(\frac{d}{dx} - \lambda\right)^m y\right](x)e^{-\lambda x}.$$

It follows that

$$\left(\frac{d}{dx} - \lambda\right)^m y = 0 \iff h^{(m)} = 0 \iff h \text{ is a polynomial of degree } \leq m - 1,$$

as claimed. Note that the implication from left to right also follows by substituting (5.3) in (4.14), as already seen in the first part of the proof for the case of $k$ distinct roots. By writing now $L$ in the form (5.23) and using the commutativity of the various factors and (5.24), we see that any function of the form (5.22) satisfies $Ly_h = 0$. 

Theorem 5.6 identifies another basis of the vector space $V$ of solutions of $Ly = 0$ on $\mathbb{R}$, namely the well-known basis given explicitly by the $n$ functions

$$e^{\lambda_j x}, \quad xe^{\lambda_j x}, \quad x^2e^{\lambda_j x}, \ldots, \quad x^{m_j - 1}e^{\lambda_j x} \quad (1 \leq j \leq k).$$

Indeed, these functions belong to $V$ (by the converse part of the theorem), and every element of $V$ is a linear combination of them (by (5.22)). Since $V$ has dimension $n$ (by Corollary 4.5), these functions must be linearly independent. As a corollary one gets the following result, which gives a real basis of $V_\mathbb{R}$ when $L$ has real coefficients and generalizes Corollary 3.7.
Corollary 5.7. Let \( L \) have real coefficients, and let
\[
\lambda_1, \lambda_2, \ldots, \lambda_p, \alpha_1 \pm i\beta_1, \alpha_2 \pm i\beta_2, \ldots, \alpha_q \pm i\beta_q
\]
be the distinct roots of the characteristic polynomial \( p(\lambda) \), where \( \lambda_j \) (\( 1 \leq j \leq p \)) is a real root of multiplicity \( n_j \), and \( \alpha_j \pm i\beta_j \) (\( 1 \leq j \leq q \)) is a pair of complex conjugate roots both of multiplicity \( m_j \), with
\[
n_1 + n_2 + \cdots + n_p + 2m_1 + 2m_2 + \cdots + 2m_q = n.
\]
Any real-valued solution of the homogeneous equation \( Ly = 0 \) can be written in the form
\[
y(x) = \sum_{j=1}^{p} P_j(x)e^{\lambda_j x} + \sum_{j=1}^{q} e^{\alpha_j x}(Q_j(x)\cos \beta_j x + R_j(x)\sin \beta_j x),
\]
where \( P_j \) (\( 1 \leq j \leq p \)) is a real polynomial of degree \( \leq n_j - 1 \), and \( Q_j, R_j \) (\( 1 \leq j \leq q \)) are real polynomials of degree \( \leq m_j - 1 \). Conversely, any function of this form is a real solution of \( Ly = 0 \).

Proof. A basis of \( V \) is given by the \( n \) functions
\[
e^{\lambda_j x}, \ x e^{\lambda_j x}, \ldots, \ x^{n_j-1} e^{\lambda_j x} \quad (1 \leq j \leq p),
\]
\[
e^{(\alpha_j \pm i\beta_j)x}, \ x e^{(\alpha_j \pm i\beta_j)x}, \ldots, \ x^{m_j-1} e^{(\alpha_j \pm i\beta_j)x} \quad (1 \leq j \leq q).
\]
Using
\[
e^{\alpha_j x} \cos \beta_j x = \frac{e^{(\alpha_j + i\beta_j)x} + e^{(\alpha_j - i\beta_j)x}}{2}, \quad e^{\alpha_j x} \sin \beta_j x = \frac{e^{(\alpha_j + i\beta_j)x} - e^{(\alpha_j - i\beta_j)x}}{2i},
\]
it is easy to verify that the functions
\[
x^k e^{\alpha_j x} \cos \beta_j x, \quad x^k e^{\alpha_j x} \sin \beta_j x \quad (0 \leq k \leq m_j - 1, \ 1 \leq j \leq q)
\]
solve \( Ly = 0 \) and are linearly independent in \( V \). It follows easily that the \( n \) functions
\[
e^{\lambda_j x}, \ x e^{\lambda_j x}, \ldots, \ x^{n_j-1} e^{\lambda_j x} \quad (1 \leq j \leq p),
\]
\[
e^{\alpha_j x} \cos \beta_j x, \ x e^{\alpha_j x} \cos \beta_j x, \ldots, \ x^{m_j-1} e^{\alpha_j x} \cos \beta_j x \quad (1 \leq j \leq q),
\]
\[
e^{\alpha_j x} \sin \beta_j x, \ x e^{\alpha_j x} \sin \beta_j x, \ldots, \ x^{m_j-1} e^{\alpha_j x} \sin \beta_j x \quad (1 \leq j \leq q)
\]
are linearly independent in \( V \) and thus form a real-valued basis of \( V \). But then these functions are linearly independent in \( V_\mathbb{R} \), so since \( \dim V_\mathbb{R} = n \) (by Corollary 4.5), they form a basis of \( V_\mathbb{R} \). We conclude that any real-valued solution of \( Ly = 0 \) is a linear combination of these \( n \) functions with real coefficients, and conversely, any such linear combination is in \( V_\mathbb{R} \). \( \square \)

Example 4. Find the general real solution of the differential equation
\[
y^{(4)} - 2y'' + y = \frac{1}{e^{x^2}}. \quad (5.25)
\]
Solution. The characteristic polynomial \( p(\lambda) = \lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 \) has the roots \( \lambda_1 = 1, \lambda_2 = -1 \), both of multiplicity \( m = 2 \). The general solution of the homogeneous equation is then
\[
y_h(x) = (ax + b)e^x + (cx + d)e^{-x} \quad (a, b, c, d \in \mathbb{R}).
\]
To this we must add any particular solution of (5.25), for example the solution $y_p$ of the initial value problem

$$\begin{cases} y^{(4)} - 2y'' + y = \frac{1}{\cosh x} \\ y(0) = y'(0) = y''(0) = y'''(0) = 0. \end{cases}$$

From (5.11), (5.12) and (5.13) we find the impulsive response

$$g(x) = \left(-\frac{1}{4} + \frac{1}{4}x\right)e^x + \left(\frac{1}{4} + \frac{1}{4}x\right)e^{-x} = \frac{1}{2}x \cosh x - \frac{1}{2} \sinh x.$$

Alternatively, use the partial fraction expansion of $\frac{1}{p(\lambda)}$

$$\frac{1}{(\lambda - 1)^2} = \frac{a}{\lambda - 1} + \frac{b}{(\lambda - 1)^2},$$

compute $a = -1/4$, $b = c = d = 1/4$, and use Theorem 5.5 to get the same result.

The forcing term $\frac{1}{\cosh x}$ is continuous on $\mathbb{R}$. By (4.7) we get $\forall x \in \mathbb{R}$

$$y_p(x) = \frac{1}{2} \int_0^x \frac{(x-t) \cosh(x-t)}{\cosh^2 t} \, dt - \frac{1}{2} \int_0^x \frac{\sinh(x-t)}{\cosh^2 t} \, dt.$$

Now use the formulas $\cosh(x-t) = \cosh x \cosh t - \sinh x \sinh t$, $\sinh(x-t) = \sinh x \cosh t - \cosh x \sinh t$. The following integrals are immediate:

$$\int \frac{1}{\cosh^2 t} \, dt = \tanh t + c, \quad \int \frac{\sinh t}{\cosh^2 t} \, dt = -\frac{1}{2\cosh^2 t} + c.$$

Integrating by parts we compute

$$\int t \frac{1}{\cosh^2 t} \, dt = t \tanh t - \log(\cosh t) + c, \quad \int t \frac{\sinh t}{\cosh^2 t} \, dt = -\frac{t}{2\cosh^2 t} + \frac{1}{2} \tanh t + c.$$

Thus

$$y_p(x) = \frac{1}{2} x \cosh x \log(\cosh x) - \frac{1}{4} x \sinh x.$$

The final result can be written as

$$y(x) = y_h(x) + \frac{1}{2} x \cosh x \log(\cosh x),$$

by noting that the term $-\frac{1}{4} x \sinh x$ is a solution of the homogeneous equation.

**Exercises**

1. Prove Corollary 5.1.

2. Prove Proposition 5.2.

3. Solve the following initial value problems:

   (a) $\begin{cases} y''' - 3y'' + 3y' - y = \frac{e^x}{\cosh^2 x} \\ y(0) = y'(0) = y''(0) = 0. \end{cases}$

   (b) $\begin{cases} y''' - 2y'' - y' + 2y = \frac{e^x}{\cosh^2 x} \\ y(0) = y'(0) = y''(0) = 0. \end{cases}$

   (c) $\begin{cases} y''' + y' = \cot x \\ y(\frac{\pi}{2}) = y'(\frac{\pi}{2}) = y''(\frac{\pi}{2}) = 0. \end{cases}$

   (d) $\begin{cases} y''' - y'' - y' + y = \frac{1}{\cosh^2 x} \\ y(0) = y'(0) = y''(0) = 0. \end{cases}$
4. Find the general real solution of each of the following differential equations:

(a) \( y''' + y = 0 \)
(b) \( y^{(4)} + y = 0 \)
(c) \( y^{(4)} + 2y''' + 2y'' + 2y' + y = 0 \)
(d) \( y^{(5)} + y = 0 \) (remind that \( \cos \frac{\pi}{5} = \frac{\sqrt{5}+1}{2}, \ \sin \frac{\pi}{5} = \frac{1}{4} \sqrt{10 - 2\sqrt{5}} \))
(e) \( y^{(6)} + y = 0 \)
(f) \( y^{(8)} + 8y^{(6)} + 24y^{(4)} + 32y'' + 16y = 0 \).

5. Compute the following analytic function in terms of elementary functions:

\[ y(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots. \]

(Hint: using differentiation term by term, show that the function \( y \) solves the differential equation \( y''' - y = 0 \) with the initial conditions \( y(0) = 1, y'(0) = y''(0) = 0 \).)

6. Prove that if \( k \in \mathbb{N}^+ \) then

\[ \sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!} = \frac{1}{k} \sum_{j=0}^{k-1} e^{\alpha_j x}, \]

where \( \alpha_j \) are the \( k \)-th roots of 1

\[ \alpha_j = \sqrt[k]{1} = e^{i \frac{2\pi j}{k}} \quad (j = 0, 1, \ldots, k-1). \]

(Hint: prove that the function \( y(x) = \sum_{n=0}^{\infty} \frac{x^{kn}}{(kn)!} \) solves the differential equation \( y^{(k)} - y = 0 \) with the initial conditions \( y(0) = 1, y'(0) = y''(0) = \cdots = y^{(k-1)}(0) = 0 \).)

6 The method of undetermined coefficients

The computation of the impulsive response \( g \) and of the convolution integral in (4.7) may be rather laborious, in general. The problem of finding a particular solution of (4.1) can be solved in a more efficient way in the case in which the forcing term \( f \) has a particular form, namely when it is a polynomial, an exponential, or a product of terms of this kind. One looks then for a particular solution which is similar to \( f \). The unknown coefficients in this tentative solution are determined by substitution into the differential equation. The justification of this method, known as the method of undetermined coefficients, and the precise form of the solution to be looked for, are given by the following result and its corollary.

**Theorem 6.1.** The differential equation

\[ Ly = \left[ \left( \frac{d}{dx} \right)^n + a_1 \left( \frac{d}{dx} \right)^{n-1} + \cdots + a_{n-1} \frac{d}{dx} + a_n \right] y = P(x) e^{\lambda_0 x}, \quad (6.1) \]

where \( a_1, \ldots, a_n, \lambda_0 \in \mathbb{C} \) and \( P \) is a complex polynomial of degree \( m \), has a particular solution of the form

\[ y(x) = x^r Q(x) e^{\lambda_0 x}, \quad (6.2) \]
where \( Q \) is a complex polynomial of degree \( m \), and

\[
r = \begin{cases} 
0 & \text{if } p(\lambda_0) \neq 0 \\
\text{multiplicity of } \lambda_0 & \text{if } p(\lambda_0) = 0.
\end{cases}
\]

**Proof.** Let \( g \) be the impulsive response of \( L \). By Theorem 4.1, (6.1) has the particular solution

\[
y_p(x) = \int_0^x g(x-t)P(t)e^{\lambda_0 t} \, dt.
\]

Let \( P(x) = \sum_{j=0}^m c_j x^j \). Recalling (2.5), (5.4) and (5.5), we can rewrite \( y_p \) in convolution form (for, e.g., \( x \geq 0 \)) as

\[
y_p = \sum_{j=0}^m j! c_j \theta g_{\lambda_1, m_1} \ast \cdots \ast \theta g_{\lambda_k, m_k} \ast \theta g_{\lambda_0, j+1}.
\]  

(6.3)

Suppose first \( p(\lambda_0) \neq 0 \). Then the function \( \theta g_{\lambda_1, m_1} \ast \cdots \ast \theta g_{\lambda_k, m_k} \ast \theta g_{\lambda_0, j+1} \) in (6.3) is the impulsive response of the differential operator \( (\frac{d}{dx} - \lambda_1)^{m_1} \ast \cdots \ast (\frac{d}{dx} - \lambda_k)^{m_k} \frac{d}{dx} - \lambda_0 \), with the extra root \( \lambda_0 \) of multiplicity \( j + 1 \) (see (5.5)). By Theorem 5.3 there are polynomials \( G_{j,1}, \ldots, G_{j,k}, G_{j,0} \), of degrees \( m_1 - 1, \ldots, m_k - 1, j \), respectively, such that

\[
\theta g_{\lambda_1, m_1} \ast \cdots \ast \theta g_{\lambda_k, m_k} \ast \theta g_{\lambda_0, j+1} = G_{j,1}(x)e^{\lambda_1 x} + \cdots + G_{j,k}(x)e^{\lambda_k x} + G_{j,0}(x)e^{\lambda_0 x},
\]

for \( x \geq 0 \). The same expression is obtained for \( -\theta g_{\lambda_1, m_1} \ast \cdots \ast \theta g_{\lambda_k, m_k} \ast \theta g_{\lambda_0, j+1} \) when \( x \leq 0 \). The first \( k \) terms in the right-hand side of this formula are solutions of the homogeneous equation \( Ly = 0 \) (by Theorem 5.6). It follows from (6.3) that (6.1) has a particular solution of the form

\[
y(x) = \sum_{j=0}^m j! c_j G_{j,0}(x) e^{\lambda_0 x} := Q(x)e^{\lambda_0 x},
\]

where \( Q \) is a polynomial of degree \( m = \max_{0 \leq j \leq m} j \).

Suppose now \( p(\lambda_0) = 0 \), for example let \( \lambda_0 = \lambda_1 \). We observe that the function \( \theta g_{\lambda_1, m_1} \ast \theta g_{\lambda_1, j+1} \), that occurs in (6.3), is just the impulsive response \( \theta g_{\lambda_1, m_1+j+1} \) of the differential operator \( (\frac{d}{dx} - \lambda_1)^{m_1} (\frac{d}{dx} - \lambda_1)^{j+1} = (\frac{d}{dx} - \lambda_1)^{m_1+j+1} \) (for \( x \geq 0 \)). In other words, we have the general formula

\[
\theta g_{\lambda_1, m_1} \ast \theta g_{\lambda_1, j+1} = \theta g_{\lambda_1, m_1+j+1} \quad (\lambda \in \mathbb{C}, \ p, q \in \mathbb{N}^+),
\]

(6.4)

that can also be proved directly from (5.4) for \( x \geq 0 \). For \( x \leq 0 \) we get the analogous formula \( -\theta g_{\lambda_1, m_1} \ast \theta g_{\lambda_1, j+1} = -\theta g_{\lambda_1, m_1+j+1} \). By (6.3) we get for \( x \geq 0 \)

\[
y_p = \sum_{j=0}^m j! c_j \left( \theta g_{\lambda_1, m_1+j+1} \ast \theta g_{\lambda_2, m_2} \ast \cdots \ast \theta g_{\lambda_k, m_k} \right).
\]

Again the expression in round brackets in this formula is the impulsive response of the differential operator \( (\frac{d}{dx} - \lambda_1)^{m_1+j+1} \cdots (\frac{d}{dx} - \lambda_k)^{m_k} \), which by Theorem 5.3 is of the form

\[
G_{j,1}(x)e^{\lambda_1 x} + G_{j,2}(x)e^{\lambda_2 x} + \cdots + G_{j,k}(x)e^{\lambda_k x},
\]

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for polynomials $G_{j,1}, \ldots, G_{j,k}$, with $\deg G_{j,1} = m_1 + j$, $\deg G_{j,l} = m_l - 1$ ($2 \leq l \leq k$). Since the term $G_{j,k}e^{\lambda_k} + \cdots + G_{j,2}e^{\lambda_2}$ is a solution of $Ly = 0$, we conclude that (6.1) has a particular solution given by

$$y(x) = \sum_{j=0}^{m} j! \, c_j \, G_{j,1}(x) \, e^{\lambda_1 x} := Q_1(x)e^{\lambda_1 x},$$

where $Q_1$ has degree $m_1 + m = \max_{0 \leq j \leq m}(m_1 + j)$. By writing out $Q_1$ as

$$Q_1(x) = (d_0 + d_1 x + \cdots + d_{m_1-1} x^{m_1-1}) + d_{m_1} x^{m_1} + \cdots + d_{m_1+m} x^{m_1+m},$$

we see that the polynomial in round brackets, multiplied by $e^{\lambda_1 x}$, is a solution of $Ly = 0$. Finally, we obtain a particular solution of (6.1) of the form

$$x^{m_1} (d_{m_1} + d_{m_1+1} x + \cdots + d_{m_1+m} x^m) e^{\lambda_0 x} = x^r Q(x)e^{\lambda_0 x},$$

where $\deg Q = m$, and $r = m_1$ is the multiplicity of $\lambda_0$ in $p(\lambda)$.

\hfill $\square$

**Remark 7.** The polynomial $Q$ in (6.2) is uniquely determined by the requirement that $y$ given by (6.2) satisfy (6.1). See, for instance, [3], pp. 97-98.

By applying the theorem with $L$ and $P$ real, and separating out the real and imaginary parts in the complex solution (6.2), we obtain the following real version of the method of undetermined coefficients.

**Corollary 6.2.** If $L$ in (6.1) has real coefficients and $P$ is a real polynomial of degree $m$, then the differential equation $Ly = P(x)e^{\alpha x} \cos \beta x$, where $\alpha, \beta \in \mathbb{R}$, has a particular solution of the form

$$y(x) = x^r e^{\alpha x} \left( Q_1(x) \cos \beta x + Q_2(x) \sin \beta x \right),$$

(6.5)

where $Q_1, Q_2$ are real polynomials of degree $\leq m$ (but at least one of them has degree $m$), and

$$r = \begin{cases} 0 & \text{if } p(\alpha + i\beta) \neq 0 \\ \text{multiplicity of } \alpha + i\beta & \text{if } p(\alpha + i\beta) = 0. \end{cases}$$

A similar result holds for the differential equation $Ly = P(x)e^{\alpha x} \sin \beta x$ ($\beta \neq 0$).

**Proof.** Let $\beta \neq 0$. By letting $\lambda_0 = \alpha + i\beta$ in Theorem 6.1 we see that the equation $Ly = P(x)e^{(\alpha+i\beta)x}$ has a particular solution $y_0$ of the form

$$y_0(x) = x^r Q(x) e^{(\alpha+i\beta)x} = x^r \left( Q_1(x) + iQ_2(x) \right) e^{\alpha x} (\cos \beta x + i \sin \beta x),$$

where $Q_1, Q_2$ are real polynomials of degree $\leq m$ and at least one of them has degree $m$ (since $Q$ has degree $m$). Separating out the real and imaginary parts we get

$$y_0(x) = x^r e^{\alpha x} \left\{ Q_1(x) \cos \beta x - Q_2(x) \sin \beta x + i \left[ Q_2(x) \cos \beta x + Q_1(x) \sin \beta x \right] \right\}.$$
By linearity and the fact that $L$ has real coefficients, it follows that the function
\[ y_1(x) = \text{Re} \, y_e(x) = x^r e^{\alpha x} \left[ Q_1(x) \cos \beta x - Q_2(x) \sin \beta x \right] \]
solves
\[ Ly = P(x) e^{\alpha x} \cos \beta x, \]
whereas the function
\[ y_2(x) = \text{Im} \, y_e(x) = x^r e^{\alpha x} \left[ Q_2(x) \cos \beta x + Q_1(x) \sin \beta x \right] \]
solves
\[ Ly = P(x) e^{\alpha x} \sin \beta x. \]
We thus obtain (6.5) (with $-Q_2$ in place of $Q_2$). Note that if $(Q_1, Q_2)$ are the polynomials in the particular solution (6.5) with forcing term $P(x) e^{\alpha x} \cos \beta x$, then the polynomials $(-Q_2, Q_1)$ give the particular solution with forcing term $P(x) e^{\alpha x} \sin \beta x$. Also note that for $\beta \neq 0$ the polynomial $Q$ is generally complex-valued (i.e., $Q_2 \neq 0$) even though $P$ and $L$ have real coefficients. If $\beta = 0$ the result (6.5) for a particular solution of $Ly = P(x) e^{\alpha x}$ (namely $y(x) = x^r e^{\alpha x} Q_1(x)$ with $Q_1$ of degree $m$) becomes identical with (6.2), the polynomial $Q$ being real-valued if $\lambda_0 \in \mathbb{R}$. \qed

We observe that, for calculational purposes, it is often simpler to use the complex formalism, namely in order to find a particular solution of $Ly = P(x) e^{\alpha x} \cos \beta x$ (resp. $Ly = P(x) e^{\alpha x} \sin \beta x$) it is generally easier to solve $Ly = P(x) e^{(\alpha + i \beta)x}$ by Theorem 6.1, and then take the real (resp. imaginary) part of the complex solution thus obtained.

**Example 5.** Find a particular solution of the differential equation
\[ y'' - 4y' + 5y = x^2 e^{2x} \cos x. \]  

**Solution.** The characteristic polynomial $p(\lambda) = \lambda^2 - 4\lambda + 5$ has roots $\lambda = 2 \pm i$, both of multiplicity 1, and the general real solution of the homogeneous equation is
\[ y_{om}(x) = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x \quad (c_1, c_2 \in \mathbb{R}). \]
As $P(x) = x^2$ is a polynomial of degree 2, Corollary 6.2 implies that (6.6) has a particular solution of the form
\[ y_1(x) = x e^{2x} \left[ (ax^2 + bx + c) \cos x + (dx^2 + cx + f) \sin x \right] \]  
(6.7)
where $a, b, c, d, e, f \in \mathbb{R}$. Substituting (6.7) in (6.6) we find, with a little tedious calculation, $a = c = e = 0$, $b = -f = 1/4$, $d = 1/6$.

Using the complex formalism simplifies the computation, as we now show. Instead of solving (6.6), we solve the complex equation
\[ y'' - 4y' + 5y = x^2 e^{(2+i)x}. \]  
(6.8)
According to Theorem 6.1, this equation has a particular solution of the form
\[ y(x) = x e^{(2+i)x} \left( Ax^3 + Bx + C \right) = e^{(2+i)x} \left( Ax^3 + Bx^2 + Cx \right) \]
where $A, B, C$ are complex constant to be determined. We have
\[ y'(x) = e^{(2+i)x} \left[ (2 + i)Ax^3 + Bx^2 + Cx \right] + 3Ax^2 + 2Bx + C, \]
\[ y''(x) = e^{(2+i)x} \left[ (2 + i)^2 Ax^3 + Bx^2 + Cx \right] + 2(2 + i) \left( 3Ax^2 + 2Bx + C \right) + 6Ax + 2B \].
Substituting in (6.8) we get the equality
\[ e^{(2+i)x} \left\{ \left[ (2 + i)^2 - 4(2 + i) + 5 \right] (Ax^3 + Bx^2 + Cx) + 2(2 + i) \left( 3Ax^2 + 2Bx + C \right) \right. \]
\[ - 4 \left( 3Ax^2 + 2Bx + C \right) + 6Ax + 2B \right\} = x^2 e^{(2+i)x}. \]

Now the term in square brackets vanishes, being equal to \( p(2 + i) = 0 \), and we get

\[ 6iAx^2 + 2(3A + 2iB)x + 2B + 2iC = x^2, \]

whence
\[
\begin{align*}
6iA &= 1 \\
3A + 2iB &= 0 \\
B + iC &= 0
\end{align*}
\Rightarrow \begin{align*}
A &= -i/6 \\
B &= 1/4 \\
C &= i/4.
\end{align*}
\]

It follows that (6.8) has the particular solution
\[
y(x) = xe^{(2+i)x} \left( -\frac{1}{6}ix^2 + \frac{1}{3}x + \frac{1}{4}i \right) \\
= x e^{2x} \left( \cos x + i \sin x \right) \left[ \frac{1}{3}x + i \left( \frac{1}{6} - \frac{1}{6}x^2 \right) \right] \\
= xe^{2x} \left\{ \frac{1}{4}x \cos x + \left( \frac{1}{6}x^2 - \frac{1}{3} \right) \sin x + i \left( \frac{1}{6} - \frac{1}{6}x^2 \right) \cos x + \frac{1}{3}x \sin x \right\}.
\]

Since \( x^2 e^{2x} \cos x = \text{Re} \left( x^2 e^{(2+i)x} \right) \), the real solution sought of (6.6) is precisely
\[
y_1(x) = \text{Re} y(x) = xe^{2x} \left[ \frac{1}{4}x \cos x + \left( \frac{1}{6}x^2 - \frac{1}{3} \right) \sin x \right],
\]
in agreement with the calculation using the real formalism. By Corollary 6.2, it also follows that the function
\[
y_2(x) = \text{Im} y(x) = xe^{2x} \left[ \left( \frac{1}{6} - \frac{1}{6}x^2 \right) \cos x + \frac{1}{3}x \sin x \right]
\]
solves
\[
y'' - 4y' + 5y = \text{Im} \left( x^2 e^{(2+i)x} \right) = x^2 e^{2x} \sin x.
\]

**Exercises**

1. Find a particular solution of the differential equation
\[ y^{(4)} - y = f(x) \]
for each of the following forcing terms:

\[
\begin{align*}
(a) \ f(x) &= e^x \\
(b) \ f(x) &= \cos x \\
(c) \ f(x) &= \sin x \\
(d) \ f(x) &= e^x \cos x \\
(e) \ f(x) &= xe^x \sin x \\
(f) \ f(x) &= x^2 e^x \cos x
\end{align*}
\]

2. Find a particular solution of the differential equation
\[ y''' - 3y'' + 4y' - 2y = x^2 e^x \sin x. \]
References


