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Spherical Harmonic Analysis for Spinors on $\mathbf{H}^n(\mathbb{C})$

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Abstract. Recent results on the harmonic analysis of spinor fields on the complex hyperbolic space $\mathbf{H}^n(\mathbb{C})$ are reviewed. We discuss the action of the invariant differential operators on the Poisson transforms, the theory of spherical functions and the spherical transform. The inversion formula, the Paley–Wiener theorem, and the Plancherel theorem for the spherical transform are obtained by reduction to Jacobi analysis on $L^2(\mathbb{R})$.

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1. Introduction

Scalar harmonic analysis on noncompact Riemannian symmetric spaces G/K is by now well understood. Harmonic analysis on sections of homogeneous vector bundles on G/K is not so well developed.

Let E^τ be the vector bundle over G/K associated to an irreducible representation (K-type) (τ, V_τ) of K. We identify the space of cross-sections of E^τ with the space $\Gamma(G,\tau)$ of V_τ-valued functions on G satisfying $f(xk)=\tau(k^{-1})f(x)$ for all $x\in G$, $k\in K$. We denote by $C^\infty(G,\tau)$, $C_0^\infty(G,\tau)$, and $L^2(G,\tau)$ the elements of $\Gamma(G,\tau)$ that are, respectively, smooth, smooth with compact support, and square integrable with respect to the inner product

$$(f,g)=\int_G \mathrm{d}x\langle f(x),g(x)\rangle.$$

The first approach to the analysis on $L^2(G,\tau)$ is the abstract approach from representation theory. It gives an abstract Plancherel theorem for $L^2(G,\tau)$, i.e., a direct integral decomposition of this space into G-irreducible modules via an appropriate Fourier transform. Let us briefly recall this approach.

Let L and R denote the left and right regular representations of G on $L^2(G)$:

$$L(g)f(x)=f(g^{-1}x),\qquad R(g)f(x)=f(xg)\quad (f\in L^2(G),x,g\in G).$$

We regard G as the homogeneous space $G \times G/\mathrm{diag}(G)$. The Plancherel theorem for $(L \otimes R, L^2(G))$ states that

$$L^2(G) = \int_{\widehat{G}}^{\oplus} H_\pi \otimes H_{\pi^*} \, \mathrm{d}\nu(\pi),$$

where $\widehat{G}$ stands for the unitary dual of G, ν is the Plancherel measure supported on the set of tempered representations, H_π is the Hilbert space of the representation π, and π^* is the contragredient representation of π. Now the space $L^2(G, \tau)$ can naturally be identified with the subspace $(L^2(G) \otimes V_\tau)^{(R\otimes\tau)(K)}$ of $(R \otimes \tau)(K)$-invariant vectors in $L^2(G) \otimes V_\tau$. This gives the decomposition

$$\begin{aligned} L^2(G,\tau) &= \int_{\widehat{G}}^{\oplus} H_\pi \otimes (H_{\pi^*} \otimes V_\tau)^{(\pi^*\otimes\tau)(K)} \, \mathrm{d}\nu(\pi) \\ &= \int_{\widehat{G}}^{\oplus} H_\pi \otimes \mathrm{Hom}_K(H_\pi, V_\tau) \, \mathrm{d}\nu(\pi), \end{aligned} \tag{1.1}$$

where we have used the classical identification of $(H_{\pi^*} \otimes V_\tau)^{(\pi^*\otimes\tau)(K)}$ with $\mathrm{Hom}_K(H_\pi, V_\tau)$, the space of K-intertwining operators between the K-modules H_π and V_τ. Note that this space is nonzero only if the representation τ occurs in the restriction of π to K (with multiplicity $m_\pi > 0$). In (1.1) G acts on the left by the induced representation $\mathrm{ind}_K^G(\tau) = L$, and on the right on $H_\pi \otimes \mathrm{Hom}_K(H_\pi, V_\tau)$ by $\pi \otimes 1$, so one obtains

$$L = \int_{\widehat{G}}^{\oplus} \pi \otimes 1 \, \mathrm{d}\nu(\pi).$$

This decomposition of $L^2(G, \tau)$ is implemented by the following (group-theoretic) Fourier transform (see, e.g., [2]). For $f \in C_0^\infty(G, \tau)$ and $T \in \mathrm{Hom}_K(V_\tau, H_\pi)$, define

$$\mathcal{H}_\pi(f)(T) = \int_G \pi(x)\, T f(x) \, \mathrm{d}x \in H_\pi.$$

The notation $\tilde{f}(\pi)(T)$ in place of $\mathcal{H}_\pi(f)(T)$ is also used. Note that $\mathcal{H}_\pi(f)$ can be viewed as an element of $H_\pi \otimes \mathrm{Hom}_K(V_\tau, H_\pi)^* \simeq H_\pi \otimes \mathrm{Hom}_K(H_\pi, V_\tau)$, so that the mapping $f \to \mathcal{H}_\pi(f)$ maps indeed $C_0^\infty(G, \tau) \subset L^2(G, \tau)$ into the space $H_\pi \otimes \mathrm{Hom}_K(H_\pi, V_\tau)$. It is easy to check that this map is G-equivariant. If $\{T_j\}_{j=1}^{m_\pi}$ is an orthonormal basis of $\mathrm{Hom}_K(V_\tau, H_\pi)$ with respect to the scalar product

$$\langle T_1, T_2 \rangle = \frac{1}{\dim \tau} \mathrm{Tr}(T_1 T_2^*),$$

with dual basis $\{T_j^*\}_{j=1}^{m_\pi}$, we can expand

$$\mathcal{H}_\pi(f) = \sum_j \mathcal{H}_\pi(f)(T_j) \otimes T_j^*,$$

and the norm squared of $\mathcal{H}_\pi(f)$ is

$$\|\mathcal{H}_\pi(f)\|^2 = \sum_j \|\mathcal{H}_\pi(f)(T_j)\|^2_{H_\pi}.$$

Then we get the following Plancherel and inversion formulas (see, e.g., [2, 5, 14]): for all $f \in C_0^\infty(G,\tau)$ we have

$$\begin{aligned}
\|f\|^2_{L^2(G,\tau)} &= \frac{1}{d_\tau}\int_{\widehat{G}} \|\mathcal{H}_\pi(f)\|^2 \,\mathrm{d}\nu(\pi),\\
f(x) &= \frac{1}{d_\tau}\int_{\widehat{G}} \sum_j T_j^* \pi(x^{-1})\mathcal{H}_\pi(f)(T_j)\,\mathrm{d}\nu(\pi)\\
&= \frac{1}{d_\tau}\int_{\widehat{G}} (f * \varphi_\pi^\tau)(x)\,\mathrm{d}\nu(\pi) \quad (x \in G),
\end{aligned}$$

where $d_\tau = \dim\tau$, φ_π^τ is the $\mathrm{End}(V_\tau)$-valued τ-spherical function given by

$$\varphi_\pi^\tau(x) = \sum_j T_j^* \pi(x^{-1}) T_j,$$

and the convolution is defined by

$$(f * \varphi_\pi^\tau)(x) = \int_G \varphi_\pi^\tau(y^{-1}x) f(y)\,\mathrm{d}y.$$

The second approach to the analysis on vector bundles uses the theory of τ-spherical functions on G to develop spherical harmonic analysis on G/K, i.e., the spherical transform theory of τ-radial functions on G. A function $F\colon G \to \mathrm{End}(V_\tau)$ is called *τ-radial* if

$$F(k_1 x k_2) = \tau(k_2^{-1}) F(x) \tau(k_1^{-1}), \quad \forall x \in G, \forall k_1, k_2 \in K. \tag{1.2}$$

For each τ-radial function F and for v in V_τ, the vector-valued function $x \to F(x)v$ is in $\Gamma(G,\tau)$ and is called a *radial* cross-section of E^τ. Thus, τ-radial functions generalize to vector bundles the notion of radial (i.e., K-invariant) scalar functions on G.

We denote by $C^\infty(G,\tau,\tau)$, $C_0^\infty(G,\tau,\tau)$, and $L^2(G,\tau,\tau)$ the spaces of τ-radial functions that are respectively smooth, smooth with compact support, and square integrable with respect to the inner product

$$\langle F_1, F_2 \rangle = \int_G \mathrm{d}x\, \mathrm{tr}\big[F_1(x) F_2(x)^*\big],$$

where $*$ denotes adjoint. The space $C_0^\infty(G,\tau,\tau)$ becomes a convolution algebra under the convolution product

$$(F_1 * F_2)(x) = \int_G \mathrm{d}y\, F_1(y^{-1}x) F_2(y).$$

This algebra may or may not be commutative, depending on the group G and on the K-type under consideration. A formal theory of the spherical transform on $C_0^\infty(G,\tau,\tau)$ can be developed both in the commutative [4, 14] and noncommutative [5] settings, in analogy with the Fourier transform theory outlined above for the nonradial case.

If G/K is a noncompact symmetric space of rank one, i.e., a hyperbolic space $\mathbf{H}^n(\mathbb{F})$ ($\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$) or the Cayley plane $\mathbf{H}^2(\mathbb{O})$, we can make this approach more explicit using the theory developed in the 70's by Flensted–Jensen and Koornwinder for the scalar case (Jacobi functions and analysis on rank-one symmetric spaces, see [13] for a survey). Namely, scalar spherical functions on G/K of rank one can be realized as Jacobi functions, and the spherical transform of radial functions coincides with a Jacobi transform.

For homogeneous vector bundles over hyperbolic spaces the link between spherical harmonic analysis and Jacobi analysis persists, although the theory becomes considerably more complicated. This analytic approach makes it possible to obtain not only the inversion and Plancherel formulas for the spherical transform, but also the concrete Paley–Wiener theorem. The image of $C_0^\infty(G,\tau,\tau)$ under spherical transformation is known, in principle, by the work of Campoli [3] in the rank one case, and Arthur [1] in the general case. To state concretely the so-called Arthur–Campoli relations requires, however, a complete knowledge of all the irreducible subquotients of the principal series of G, which is not available yet. Therefore a case-by-case discussion is needed, based on a specific choice of the vector bundle E^τ.

The cases of differential forms and Dirac spinors on $\mathbf{H}^n(\mathbb{R})$ were considered in [14, 7], and the case of differential forms on $\mathbf{H}^n(\mathbb{C})$ was studied in [15]. The case of vector bundles E^τ over $\mathbf{H}^n(\mathbb{H})$ associated with irreducible representations τ of $K=\mathrm{Sp}(n)\times\mathrm{Sp}(1)$ which are trivial on $\mathrm{Sp}(n)$ was discussed in [9].

In this paper we review the results obtained recently in [6] for the case of Dirac spinors on the complex hyperbolic space $\mathbf{H}^n(\mathbb{C})$.

The material will be organized as follows. In Section 2 we discuss spinors on $\mathbf{H}^n(\mathbb{C})$. The spinor bundle is realized concretely as an exterior bundle constructed from a maximal isotropic subspace V of the complexification $\mathfrak{p}^{\mathbb{C}}$ of $\mathfrak{p}\simeq T_o(G/K)$. Here the groups G and K are respectively $\mathrm{SU}(n,1)$ and $\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$, for n odd, and two double covers of these groups, for n even. The isotropy group K acts irreducibly on the homogeneous component $\bigwedge^j(V)$ of the exterior algebra $\bigwedge(V)$ by the K-type $\tau_j=\tau_0\otimes\bigwedge^j(\mathrm{Ad}|_V)$, where τ_0 is a one-dimensional K-type. The Dirac operator D can then be written as $D=\mathrm{d}+\delta$, where d and δ are first-order differential operators acting respectively as raising and lowering operators for the index j. The algebra $\mathbb{D}(G,\tau_j)$ of invariant differential operators on the bundle $C^\infty(G,\tau_j)$ is generated by $\mathrm{d}\delta$ and $\delta\mathrm{d}$.

In Section 3 we identify the continuous part and the discrete part in the Plancherel decomposition for $L^2(G,\tau_j)$, given respectively in [8] and [10]. Then we define the Poisson transform and give its geometric interpretation as a map sending spinors defined on the boundary of $\mathbf{H}^n(\mathbb{C})$ into spinors defined on the interior. The action of D, d, δ, and $\bar{*}$ (the conjugate-linear Hodge operator) on the Poisson transforms is given.

In Section 4 the spinor τ_j-spherical functions Φ are computed in terms of Jacobi functions. We use the radial parts of the differential equations satisfied by Φ, namely $D^2\Phi = \chi_\Phi(D^2)\Phi$, and either $\mathrm{d}\Phi = 0$ or $\delta\Phi = 0$, where χ_Φ is the character of $\mathbb{D}(G, \tau_j)$ corresponding to Φ. We show that the solutions to $D^2\Phi = 0$ are in L^2 if and only if the K-type τ_j equals $\tau_{n/2}$.

In Section 5 we write down the spherical transform of a τ_j-radial function F in terms of suitable Jacobi transforms of the scalar components of F. Then we give the inversion formula, the Paley–Wiener theorem and the Plancherel theorem on $L^2(G, \tau_j, \tau_j)$.

2. Spinors on $\mathbf{H}^n(\mathbb{C})$

The complex hyperbolic space is the manifold

$$\mathbf{H}^n(\mathbb{C}) = \{z \in \mathbb{C}^{n+1} : L(z, z) = -1\}/\mathrm{U}(1),$$

where L is the Hermitian Lorentz form in $\mathbb{C}^{n+1}$

$$L(z, w) = z_1\overline{w_1} + \cdots + z_n\overline{w_n} - z_{n+1}\overline{w_{n+1}}.$$

The group $G = \mathrm{SU}(n, 1)$ acts transitively on $\mathbf{H}^n(\mathbb{C})$ and the isotropy subgroup at the origin $x_0 = (0, \ldots, 0, 1)\mathrm{U}(1)$ is $K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, which is a maximal compact subgroup of G. Thus we may view $\mathbf{H}^n(\mathbb{C})$ as the Riemannian symmetric space of noncompact type G/K. Let $\mathfrak{g} = \mathfrak{su}(n, 1) = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, where

$$\mathfrak{p} = \left\{ p(z) \equiv \begin{pmatrix} 0_n & z \\ {}^t\bar{z} & 0 \end{pmatrix}, z \in \mathbb{C}^n \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} B & 0 \\ 0 & b \end{pmatrix}, B \in \mathfrak{u}(n), b \in \mathfrak{u}(1), b + \mathrm{tr}\, B = 0 \right\}.$$

Let $\{v_j\}_{j=1}^n$ be the canonical basis of $\mathbb{C}^n$. We define the real basis $\{e_j, \tilde{e}_j\}_{j=1}^n$ of $\mathfrak{p} \simeq \mathbb{R}^{2n}$ by $e_j = p(v_j)$, and $\tilde{e}_j = p(i\, v_j)$, where $i = \sqrt{-1}$. This basis is orthonormal with respect to $cB|_{\mathfrak{p}\times\mathfrak{p}}$, where B is the Killing form on $\mathfrak{g} = \mathfrak{su}(n, 1)$, and the normalization constant $c > 0$ is chosen so that

$$cB(X, Y) = \tfrac{1}{2}\mathrm{tr}(XY), \quad \forall X, Y \in \mathfrak{su}(n, 1). \tag{2.1}$$

With this choice the scalar product $\langle\, ,\rangle = cB|_{\mathfrak{p}\times\mathfrak{p}}$ on $\mathfrak{p}$ induces, by left translation, the standard metric of sectional curvature $-4 \leqslant \kappa \leqslant -1$ on $\mathbf{H}^n(\mathbb{C})$. The scalar curvature of this metric is given by $S = -4n(n + 1)$.

We fix the maximal Abelian subspace $\mathfrak{a} = \mathbb{R}e_1 \subset \mathfrak{p}$, and let $A = \exp(\mathfrak{a})$, and M be the centralizer of A in K, with Lie algebra $\mathfrak{m}$ given by

$$\mathfrak{m} = \left\{ \begin{pmatrix} b & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & b \end{pmatrix}, B \in \mathfrak{u}(n-1), b \in \mathfrak{u}(1), 2b + \operatorname{tr} B = 0 \right\}.$$

Let $\mathfrak{h}$ be the diagonal matrices in $\mathfrak{su}(n,1)$, then $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ is a compact Cartan subalgebra. Let $\mathfrak{b} \subset \mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{m}$ consisting of the diagonal elements. For $1 \leqslant j \leqslant n+1$, let ε_j be the linear functional on $\mathfrak{h}^{\mathbb{C}}$ defined by $\varepsilon_j(\operatorname{diag}(h_1, \dots, h_{n+1})) = h_j$. Then $\varepsilon_1 + \dots + \varepsilon_{n+1} = 0$, and each linear functional on $\mathfrak{h}^{\mathbb{C}}$ can be written nonuniquely as $\sum_1^{n+1} c_j \varepsilon_j$. We require $\sum_1^{n+1} c_j = 0$, so that the realization as $\sum c_j \varepsilon_j$ is unique. The restriction of ε_j to $\mathfrak{b}^{\mathbb{C}}$ will be denoted by the same symbol ε_j.

If $H \in \mathfrak{h}^{\mathbb{C}}$ and E_{ij} denotes the $n+1 \times n+1$ matrix with entry 1 at place (i, j), and zero entries elsewhere, we have

$$[H, E_{ij}] = \big(\varepsilon_i(H) - \varepsilon_j(H)\big) E_{ij}.$$

Then the roots $\Delta_{\mathfrak{g}}$, $\Delta_{\mathfrak{k}}$, and $\Delta_{\mathfrak{m}}$ of the pairs $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, and $(\mathfrak{m}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ are the linear functionals $\varepsilon_i - \varepsilon_j$, with $1 \leqslant i \neq j \leqslant n+1$ for $\Delta_{\mathfrak{g}}$, $1 \leqslant i \neq j \leqslant n$ for $\Delta_{\mathfrak{k}}$, and $2 \leqslant i \neq j \leqslant n$ for $\Delta_{\mathfrak{m}}$. All roots are real valued on $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}$ and define members of $\mathfrak{h}_{\mathbb{R}}^*$ by restriction. In the usual lexicographic ordering the positive roots are given by $\varepsilon_i - \varepsilon_j$ with $i < j$. The positive (open) Weyl chamber of $\Delta_{\mathfrak{k}}^+$ in $\mathfrak{h}_{\mathbb{R}}^*$ is

$$\mathfrak{C}_{\mathfrak{k}}^+ = \left\{ \sum_1^{n+1} c_k \varepsilon_k : c_1 > c_2 > \dots > c_n \right\}. \tag{2.2}$$

The positive noncompact roots are given by

$$\Delta_{\mathfrak{p}}^+ = \{\alpha_j = \varepsilon_j - \varepsilon_{n+1}, 1 \leqslant j \leqslant n\}. \tag{2.3}$$

The noncompact roots $\pm\alpha_j$ are the weights of the complexified isotropy representation $k \to \operatorname{Ad}(k)|_{\mathfrak{p}^{\mathbb{C}}}$ of K on $\mathfrak{p}^{\mathbb{C}}$. Set

$$e_j^{\pm} = \frac{e_j \pm i\tilde{e}_j}{\sqrt{2}} \in \mathfrak{p}^{\mathbb{C}}.$$

Then the vectors

$$e_j^+ = \sqrt{2}\, E_{n+1,j} \equiv \sqrt{2}\, E_{-\alpha_j}, \qquad e_j^- = \sqrt{2}\, E_{j,n+1} \equiv \sqrt{2}\, E_{\alpha_j},$$

are root vectors for $\mp\alpha_j$ respectively, i.e., $\operatorname{ad}(H)e_j^{\pm} = \mp\alpha_j(H)e_j^{\pm}$ for all $H \in \mathfrak{h}^{\mathbb{C}}$. We have the decomposition

$$\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-, \qquad \mathfrak{p}^{\pm} = \sum_{j=1}^{n} \mathbb{C} E_{\pm\alpha_j}. \tag{2.4}$$

The (Abelian) subspaces $\mathfrak{p}^{\pm}$ are invariant and irreducible under $\mathrm{Ad}(K)$. Thus $\mathrm{Ad}|_{\mathfrak{p}^{\mathbb{C}}} = \mathrm{Ad}_{+} \oplus \mathrm{Ad}_{-}$, where $\mathrm{Ad}_{\pm} = \mathrm{Ad}|_{\mathfrak{p}^{\pm}}$ have highest weights $\lambda_{+} = \alpha_1$ and $\lambda_{-} = -\alpha_n$. More generally, the representation $\bigwedge^j(\mathrm{Ad}_{+})$ is irreducible with highest weight $\lambda_j = \alpha_1 + \cdots + \alpha_j$.

Now let us introduce the spinor representation of $\mathfrak{k}$. We identify $\mathfrak{p}$ with $\mathbb{R}^{2n}$ by means of the orthonormal basis $\{u_a\}_{a=1}^{2n}$, where $u_{2j-1} = e_j$, $u_{2j} = \tilde{e}_j$ and $1 \leqslant j \leqslant n$. For each $Y \in \mathfrak{k}$ the operator $\alpha(Y) = \mathrm{ad}\, Y|_{\mathfrak{p}}$ is in $\mathfrak{so}(\mathfrak{p}) \simeq \mathfrak{so}(2n)$, and the map $\alpha\colon \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ is a Lie algebra homomorphism. (This is just the differential of the real isotropy representation α of K on $\mathfrak{p}$, $\alpha(k) = \mathrm{Ad}(k)|_{\mathfrak{p}}$.) Let $\mathrm{Cl}(\mathfrak{p}) \simeq \mathrm{Cl}_{2n}$ be the real Clifford algebra of $\mathfrak{p} \simeq \mathbb{R}^{2n}$, and let $\mathrm{Cl}(\mathfrak{p}^{\mathbb{C}}) \simeq \mathrm{Cl}(\mathfrak{p}) \otimes \mathbb{C} \simeq \mathbb{C}l_{2n}$ be its complexification. Let (γ, S) be a space of spinors for $\mathrm{Cl}(\mathfrak{p}^{\mathbb{C}})$, i.e., an irreducible representation γ of $\mathrm{Cl}(\mathfrak{p}^{\mathbb{C}})$ on $S \simeq \mathbb{C}^{2^n} \simeq \bigwedge(\mathbb{C}^n)$. We realize (γ, S) concretely as follows (see [11], Chapter 6).

We denote by β the complex symmetric bilinear form on $\mathfrak{p}^{\mathbb{C}} \simeq \mathbb{C}^{2n}$ given by $\beta = cB|_{\mathfrak{p}^{\mathbb{C}} \times \mathfrak{p}^{\mathbb{C}}}$, and by $\langle\, ,\rangle$ the inner product on $\mathfrak{p}$ given by $cB|_{\mathfrak{p}\times\mathfrak{p}}$ and also the associated Hermitian inner product on $\mathfrak{p}^{\mathbb{C}}$ (cf. (2.1)).

Fix a pair V, V^* of dual maximal isotropic subspaces of $\mathfrak{p}^{\mathbb{C}}$ relative to β. Then V^* can be identified with the dual space of V via the form β, and $\mathfrak{p}^{\mathbb{C}} = V \oplus V^*$. The space of spinors for $\mathrm{Cl}(\mathfrak{p}^{\mathbb{C}})$ can then be realized as the exterior algebra over V:

$$S = \bigwedge(V) = \bigoplus_{j=0}^{n} \bigwedge\nolimits^{j}(V).$$

Since the subspaces $\mathfrak{p}^{\pm}$ in (2.4) are maximal isotropic relative to the Killing form, we choose $V = \mathfrak{p}^{+} = \mathrm{span}\{e_j^-, 1 \leqslant j \leqslant n\}$. (The choice $V = \mathfrak{p}^{-}$ would do as well.) Then $\{e_j^-\}_{j=1}^n$ is a $\langle\, ,\rangle$-orthonormal basis of V, $\{e_j^+\}_{j=1}^n$ is the dual basis in V^*, and the set $\{e_j^+, e_j^-\}$ is a β-isotropic basis of $\mathfrak{p}^{\mathbb{C}}$ (cf. [11], p. 617). Given $v \in V$, we write v^* for the unique element of V^* such that $v^*(w) = \langle w, v\rangle$, $\forall w \in V$. Thus $e_j^+ = (e_j^-)^*$.

Let $\epsilon(v) = v \wedge$ be the operator of exterior left multiplication by $v \in V$ in $\bigwedge(V)$. The adjoint of $\epsilon(v)$ relative to the natural Hermitian inner product on $\bigwedge(V)$ is the interior product $i(v^*)$. Then the linear map $\gamma\colon \mathfrak{p}^{\mathbb{C}} = V \oplus V^* \to \mathrm{End}(S)$ defined by

$$\gamma(v + w^*) = \sqrt{2}\,\epsilon(v) - \sqrt{2}\,i(w^*) \quad (v \in V, w^* \in V^*), \tag{2.5}$$

satisfies the Clifford algebra anticommutation rules

$$\{\gamma(x), \gamma(y)\} = -2\beta(x, y)\mathbf{1}, \quad \forall x, y \in \mathfrak{p}^{\mathbb{C}},$$

and extends to an irreducible representation $\gamma\colon \mathrm{Cl}(\mathfrak{p}^{\mathbb{C}}) \to \mathrm{End}(S)$. The map γ is called Clifford multiplication.

Now let $\mathrm{Spin}(\mathfrak{p}) \simeq \mathrm{Spin}(2n) \subset \mathbb{C}l_{2n}$ be the standard two-fold covering group of $\mathrm{SO}(\mathfrak{p}) \simeq \mathrm{SO}(2n)$. The spinor representation (s, S) of $\mathrm{Spin}(2n)$ is the restriction

$s = \gamma|_{\mathrm{Spin}(2n)}$. We have $s = s_+ \oplus s_-$, where $s_\pm$ are the so called half-spin representations. These are the fundamental representations of $\mathrm{Spin}(2n)$ with highest weights $\frac{1}{2}\lambda_1 + \cdots + \frac{1}{2}\lambda_{n-1} \pm \frac{1}{2}\lambda_n$, where $\{\lambda_j\}$ are the standard functionals on the Cartan subalgebra of $\mathfrak{so}(2n)^{\mathbb{C}}$ ([12], p. 64). The Lie algebra $\mathfrak{so}(2n)$ is isomorphic to $\mathfrak{spin}(2n)$ by the map $E_{ab} - E_{ba} \to -\frac{1}{2}u_a u_b$. The differential of s is given then by $s(E_{ab} - E_{ba}) = -\frac{1}{2}\gamma(u_a)\gamma(u_b)$.

The *spinor representation* τ of $\mathfrak{k}$ is defined by $\tau(Y) = s(\mathrm{ad}\, Y|_{\mathfrak{p}}) = s \circ \alpha(Y)$ ($Y \in \mathfrak{k}$). This is a unitary representation of $\mathfrak{k}$ on the space of spinors S. We have $\tau = \tau_+ \oplus \tau_-$, where $\tau_\pm = s_\pm \circ \alpha$. It is easy to prove that for $H \in \mathfrak{h}^{\mathbb{C}}$ the operator $\tau(H) \in \mathrm{End}(S)$ is given by

$$\tau(H) = \sum_{j=1}^{n} \alpha_j(H)\big(\epsilon(e_j^-)i(e_j^+) - \tfrac{1}{2}\big).$$

This formula implies easily that the weights of τ are all linear combinations

$$\pm\tfrac{1}{2}\alpha_1 \pm \tfrac{1}{2}\alpha_2 \pm \cdots \pm \tfrac{1}{2}\alpha_n, \tag{2.6}$$

with an even number of minus signs for τ_+, and an odd number for τ_-, all with multiplicity one. The weight vectors are given as follows. For a multiindex $I = \{i_1 < \cdots < i_k\} \subset \{1, 2, \dots, n\}$ let $e_I^- = e_{i_1}^- \wedge \cdots \wedge e_{i_k}^-$. Then we have

$$\tau(H)e_I^- = \lambda_I(H)e_I^-, \quad \text{where } \lambda_I = \tfrac{1}{2}\textstyle\sum_{j\in I}\alpha_j - \tfrac{1}{2}\sum_{j\notin I}\alpha_j. \tag{2.7}$$

By (2.2) and (2.3), we see that the only dominant weights of τ are $\delta_0, \delta_1, \dots, \delta_n$, where for $0 \leqslant j \leqslant n$ we set

$$\begin{aligned} \delta_j &= \tfrac{1}{2}\alpha_1 + \cdots + \tfrac{1}{2}\alpha_j - \tfrac{1}{2}\alpha_{j+1} - \cdots - \tfrac{1}{2}\alpha_n \\ &= \tfrac{1}{2}\varepsilon_1 + \cdots + \tfrac{1}{2}\varepsilon_j - \tfrac{1}{2}\varepsilon_{j+1} - \cdots - \tfrac{1}{2}\varepsilon_n + \left(\frac{n}{2} - j\right)\varepsilon_{n+1}. \end{aligned} \tag{2.8}$$

We also see that these weights are $\Delta_{\mathfrak{k}}^+$-extreme, i.e., $\delta_j + \alpha$ is not a weight for all $\alpha \in \Delta_{\mathfrak{k}}^+$. This implies the decomposition

$$\tau = \tau_0 \oplus \tau_1 \oplus \cdots \oplus \tau_n, \tag{2.9}$$

where τ_j is the irreducible representation of $\mathfrak{k}$ with highest weight δ_j, and each τ_j occurs with multiplicity one. The weights of τ_j are precisely all linear combinations of the form (2.6) with j plus signs. Then (2.7) implies that $V_{\tau_j} = \bigwedge^j(V)$. For example the highest weight vector of τ_j is $v_{\delta_j} = e_1^- \wedge \cdots \wedge e_j^-$.

Lifting τ to $K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ requires some care since the weights (2.6) need not be analytically integral (note that K and G are not simply connected). Indeed, it turns out that each δ_j is analytically integral only for n odd [6]. This means that for n odd the isotropy representation α of K lifts to a homomorphism $\tilde{\alpha}$ from K to $\mathrm{Spin}(2n)$, and τ lifts to K by $\tau = s \circ \tilde{\alpha}$.

For n even α does not lift to $\mathrm{Spin}(2n)$ and τ does not lift to K. To get around this problem, we replace K by a suitable double cover $\widetilde{K}$, namely the analytic subgroup of $\mathrm{Spin}(2n)$ with Lie algebra $\mathrm{ad}(\mathfrak{k})|_{\mathfrak{p}} \subseteq \mathfrak{so}(2n) \simeq \mathfrak{spin}(2n)$ (see [12], p. 444). Correspondingly, we also replace G by a double cover $\widetilde{G}$ such that $\widetilde{K} \subset \widetilde{G}$, and regard $\mathbf{H}^n(\mathbb{C})$ as the coset space $\widetilde{G}/\widetilde{K}$. In this way, α lifts to a homomorphism $\tilde{\alpha}$: $\widetilde{K} \to \mathrm{Spin}(2n)$, and τ lifts to $\widetilde{K}$ by $\tau = s \circ \tilde{\alpha}$. The spinor bundle on $\mathbf{H}^n(\mathbb{C})$, n even, is defined then as the homogeneous vector bundle $E^\tau = \widetilde{G} \times_{\widetilde{K}} V_\tau$ associated with the representation τ of $\widetilde{K}$.

Remark. The nonexistence of a spin structure on the compact dual space $U/K = \mathbf{P}^n(\mathbb{C})$ (the complex projective space) for n even can be understood as follows. The group $U = \mathrm{SU}(n+1)$ is now simply connected, thus a double cover of U is isomorphic with U itself. Since U does not contain $\widetilde{K}$ as a subgroup, we cannot write $\mathbf{P}^n(\mathbb{C})$ as $\widetilde{U}/\widetilde{K}$, and we cannot define a homogeneous vector bundle on $\mathbf{P}^n(\mathbb{C})$ associated with τ.

To simplify the notation and unify the even and odd cases, we shall drop the tildes from now on, and write G, K, M, etc. in place of $\widetilde{G}$, $\widetilde{K}$, $\widetilde{M}$, etc. for n even.

The representation τ of K on $V_\tau \simeq \bigwedge(V)$ satisfies the fundamental relation

$$\tau(k)\gamma(X)\tau(k^{-1}) = \gamma(\mathrm{Ad}(k)X), \quad \forall k \in K, \forall X \in \mathfrak{p}^{\mathbb{C}}. \tag{2.10}$$

This implies easily that the K-type τ_j in the decomposition $\tau = \bigoplus_j \tau_j$ is just the tensor product

$$\tau_j = \tau|_{\bigwedge^j(V)} = \tau_0 \otimes \bigwedge^j (\mathrm{Ad}|_V), \quad \forall j = 1, \dots, n,$$

where $k \to \tau_0(k) \in \mathbb{C}$ is the one-dimensional K-type with highest weight $\delta_0 = -\frac{1}{2}\sum_1^n \alpha_k$. To check this, recall that the highest weight of $\bigwedge^j(\mathrm{Ad}|_V)$ is $\lambda_j = \sum_1^j \alpha_k$, so that $\delta_0 + \lambda_j$ equals indeed δ_j, given by (2.8).

The branching rules for $K \supset M$ give the following result for the restriction of τ_j to M (see [8]):

$$\begin{aligned} &\tau_0|_M = \sigma_0, \qquad \tau_n|_M = \sigma_{n-1}, \\ &\tau_j|_M = \sigma_{j-1} \oplus \sigma_j \quad (1 \leqslant j \leqslant n-1), \end{aligned} \tag{2.11}$$

where σ_j is the M-type with highest weight

$$\begin{aligned} \mu_j = {}& \frac{n-1-2j}{4}(\varepsilon_1 + \varepsilon_{n+1}) + \frac{1}{2}\sum_{k=2}^{j+1} \varepsilon_k - \frac{1}{2}\sum_{k=j+2}^{n} \varepsilon_k \\ &(0 \leqslant j \leqslant n-1). \end{aligned} \tag{2.12}$$

For $1 \leqslant j \leqslant n-1$ we write $V_{\tau_j} = V_{\sigma_{j-1}} \oplus V_{\sigma_j}$. In the identification $V_{\tau_j} \simeq \bigwedge^j(V) \simeq \bigwedge^j(\mathbb{C}^n)$, the M-isotypic subspaces $V_{\sigma_{j-1}}$ and V_{σ_j} of V_{τ_j} are given then by

$$\begin{aligned} V_{\sigma_{j-1}} &= e_1^- \wedge \mathrm{span}\{e_{k_2}^- \wedge \cdots \wedge e_{k_j}^-,\ k_r \geqslant 2\} \simeq e_1^- \wedge \textstyle\bigwedge^{j-1}(\mathbb{C}^{n-1}),\\ V_{\sigma_j} &= \mathrm{span}\{e_{k_1}^- \wedge \cdots \wedge e_{k_j}^-,\ k_r \geqslant 2\} \simeq \textstyle\bigwedge^{j}(\mathbb{C}^{n-1}). \end{aligned} \tag{2.13}$$

We now come to the spinor bundle $E^\tau = \bigoplus_{j=0}^n E^{\tau_j}$. Any $f \in \Gamma(G, \tau_j)$ can be written as

$$f(x) = \sum_{k_1<\cdots<k_j} f(x)_{k_1\cdots k_j}\, e_{k_1}^- \wedge \cdots \wedge e_{k_j}^- \equiv \sum_I f_I(x)\, e_I^- \quad (x \in G).$$

The Dirac operator D: $C^\infty(G, \tau) \to C^\infty(G, \tau)$ is defined by

$$Df(x) = \sum_{a=1}^{2n} \gamma(u_a)\, u_a f(x) = \sum_{a=1}^{2n} \gamma(u_a)\, \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 f(x \exp t u_a).$$

It is easy to see that this definition is independent on the orthonormal basis $\{u_a\}$ of $\mathfrak{p}$, and (using (2.10)) that D maps indeed $C^\infty(G, \tau)$ into itself. Using the basis $\{e_k, \tilde{e}_k\}_{k=1}^n$, with

$$e_k = \frac{1}{\sqrt{2}}(e_k^+ + e_k^-) \quad \text{and} \quad \tilde{e}_k = \frac{-i}{\sqrt{2}}(e_k^+ - e_k^-),$$

and recalling (2.5), we get

$$D = \sum_{k=1}^n \big(\gamma(e_k^-) e_k^+ + \gamma(e_k^+) e_k^-\big) = \mathrm{d} + \delta,$$

where d and δ are the differential operators given by

$$\mathrm{d}\, f(x) = \sqrt{2} \sum_{k=1}^n e_k^- \wedge (e_k^+ f)(x),$$

$$\delta\, f(x) = -\sqrt{2} \sum_{k=1}^n i(e_k^+)\, (e_k^- f)(x),$$

for all $f \in C^\infty(G, \tau)$. The following properties of D, d and δ hold [6].

PROPOSITION 2.1. (I) *For all* $j = 0, \ldots, n$, *we have*

$$\begin{aligned} &\mathrm{d}\colon C^\infty(G, \tau_j) \to C^\infty(G, \tau_{j+1}), \qquad \delta\colon C^\infty(G, \tau_j) \to C^\infty(G, \tau_{j-1}),\\ &D\colon C^\infty(G, \tau_j) \to C^\infty(G, \tau_{j-1}) \oplus C^\infty(G, \tau_{j+1}), \end{aligned} \tag{2.14}$$

where $C^\infty(G, \tau_{n+1}) = C^\infty(G, \tau_{-1}) \equiv \{0\}$.

(II) $\mathrm{d}^2 = 0, \delta^2 = 0, \Rightarrow D^2 = \mathrm{d}\delta + \delta\mathrm{d}$.

(III) *We have the formal adjoints* $D^* = D$, $\mathrm{d}^* = \delta$, $\delta^* = \mathrm{d}$.

(IV) *Let* $\mathbb{D}(G, \tau_j)$ *be the algebra of* G*-invariant differential operators on* $C^\infty(G, \tau_j)$. *Then* $\mathbb{D}(G, \tau_0)$ (*resp.* $\mathbb{D}(G, \tau_n)$) *is generated by* $\delta\,\mathrm{d}$ (*resp.* $\mathrm{d}\,\delta$), *and* $\mathbb{D}(G, \tau_j)$ $(1 \leqslant j \leqslant n-1)$ *is generated by* $\mathrm{d}\,\delta$ *and* $\delta\,\mathrm{d}$. *In particular,* $\mathbb{D}(G, \tau_j)$ *is commutative for all* j.

Since the spinor bundle is an exterior bundle, we have a natural notion of duality on it. Let $*$ be the usual Hodge operator, and define the conjugate Hodge operator $\bar{*}$: $V_{\tau_j} \to V_{\tau_{n-j}}$ by $\bar{*}(\beta) = *(\bar{\beta})$, where $\bar{\beta} = \sum \overline{\beta_I}\, e_I^-$ if $\beta = \sum \beta_I\, e_I^- \in V_{\tau_j}$. Now note that the contragredient representation $\check{\tau}_j$ of τ_j is equivalent to τ_{n-j}, for all j. [This can be seen from (2.8) and the classical fact that the highest weight of $\check{\tau}_j$ equals minus the lowest weight of τ_j.] This implies easily that the conjugate-linear map $\bar{*}$ intertwines τ_j and τ_{n-j}, $\bar{*}\,\tau_j(k) = \tau_{n-j}(k)\,\bar{*}$ for all $k \in K$, and extends to a conjugate-linear isomorphism

$$\bar{*}\colon\ C^\infty(G, \tau_j) \to C^\infty(G, \tau_{n-j}),$$

by $(\bar{*}f)(x) = \bar{*}(f(x))$. As in the case of p-forms we have the relations

$$\bar{*}^2 = (-1)^{j(n-j)} \quad \text{on } V_{\tau_j}, \tag{2.15}$$

and

$$\delta = (-1)^{n(j+1)+1}\,\bar{*}\,\mathrm{d}\,\bar{*} \quad \text{on } C^\infty(G, \tau_j). \tag{2.16}$$

3. The Poisson Transform

We keep the notations of Section 2. Thus G and K denote the groups $\mathrm{SU}(n, 1)$ and $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$, for n odd, and two double covers of these groups, for n even.

Recall the maximal Abelian subspace $\mathfrak{a} = \mathbb{R}e_1$ of $\mathfrak{p}$. The corresponding analytic Lie subgroup A of G is parametrized by the elements $a_t = \exp(te_1)$. Let $\alpha \in \mathfrak{a}^*$ be defined by $\alpha(te_1) = t$. Then $\{\pm\alpha, \pm 2\alpha\}$ is a restricted root system of $(\mathfrak{g}, \mathfrak{a})$, with Weyl group $W \simeq \{\pm\mathrm{id}\}$. Hereafter we identify the complex dual $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ by means of $\lambda\alpha \leftrightarrow \lambda$.

Let $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ be the sum of the positive root subspaces, with dimensions $m_\alpha = 2n - 2$ and $m_{2\alpha} = 1$, and let $\rho = \frac{1}{2}(m_\alpha\alpha + m_{2\alpha}2\alpha) = n\alpha$ be the half-sum of the positive roots, counted with multiplicity. Let $N = \exp\mathfrak{n}$, let $G = KAN$ be the corresponding Iwasawa decomposition of G, and write $x = \mathbf{k}(x)\exp[H(x)]n(x)$, for all $x \in G$, where $H(x) \in \mathfrak{a}$, $\mathbf{k}(x) \in K$, and $n(x) \in N$. Let M be the centralizer of A in K.

For $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}$, let $\pi_{\sigma,\lambda}$ denote the principal series $\mathrm{ind}_{MAN}^G(\sigma \otimes \mathrm{e}^{i\lambda} \otimes 1)$ of G, acting on $H_{\sigma,\lambda} \simeq L^2(K, \sigma)$, the space of square integrable V_σ-valued functions on K satisfying $f(km) = \sigma(m^{-1})f(k)$, for all $k \in K$, $m \in M$. For a K-type δ we denote by $\widehat{M}(\delta)$ the set of M-types σ that occur in $\delta|_M$. By (2.11) we have

$$\begin{aligned} &\widehat{M}(\tau_0) = \{\sigma_0\}, \qquad \widehat{M}(\tau_n) = \{\sigma_{n-1}\}, \\ &\widehat{M}(\tau_j) = \{\sigma_{j-1}, \sigma_j\} \quad (1 \leqslant j \leqslant n-1). \end{aligned}$$

By Frobenius Reciprocity, each τ_j occurs in $\pi_{\sigma,\lambda}|_K$ ($\sigma \in \widehat{M}(\tau_j)$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$) with multiplicity one. The continuous part in the Plancherel decomposition of $L^2(G, \tau) =$

$\bigoplus_j L^2(G,\tau_j)$ (i.e., in the right-hand side of (1.1)) can then be worked out using Harish-Chandra's Plancherel theorem for real rank one semisimple Lie groups, and is given by [8, 6]

$$L_c^2(G,\tau) \simeq \bigoplus_{j=0}^{n} \bigoplus_{\sigma\in\widehat{M}(\tau_j)} \int_{\mathbb{R}^+}^{\oplus} H_{\sigma,\lambda}\, p_\sigma(\lambda)\, \mathrm{d}\lambda,$$

where $\mathrm{d}\lambda$ denotes Lebesgue measure on $\mathfrak{a}_+^* \simeq \mathbb{R}^+$ (the positive Weyl chamber in $\mathfrak{a}^*$), and $p_\sigma(\lambda)$ is the Plancherel density associated with $\pi_{\sigma,\lambda}$.

The discrete part $L_d^2(G,\tau_j)$ of $L^2(G,\tau_j)$ can be obtained from the work of Goette and Semmelmann [10]. It is zero for $j \neq n/2$, since in this case τ_j does not occur in the restriction of π to K for any discrete series representation π of G. For $j = n/2$ (n even) the K-type $\tau_{n/2}$ is contained with multiplicity one in the discrete series representation (π, H_π) of G with Harish-Chandra parameter $\delta_{\mathfrak{k}} = \frac{1}{2}\sum_{\alpha\in\Delta_{\mathfrak{k}}^+}\alpha$. (Actually $\tau_{n/2}$ is precisely the minimal K-type of π, see [6].) Thus $L_d^2(G,\tau_{n/2}) = H_\pi$. More generally, Goette and Semmelmann prove that the point spectrum of the Dirac operator on a noncompact (irreducible) Riemannian symmetric space X is nonempty if and only if X is isometric to $\mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$ with $p+q$ odd, and $\mathrm{spec}_p(D) = \{0\}$ in this case.

We now come to the Poisson transform. To ease the notation, we temporarily drop the index j in τ_j, and denote by (τ, V_τ) any of the K-types τ_j $(j = 0,\dots,n)$ in (2.9).

For each $\sigma \in \widehat{M}(\tau)$ and $\lambda \in \mathbb{C}$, the space $\mathrm{Hom}_K(V_\tau, H_{\sigma,\lambda}) \simeq \mathrm{Hom}_K(V_\tau, L^2(K,\sigma))$ is one-dimensional and isomorphic to the space $\mathrm{Hom}_M(V_\tau, V_\sigma)$ of M-intertwining operators from V_τ to V_σ (by Frobenius Reciprocity). Let us think of the representation space V_σ of σ as the σ-isotypic subspace of V_τ. Then the isometric generator J_σ^τ of $\mathrm{Hom}_K(V_\tau, L^2(K,\sigma))$ is given by

$$(J_\sigma^\tau v)(\cdot) = \sqrt{\frac{d_\tau}{d_\sigma}}\, P_\sigma \tau(\cdot)^{-1} v, \quad v \in V_\tau, \tag{3.1}$$

where $P_\sigma \in \mathrm{Hom}_M(V_\tau, V_\sigma)$ is the orthogonal projection from V_τ onto V_σ, and d_τ, d_σ are the dimensions of τ and σ. The adjoint operator $P_\sigma^\tau = (J_\sigma^\tau)^* \in \mathrm{Hom}_K(L^2(K,\sigma), V_\tau)$ is given by

$$P_\sigma^\tau f = \sqrt{\frac{d_\tau}{d_\sigma}} \int_K \mathrm{d}k\, \tau(k) f(k). \tag{3.2}$$

Let $C^\infty(G, \sigma\otimes \mathrm{e}^{i\lambda}\otimes 1)$ denote the space of smooth maps in $H_{\sigma,\lambda}$, i.e., smooth V_σ-valued functions on G satisfying $f(xma_tn) = \mathrm{e}^{-(i\lambda+\rho)t}\sigma(m^{-1})f(x)$, for all $x\in G, m\in M, a_t\in A$, and $n\in N$. The *Poisson transform* on $C^\infty(G,\sigma\otimes e^{i\lambda}\otimes \mathbf{1})$ is the map

$$\mathcal{P}_{\sigma,\lambda}^\tau\colon C^\infty(G,\sigma\otimes \mathrm{e}^{i\lambda}\otimes \mathbf{1}) \to C^\infty(G,\tau)$$

given by

$$\begin{aligned}(\mathcal{P}^{\tau}_{\sigma,\lambda} f)(x) &= P^{\tau}_{\sigma}\, \pi_{\sigma,\lambda}(x^{-1})\, f \\ &= \sqrt{\frac{d_\tau}{d_\sigma}} \int_K \mathrm{d}k\, \mathrm{e}^{-(i\lambda+\rho)(H(xk))}\, \tau(k)\, f(\mathbf{k}(xk)).\end{aligned} \tag{3.3}$$

This map is continuous, linear, and G-equivariant. Its geometric interpretation is as follows. The elements of $C^\infty(G, \sigma \otimes \mathrm{e}^{i\lambda} \otimes \mathbf{1})$ can be considered as cross sections of the homogeneous vector bundle over the boundary G/MAN associated to the representation $\sigma \otimes \mathrm{e}^{i\lambda} \otimes \mathbf{1}$ of MAN. Equivalently, note that if $f \in C^\infty(G, \sigma \otimes \mathrm{e}^{i\lambda} \otimes \mathbf{1})$ then $f|_K \in C^\infty(K, \sigma)$. Therefore for each $\lambda \in \mathbb{C}$, the Poisson transform $\mathcal{P}^{\tau}_{\sigma,\lambda}$ maps cross sections of the homogeneous vector bundle E^σ over the boundary K/M associated to σ into cross sections of the bundle E^τ over G/K.

In the present case the boundary $K/M \simeq \mathrm{S}^{2n-1}$ admits a (unique) spin structure, and one can see that the bundle $E^{\sigma_0} \oplus E^{\sigma_1} \oplus \cdots \oplus E^{\sigma_{n-1}}$ is precisely the spinor bundle on K/M. Letting $\mathcal{P}^{j}_{k,\lambda} = \mathcal{P}^{\tau_j}_{\sigma_k,\lambda}$ (where $k = 0$ for $j = 0$, $k = n-1$ for $j = n$, and $k \in \{j-1, j\}$ for $1 \leqslant j \leqslant n-1$), we see that $\mathcal{P}^{j}_{k,\lambda}$ sends spinors of M-type σ_k on K/M into spinors of K-type τ_j on G/K.

In general the Poisson transforms $\mathcal{P}^{j}_{k,\lambda} f$ are eigenfunctions of the invariant differential operators on $C^\infty(G, \tau_j)$, such as D^2, $\mathrm{d}\delta$ and $\delta\mathrm{d}$. The first-order operators D, d and δ are not in $\mathbb{D}(G, \tau_j)$ since they are G-equivariant (cf. (2.14)). Their action on the Poisson transforms is given by the following result.

PROPOSITION 3.1 ([6], Prop. 7.2). *For all* $f \in C^\infty(G, \sigma_j \otimes \mathrm{e}^{i\lambda} \otimes \mathbf{1})$, *where* $0 \leqslant j \leqslant n-1$ *and* $\lambda \in \mathbb{C}$, *we have*

$$\begin{aligned} &\mathrm{d}\,\mathcal{P}^{j}_{j,\lambda} f = c_j(\lambda)\mathcal{P}^{j+1}_{j,\lambda} f, \quad \delta\,\mathcal{P}^{j}_{j,\lambda} f = 0, \implies D\,\mathcal{P}^{j}_{j,\lambda} f = c_j(\lambda)\mathcal{P}^{j+1}_{j,\lambda} f, \\ &\delta\,\mathcal{P}^{j+1}_{j,\lambda} f = d_j(\lambda)\mathcal{P}^{j}_{j,\lambda} f, \quad \mathrm{d}\,\mathcal{P}^{j+1}_{j,\lambda} f = 0, \implies D\,\mathcal{P}^{j+1}_{j,\lambda} f = d_j(\lambda)\mathcal{P}^{j}_{j,\lambda} f, \end{aligned} \tag{3.4}$$

where the numbers $c_j(\lambda)$, $d_j(\lambda)$ *are given by*

$$\begin{aligned} c_j(\lambda) &= \sqrt{\frac{j+1}{n-j}}\left(\frac{n-1}{2} - j - i\lambda\right), \\ d_j(\lambda) &= \sqrt{\frac{n-j}{j+1}}\left(\frac{n-1}{2} - j + i\lambda\right). \end{aligned} \tag{3.5}$$

Thus the effect of the operators D, d and δ is to relate Poisson transforms with the same M-type and with the correct K-type (consistent with (2.14) and (2.11)).

Let us denote by the same symbol $\bar{*}$ the conjugate-linear mapping on the exterior algebra $\bigwedge(\mathbb{C}^{n-1})$,

$$\bar{*}\colon V_{\sigma_j} \simeq \textstyle\bigwedge^j(\mathbb{C}^{n-1}) \to V_{\sigma_{n-1-j}} \simeq \bigwedge^{n-j-1}(\mathbb{C}^{n-1}),$$

defined as before by $\bar{*}(\beta) = *(\bar{\beta})$ for $\beta \in V_{\sigma_j} \simeq \bigwedge^j(\mathbb{C}^{n-1})$ (cf. (2.13)).

It follows from (2.12) that $\check{\sigma}_j \sim \sigma_{n-j-1}$. This implies that $\bar{*}\,\sigma_j(m) = \sigma_{n-j-1}(m)\,\bar{*}$ for all $m \in M$, and $\bar{*}$ extends to a conjugate-linear isomorphism

$$\bar{*}\colon C^\infty(K, \sigma_j) \to C^\infty(K, \sigma_{n-1-j}),$$

by $(\bar{*} f)(k) = \bar{*}(f(k))$. It can be checked that $\bar{*}$ intertwines $\pi_{\sigma_j,\lambda}$ with $\pi_{\sigma_{n-j-1},-\bar{\lambda}}$, i.e.,

$$\bar{*}\,\pi_{\sigma_j,\lambda}(x) = \pi_{\sigma_{n-j-1},-\bar{\lambda}}(x)\,\bar{*} \quad (x \in G). \tag{3.6}$$

The action of $\bar{*}$ on the Poisson transforms is given then as follows.

PROPOSITION 3.2 ([6], Prop. 7.3). *For all* $f \in C^\infty(G, \sigma_j \otimes \mathrm{e}^{i\lambda} \otimes \mathbf{1})$ $(0 \leqslant j \leqslant n-1,\ \lambda \in \mathbb{C})$, *we have*

$$\bar{*}\,\mathcal{P}^{j}_{j,\lambda} f = (-1)^j\, \mathcal{P}^{n-j}_{n-j-1,-\bar{\lambda}}\,\bar{*} f, \qquad \bar{*}\,\mathcal{P}^{j+1}_{j,\lambda} f = \mathcal{P}^{n-j-1}_{n-j-1,-\bar{\lambda}}\,\bar{*} f. \tag{3.7}$$

The duality allows one to prove only half of the relations (3.4). For example suppose we have proved $\mathrm{d}\,\mathcal{P}^{j+1}_{j,\lambda} f = 0$. Then by (2.16) and (3.7) we see that $\delta\,\mathcal{P}^{j}_{j,\lambda} f$ is proportional to $\bar{*}\mathrm{d}(\mathcal{P}^{n-j}_{n-j-1,-\bar{\lambda}}\bar{*} f) = 0$. Similarly, suppose we have proved $\mathrm{d}\,\mathcal{P}^{j}_{j,\lambda} f = c_j(\lambda)\mathcal{P}^{j+1}_{j,\lambda} f$. Then by (2.15), (2.16) and (3.7) we get

$$\delta\,\mathcal{P}^{j+1}_{j,\lambda} f = -\overline{c_{n-j-1}(-\bar{\lambda})}\,\mathcal{P}^{j}_{j,\lambda} f.$$

This gives the relation $d_j(\lambda) = -\overline{c_{n-j-1}(-\bar{\lambda})}$, which is easily checked from (3.5). By (3.4) we see that the Poisson transforms $\mathcal{P}^{j}_{j,\lambda} f$ and $\mathcal{P}^{j+1}_{j,\lambda} f$ are both eigenfunctions of D^2, with eigenvalue

$$c_j(\lambda) d_j(\lambda) = \lambda^2 + \left(\frac{n-1}{2} - j\right)^2. \tag{3.8}$$

Another way to check this value is to express the iterated Dirac operator D^2 on G/K in terms of the Casimir operator $\Omega_{\mathfrak{g}}$ of $\mathfrak{g}$ by the so-called Parthasarathy formula,

$$D^2 = -\Omega_{\mathfrak{g}} + \frac{S}{8}\mathrm{Id},$$

where S is the scalar curvature of G/K, and then use the explicit formula

$$\pi_{\sigma,\lambda}(-\Omega_{\mathfrak{g}}) = \lambda^2 + \rho^2 - c(\sigma),$$

where $c(\sigma) > 0$ is the Casimir value of σ (see, e.g., [8]).

4. τ-Spherical Functions

Let (τ, V_τ) be any of the K-types τ_j $(j = 0, \ldots, n)$ in (2.9), and consider τ-radial functions on G (cf. (1.2)). Since $\tau|_M$ is multiplicity free, the convolution algebra

$C_0^\infty(G, \tau, \tau)$ is commutative [4, 14]. A function $\Phi \in C^\infty(G, \tau, \tau)$ with $\Phi(e) = \mathrm{Id}$ is called *τ-spherical* if Φ is an eigenfunction for $\mathbb{D}(G, \tau)$, i.e., there is a character χ_Φ of $\mathbb{D}(G, \tau)$ such that

$$D\Phi(\cdot)v = \chi_\Phi(D)\Phi(\cdot)v,$$

for all $D \in \mathbb{D}(G, \tau)$ and all $v \in V_\tau$. This is equivalent to the condition that the map

$$F \to \mathcal{H}_\Phi(F) = \frac{1}{d_\tau}\int_G \mathrm{d}x\, \mathrm{tr}\big[F(x)\Phi(x^{-1})\big] \tag{4.1}$$

is a character of $C_0^\infty(G, \tau, \tau)$. We have two other equivalent characterizations of τ-spherical functions as eigenfunctions with respect to convolution with $C_0^\infty(G, \tau, \tau)$, and as solutions of functional equations [4, 14].

For $\sigma \in \widehat{M}(\tau)$ and $\lambda \in \mathbb{C}$ define $\Phi_\sigma^\tau(\lambda, \cdot) \in C^\infty(G, \tau, \tau)$ by

$$\Phi_\sigma^\tau(\lambda, x) = P_\sigma^\tau \circ \pi_{\sigma,\lambda}(x^{-1}) \circ J_\sigma^\tau \quad (x \in G),$$

where J_σ^τ and P_σ^τ are given by (3.1) and (3.2). It is well known that $\Phi_\sigma^\tau(\lambda, \cdot)$ is τ-spherical for any $\lambda \in \mathbb{C}$. Conversely, every τ-spherical function can be written as $\Phi_\sigma^\tau(\lambda, \cdot)$, for suitable $\lambda \in \mathbb{C}$ and $\sigma \in \widehat{M}(\tau)$. (This is a consequence of Harish-Chandra's Subquotient Theorem and the fact that $\tau|_M$ is multiplicity free.) Since the nontrivial element w of the Weyl group acts trivially on $\widehat{M}$, the functions Φ_σ^τ are all even in λ. Note that for each $v \in V_\tau$, the function $\Phi_\sigma^\tau(\lambda, \cdot)v$ is just the Poisson transform $\Phi_\sigma^\tau(\lambda, \cdot)v = \mathcal{P}_{\sigma,\lambda}^\tau(J_\sigma^\tau v)$ (cf. (3.3)).

Now fix the K-type $\tau = \tau_j$ $(0 \leqslant j \leqslant n)$, and consider the τ_j-spherical functions $\Phi_k^j(\lambda, \cdot) \equiv \Phi_{\sigma_k}^{\tau_j}(\lambda, \cdot)$. Proposition 3.1 and (3.8) imply that they satisfy the following system of differential equations:

$$D^2\Phi_{j-1}^j(\lambda, \cdot) = \left[\lambda^2 + \left(\frac{n+1}{2} - j\right)^2\right]\Phi_{j-1}^j(\lambda, \cdot), \tag{4.2}$$

$$\mathrm{d}\Phi_{j-1}^j(\lambda, \cdot) = 0,$$

$$D^2\Phi_j^j(\lambda, \cdot) = \left[\lambda^2 + \left(\frac{n-1}{2} - j\right)^2\right]\Phi_j^j(\lambda, \cdot), \qquad \delta\Phi_j^j(\lambda, \cdot) = 0, \tag{4.3}$$

for all $j = 1, \ldots, n-1$. For $j = 0, n$ the equations $\mathrm{d}\Phi_{n-1}^n = 0$ and $\delta\Phi_0^0 = 0$ are identically satisfied, and we are left with the second-order equations obtained by letting $j = n$ in (4.2) and $j = 0$ in (4.3).

We now need to take the radial part of these equations. By the Cartan decomposition $G = KAK$, it follows that any τ_j-radial function F is completely determined from its restriction to A. Since $F(a) \in \mathrm{End}_M(V_{\tau_j})$ $\forall a \in A$, Schur's lemma implies that $F(a)$ must be a scalar on each M-isotypic subspace V_σ of V_{τ_j}. Thus for each $\sigma \in \widehat{M}(\tau_j)$ there exists a function $f_\sigma\colon \mathbb{R} \to \mathbb{C}$, such that

$$F(a_t) = \sum\nolimits_{\sigma \in \widehat{M}(\tau_j)}^{\oplus} f_\sigma(t)\mathrm{Id}_{V_\sigma}.$$

The functions f_σ are called the *scalar components* of F. One can prove they are even functions of t, $\forall\sigma \in \widehat{M}(\tau_j)$, $\forall j$. Thus for $j = 0$ and $j = n$ a τ_j-radial function F has only one scalar component, whereas for $1 \leqslant j \leqslant n-1$ it has two scalar components (f_{j-1}, f_j) with $f_k = f_{\sigma_k}$. We call f_{j-1} (resp. f_j) the upper (resp. lower) component.

The following result gives the radial part of the homogeneous equations.

PROPOSITION 4.1 ([6], Prop. 7.5). *Let* $F \in C^\infty(G, \tau_j, \tau_j)$ $(1 \leqslant j \leqslant n-1)$, *with scalar components* (f_{j-1}, f_j). *Then*

$$\mathrm{d}F = 0 \Longleftrightarrow f_{j-1} = \frac{1}{2j}\mathrm{sh}(t)\left[\frac{\mathrm{d}}{\mathrm{d}t} + \left(\frac{n+1}{2} - j\right)\mathrm{th}\,t + 2j\,\mathrm{cth}\,t\right]f_j, \quad (4.4)$$

$$\begin{aligned}\delta F = 0 \Longleftrightarrow {}& f_j \\ &= \frac{1}{2(n-j)}\mathrm{sh}(t)\left[\frac{\mathrm{d}}{\mathrm{d}t} + \left(j - \frac{n-1}{2}\right)\mathrm{th}\,t + 2(n-j)\mathrm{cth}\,t\right]f_{j-1}. \end{aligned}\quad (4.5)$$

To get the radial part of the second-order equations, we use the Parthasarathy formula $D^2 = -\Omega_{\mathfrak{g}} + (S/8)\mathrm{Id}$, and the radial part of the Casimir operator acting on τ-spherical functions. The result is as follows ([6], Props. 4.7 and 4.9).

PROPOSITION 4.2. *The scalar component* $\phi = \phi_0$ *of* Φ_0^0 *satisfies the differential equation*

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + [(2n-1)\,\mathrm{cth}\,t + \mathrm{th}\,t]\frac{\mathrm{d}}{\mathrm{d}t} + \frac{(n+1)^2}{4\,\mathrm{ch}^2\,t} + \lambda^2 + n^2\right\}\phi(\lambda, t) = 0. \quad (4.6)$$

Similarly, the scalar component $\phi = \phi_{n-1}$ *of* Φ_{n-1}^n *satisfies* (4.6).

PROPOSITION 4.3. *The first differential equation in* (4.2) *corresponds to the following system of differential equations for the scalar components* (ϕ_{j-1}, ϕ_j) *of* Φ_{j-1}^j:

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + [(2n-1)\,\mathrm{cth}\,t + \mathrm{th}\,t]\frac{\mathrm{d}}{\mathrm{d}t} + \frac{(n-2j-1)^2}{4\,\mathrm{ch}^2\,t} - \frac{4(n-j)}{\mathrm{sh}^2\,t} + \right.\\ &\left. {}+\lambda^2 + n^2\right\}\phi_{j-1}(\lambda, t) + \frac{4(n-j)\,\mathrm{ch}\,t}{\mathrm{sh}^2\,t}\phi_j(\lambda, t) = 0, \end{aligned}\quad (4.7)$$

$$\begin{aligned}&\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + [(2n-1)\,\mathrm{cth}\,t + \mathrm{th}\,t]\frac{\mathrm{d}}{\mathrm{d}t} + \frac{(n-2j+1)^2}{4\,\mathrm{ch}^2\,t} - \frac{4j}{\mathrm{sh}^2\,t} + \right.\\ &\left. {}+\lambda^2 + n^2 + n - 2j\right\}\phi_j(\lambda, t) + \frac{4j\,\mathrm{ch}\,t}{\mathrm{sh}^2\,t}\phi_{j-1}(\lambda, t) = 0. \end{aligned}\quad (4.8)$$

Similarly, the scalar components $(\tilde{\phi}_{j-1}, \tilde{\phi}_j)$ *of* Φ_j^j *satisfy* (4.7)–(4.8), *with the replacements* $\phi_{j-1} \to \tilde{\phi}_{j-i}$, $\phi_j \to \tilde{\phi}_j$, *and* $\lambda^2 \to \lambda^2 - n + 2j$.

We can now solve these equations in terms of Jacobi functions. For $\alpha, \beta, \lambda \in \mathbb{C}$, $\alpha \neq -1, -2, \ldots$, and $t \in \mathbb{R}$, let $\varphi_\lambda^{(\alpha,\beta)}(t)$ be the Jacobi function defined by

$$\varphi_\lambda^{(\alpha,\beta)}(t) = F\left(\frac{\alpha+\beta+1+i\lambda}{2}, \frac{\alpha+\beta+1-i\lambda}{2}, \alpha+1, -\operatorname{sh}^2 t\right),$$

where $F(a,b,c,z)$ is the hypergeometric function. It is well known (see, e.g., [13]) that $\varphi = \varphi_{\pm\lambda}^{(\alpha,\beta)}$ is the unique C^∞ solution of the equation

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2} + [(2\alpha+1)\operatorname{cth} t + (2\beta+1)\operatorname{th} t]\frac{\mathrm{d}}{\mathrm{d}t} + \lambda^2 + (\alpha+\beta+1)^2\right\}\varphi = 0,$$

which is even and normalized by $\varphi(0) = 1$. From (4.6) we easily get

THEOREM 4.4 ([6], Thm. 4.10). *The scalar components* ϕ_0 *of* Φ_0^0 *and* ϕ_{n-1} *of* Φ_{n-1}^n *are given by*

$$\phi_0(\lambda, t) = \phi_{n-1}(\lambda, t) = (\operatorname{ch} t)^{(n+1)/2}\, \varphi_\lambda^{(n-1,(n+1)/2)}(t).$$

To get the scalar components (ϕ_{j-1}, ϕ_j) of Φ_{j-1}^j we substitute (4.4) in (4.8) and obtain the following result.

THEOREM 4.5 ([6], Thm. 4.13). *The scalar components* (ϕ_{j-1}, ϕ_j) *of* Φ_{j-1}^j $(1 \leqslant j \leqslant n-1)$ *are given by*

$$\phi_{j-1}(\lambda, t) = (\operatorname{ch} t)^{j-\frac{n-1}{2}}\left\{\frac{n}{j}\,\varphi_\lambda^{(n-1,j-\frac{n-1}{2})}(t) - \frac{n-j}{j}\,\varphi_\lambda^{(n,j-\frac{n+1}{2})}(t)\right\}, \tag{4.9}$$

$$\phi_j(\lambda, t) = (\operatorname{ch} t)^{j-\frac{n+1}{2}}\,\varphi_\lambda^{(n,j-\frac{n+1}{2})}(t). \tag{4.10}$$

The scalar components of $\Phi_j^j(\lambda, \cdot)$ can be obtained similarly by substituting (4.5) in (4.7), or, more simply, using the duality. Given $F \in C^\infty(G, \tau_j, \tau_j)$, define the τ_{n-j}-radial function $\widetilde{F}$ by

$$\widetilde{F}(x)\xi = (-1)^{j(n-j)}\,\bar{*}\,F(x)\,\bar{*}\,\xi \quad (\xi \in V_{\tau_{n-j}}).$$

It is easy to see that if F has scalar components (f_{j-1}, f_j), then $\widetilde{F}$ has scalar components $(\overline{f_j}, \overline{f_{j-1}})$. Now let $F = \Phi_j^j(\lambda, \cdot)$. By taking $f = J_{\sigma_j}^{\tau_j}\,\bar{*}\,\beta$ $(\beta \in V_{\tau_{n-j}})$ in the first equation in (3.7), we find

$$\widetilde{\Phi_j^j(\lambda, \cdot)} = \Phi_{n-j-1}^{n-j}(-\bar{\lambda}, \cdot).$$

This implies easily that

$$\tilde{\phi}_{j-1} = \phi_{n-j}, \qquad \tilde{\phi}_j = \phi_{n-j-1},$$

i.e., the upper component of Φ_j^j equals the lower component of Φ_{n-j-1}^{n-j}, and vice-versa. Letting $j \to n-j$ in (4.9)–(4.10) gives

THEOREM 4.6. *The scalar components* $(\tilde{\phi}_{j-1}, \tilde{\phi}_j)$ *of* Φ^j_j $(1 \leqslant j \leqslant n-1)$ *are given by*

$$\tilde{\phi}_{j-1}(\lambda, t) = (\operatorname{ch} t)^{\frac{n-1}{2}-j}\, \varphi_\lambda^{(n, \frac{n-1}{2}-j)}(t),$$

$$\tilde{\phi}_j(\lambda, t) = (\operatorname{ch} t)^{\frac{n+1}{2}-j} \left\{ \frac{n}{n-j}\, \varphi_\lambda^{(n-1, \frac{n+1}{2}-j)}(t) - \frac{j}{n-j}\, \varphi_\lambda^{(n, \frac{n-1}{2}-j)}(t) \right\}.$$

We can now check the existence of harmonic L^2 τ_j-radial spinors. For $\sigma \in \{\sigma_{j-1}, \sigma_j\}$ we have, by (3.8), $\pi_{\sigma,\lambda}(D^2) = 0 \Leftrightarrow \lambda = \pm\lambda_{j-1}, \pm\lambda_j$, where

$$\lambda_{j-1} = i\left(\frac{n+1}{2} - j\right), \qquad \lambda_j = i\left(\frac{n-1}{2} - j\right). \tag{4.11}$$

By Proposition 3.1 it is easily seen that the τ_j-spherical functions $\Phi = \Phi^j_{j-1}(\pm\lambda_{j-1}, \cdot)$ and $\Phi = \Phi^j_j(\pm\lambda_j, \cdot)$ satisfy the same differential equation, namely $\mathrm{d}\Phi = 0$ and $\delta\Phi = 0$, with the same normalization. This implies that

$$\Phi^j_{j-1}(\pm\lambda_{j-1}, \cdot) = \Phi^j_j(\pm\lambda_j, \cdot). \tag{4.12}$$

It is then enough to study the norm squared $\|\Phi^j_{j-1}(\pm\lambda_{j-1}, \cdot)\|^2_{L^2}$. Using the integral formula for the Cartan decomposition, and the explicit expression for the scalar components given above, it can be shown [6] that $\|\Phi^j_{j-1}(\pm\lambda_{j-1}, \cdot)\|^2_{L^2}$ is finite if and only if $j = n/2$. This is in agreement with the identification, made in Section 3, of the discrete part of $L^2(G, \tau_{n/2})$ as the discrete series (π, H_π) with minimal K-type $\tau_{n/2}$.

5. The Spherical Transform

Let $\tau = \tau_j$ $(j = 0, \ldots, n)$. The *spherical transform* of a τ-radial function $F \in C_0^\infty(G, \tau, \tau)$ is the collection of scalar-valued functions $\{\lambda \to \mathcal{H}^\tau_\sigma(F)(\lambda),\ \sigma \in \widehat{M}(\tau)\}$, where

$$\mathcal{H}^\tau_\sigma(F)(\lambda) = \frac{1}{d_\tau} \int_G \mathrm{d}x\, \operatorname{tr}\{F(x)\Phi^\tau_\sigma(\lambda, x^{-1})\}.$$

The spherical transform arises naturally as the Gelfand transform on the commutative convolution algebra $C_0^\infty(G, \tau, \tau)$ (cf. (4.1)). As usual, we set for short $\mathcal{H}^j_k = \mathcal{H}^{\tau_j}_{\sigma_k}$.

Recall [13] the Jacobi transform $\mathcal{J}^{(\alpha,\beta)}(f)$ of $f \in C_0^\infty(\mathbb{R})_{\mathrm{even}}$, defined by

$$\mathcal{J}^{(\alpha,\beta)}(f)(\lambda) = \int_0^\infty \mathrm{d}t\, (2\operatorname{sh} t)^{2\alpha+1}(2\operatorname{ch} t)^{2\beta+1}\varphi_\lambda^{(\alpha,\beta)}(t)\, f(t).$$

The Paley–Wiener theorem for the Jacobi transform ([13], Theorem 2.1) says that the map $f \to \mathcal{J}^{(\alpha,\beta)}(f)$ is, for all complex α, β with $\alpha \neq -1, -2, \ldots,$

and all $R > 0$, a linear isomorphism from the space $C_R^\infty(\mathbb{R})_{\text{even}}$ of smooth even functions on $\mathbb{R}$ with compact support contained in the interval $[-R, R]$, onto the space $PW_R(\mathbb{C})_{\text{even}}$ of even entire functions h on $\mathbb{C}$ verifying the condition

$$\forall N \in \mathbb{N},\ \exists C_N > 0 : |h(\lambda)| \leqslant C_N(1 + |\lambda|)^{-N} \mathrm{e}^{R|\mathrm{Im}\,\lambda|}.$$

The inversion formula and the Plancherel theorem for the Jacobi transform are given in [13], Theorems 2.3 and 2.4.

The spherical transform of $F \in C_0^\infty(G, \tau_j, \tau_j)$ can now be written in terms of suitable Jacobi transforms of its scalar components. The case of $\tau = \tau_0, \tau_n$ involves a single Jacobi transform, and the inversion formula, the Paley–Wiener theorem and the Plancherel theorem are obtained easily from the results in [13].

PROPOSITION 5.1 ([6], Prop. 5.1). *Let $\tau = \tau_0$, and let $F \in C_0^\infty(G, \tau, \tau)$, with scalar component f. Then*

$$\mathcal{H}_0^0(F) = 2^{-n-1} \mathcal{J}^{(n-1,(n+1)/2)}\left(\frac{f(t)}{(\mathrm{ch}\, t)^{(n+1)/2}}\right). \tag{5.1}$$

Similarly, the spherical transform $\mathcal{H}_{n-1}^n(F)$ of $F \in C_0^\infty(G, \tau_n, \tau_n)$, with scalar component f, is given by the right-hand side of (5.1).

For any $R > 0$ we denote by $C_R^\infty(G, \tau_j, \tau_j)$ the space of smooth τ_j-radial functions with compact support contained in the set $B_R = \{k_1 a_t k_2 : k_1, k_2 \in K, |t| \leqslant R\}$. Clearly $F \in C_R^\infty(G, \tau_j, \tau_j)$ if and only if its scalar components are in $C_R^\infty(\mathbb{R})_{\text{even}}$.

THEOREM 5.2 ([6], Thm. 5.4). *Let $\tau = \tau_0$ (resp. τ_n), and let $\mathcal{H} = \mathcal{H}_0^0$ (resp. $\mathcal{H}_{n-1}^n$), and $\Phi(\lambda, \cdot) = \Phi_0^0(\lambda, \cdot)$ (resp. $\Phi_{n-1}^n(\lambda, \cdot)$).*

(I) *The spherical transform $F \to \mathcal{H}(F)$ is inverted by*

$$F(x) = \int_0^\infty \mathrm{d}\nu(\lambda)\, \mathcal{H}(F)(\lambda)\, \Phi(\lambda, x) \quad (x \in G),$$

for all $F \in C_0^\infty(G, \tau, \tau)$, where

$$\mathrm{d}\nu(\lambda) = \frac{2^n}{\pi} |c(\lambda)|^{-2}\, \mathrm{d}\lambda,$$

$\mathrm{d}\lambda$ *being the Lebesgue measure on $\mathbb{R}$, and*

$$c(\lambda) = \frac{2^{\frac{3n+1}{2} - i\lambda}\, \Gamma(n)\Gamma(i\lambda)}{\Gamma(\frac{i\lambda + (3n+1)/2}{2})\Gamma(\frac{i\lambda + (n-1)/2}{2})}.$$

(II) *The map $F \to \mathcal{H}(F)$ is a linear isomorphism of $C_R^\infty(G, \tau, \tau)$ onto $PW_R(\mathbb{C})_{\text{even}}$.*

(III) *For all* $F \in C_0^\infty(G, \tau, \tau)$ *we have the Plancherel formula*

$$\|F\|^2_{L^2(G,\tau,\tau)} = \int_0^\infty \mathrm{d}\nu(\lambda)\, |\mathcal{H}(F)(\lambda)|^2.$$

(IV) *The map* $F \to \mathcal{H}(F)$ *extends to an isometry of* $L^2(G, \tau, \tau)$ *onto* $L^2(\mathbb{R}^+, \mathrm{d}\nu)$.

Consider now the generic case, i.e., $\tau = \tau_j$, with $1 \leqslant j \leqslant n-1$.

PROPOSITION 5.3 ([6], Prop. 5.3). *Let* $\tau = \tau_j$, *with* $1 \leqslant j \leqslant n-1$, *and let* $F \in C_0^\infty(G, \tau, \tau)$, *with scalar components* (f_{j-1}, f_j). *Then the spherical transform* $\mathcal{H}(F) = (\mathcal{H}^j_{j-1}(F), \mathcal{H}^j_j(F))$ *can be written in terms of Jacobi transforms as follows:*

$$\begin{aligned}\mathcal{H}^j_{j-1}(F) = {}& -\frac{2^{n-2j-2}}{n}\mathcal{J}^{(n,j-\frac{n+1}{2})} \times \\ & \times \left(\frac{\left[\frac{\mathrm{d}}{\mathrm{d}t} + (j - \frac{n-1}{2})\,\mathrm{th}\,t + 2(n-j)\,\mathrm{cth}\,t\right] f_{j-1} - \frac{2(n-j)}{\mathrm{sh}\,t} f_j}{\mathrm{sh}\,t\,(\mathrm{ch}\,t)^{j-\frac{n+1}{2}}} \right),\end{aligned}$$

$$\begin{aligned}\mathcal{H}^j_{j}(F) = {}& -\frac{2^{2j-n-2}}{n}\mathcal{J}^{(n,\frac{n-1}{2}-j)} \times \\ & \times \left(\frac{\left[\frac{\mathrm{d}}{\mathrm{d}t} + (\frac{n+1}{2} - j)\,\mathrm{th}\,t + 2j\,\mathrm{cth}\,t\right] f_{j} - \frac{2j}{\mathrm{sh}\,t} f_{j-1}}{\mathrm{sh}\,t\,(\mathrm{ch}\,t)^{\frac{n-1}{2}-j}} \right).\end{aligned}$$

Moreover $\mathcal{H}^j_j(F) = 0$ *if and only if* $\mathrm{d}F = 0$, *and in that case*

$$\begin{aligned}&\mathcal{H}^j_{j-1}(F)(\lambda) \\ &\quad = \frac{2^{n-2j-3}}{jn}\left[\lambda^2 + \left(\frac{n+1}{2} - j\right)^2\right]\mathcal{J}^{(n,j-\frac{n+1}{2})}\big((\mathrm{ch}\,t)^{\frac{n+1}{2}-j} f_j(t)\big)(\lambda). \qquad (5.2)\end{aligned}$$

Similarly, $\mathcal{H}^j_{j-1}(F) = 0$ *if and only if* $\delta F = 0$, *and in that case*

$$\begin{aligned}&\mathcal{H}^j_{j}(F)(\lambda) \\ &\quad = \frac{2^{2j-n-3}}{n(n-j)}\left[\lambda^2 + \left(\frac{n-1}{2} - j\right)^2\right]\mathcal{J}^{(n,\frac{n-1}{2}-j)}\big((\mathrm{ch}\,t)^{j-\frac{n-1}{2}} f_{j-1}(t)\big)(\lambda). \qquad (5.3)\end{aligned}$$

In the generic case we see that the spherical transform involves two ‘interlaced’ Jacobi transforms. As a preliminary step, one can derive the inversion formula for the τ-radial functions F satisfying either $\mathrm{d}F = 0$ or $\delta F = 0$, since in this case a single Jacobi transform is involved (cf. (5.2) and (5.3)). Eventually, one gets the following inversion formula and Paley–Wiener theorem for arbitrary τ-radial functions.

THEOREM 5.4 ([6], Thm. 5.7). *Let* $\tau = \tau_j$, *with* $1 \leqslant j \leqslant n-1$.

(I) *For all* $j \neq n/2$ *the spherical transform* $\mathcal{H}(F) = (\mathcal{H}^j_{j-1}(F), \mathcal{H}^j_j(F))$ *of* $F \in C_0^\infty(G,\tau,\tau)$ *is inverted by*

$$F(x) = \frac{1}{d\tau_j}\times$$
$$\times \int_0^\infty \left\{\mathrm{d}\nu_{j-1}(\lambda)\, \mathcal{H}^j_{j-1}(F)(\lambda)\, \Phi^j_{j-1}(\lambda, x) + \mathrm{d}\nu_j(\lambda)\, \mathcal{H}^j_j(F)(\lambda)\, \Phi^j_j(\lambda, x)\right\}, \quad (5.4)$$

for all $x \in G$, *where* $d\tau_j = \binom{n}{j}$ *and for all* $j = 0, \ldots, n-1$ *we set*

$$\mathrm{d}\nu_j(\lambda) = \mathrm{d}\sigma_j \frac{2^{n+2-2j}}{\pi} \frac{n^2}{\lambda^2 + [(n-1)/2 - j]^2} \frac{\mathrm{d}\lambda}{|c_j(\lambda)|^2}, \quad (5.5)$$

where $d\sigma_j = \binom{n-1}{j}$ *and*

$$c_j(\lambda) = \frac{2^{\frac{3n+1}{2}-j-i\lambda}\, \Gamma(n+1)\Gamma(i\lambda)}{\Gamma(\frac{i\lambda - j + (3n+1)/2}{2})\Gamma(\frac{i\lambda + j + (n+3)/2}{2})}. \quad (5.6)$$

For $j = n/2$ *the discrete term*

$$n\, 2^{1-4n}\, \mathcal{H}^{n/2}_{n/2}(F)\left(\frac{i}{2}\right) \Phi^{n/2}_{n/2}\left(\frac{i}{2}, x\right) \quad (5.7)$$

must be added to the right-hand side of (5.4).

(II) *The spherical transform* $F \to \mathcal{H}(F)$ *is a linear isomorphism of* $C_R^\infty(G,\tau,\tau)$ *onto the space*

$$PW_R = \left\{(h_1, h_2) \in PW_R(\mathbb{C})^2_{\mathrm{even}} : h_1(\pm\lambda_{j-1}) = h_2(\pm\lambda_j)\right\},$$

where λ_{j-1}, λ_j *are given by* (4.11).

Note the relation $\mathrm{d}\nu_j = \mathrm{d}\nu_{n-j-1}$ between the Plancherel measures, which follows from (3.6). In particular for $j = n/2$ we get $\mathrm{d}\nu_{n/2-1} = \mathrm{d}\nu_{n/2}$.

Let us make some remarks on the proof of this result. Equation (4.12) implies that

$$\mathcal{H}^j_{j-1}(F)(\pm\lambda_{j-1}) = \mathcal{H}^j_j(F)(\pm\lambda_j)$$

for all $F \in C_0^\infty(G,\tau,\tau)$, thus $\mathcal{H}(F) \in PW_R$ for any $F \in C_R^\infty(G,\tau,\tau)$. The main step is to prove the surjectivity statement. To this end, given $h = (h_1, h_2) \in PW_R$ one defines $F_h \in C^\infty(G,\tau,\tau)$ by the inversion formula (5.4), and proves the support preserving property for each scalar component of F_h. The integration over the real line is shifted into the upper half-plane to avoid the singularities of the functions $(c_{j-1}(-\lambda))^{-1}$ and $(c_j(-\lambda))^{-1}$ in (5.4). One is left with the two simple poles at

$$\lambda = \lambda_{j-1}, \lambda_j \left(\text{if } j < \frac{n-1}{2}\right) \quad \text{and} \quad \lambda = -\lambda_{j-1}, -\lambda_j \left(\text{if } j > \frac{n+1}{2}\right)$$

due to the factors $\{\lambda^2+[(n\pm1)/2-j]^2\}$ in the denominators of $\mathrm{d}\nu_{j-1}(\lambda)$ and $\mathrm{d}\nu_j(\lambda)$. For $j = n/2$, there is only one simple pole at $\lambda = i/2$, and for $j = (n \pm 1)/2$ one of the two poles is at the origin, so the procedure must be modified by taking the principal value of the integral.

One proves that for $j \neq n/2$ the contribution from the two residues cancels out, thanks to the condition $h_1(\pm\lambda_{j-1}) = h_2(\pm\lambda_j)$ (this is the only 'Arthur–Campoli' relation needed here). For $j = n/2$ the contribution does not cancel and must be subtracted out in order for F_h to have compact support. This amounts to adding the 'discrete term' (5.7) to the inversion formula (5.4). Here $\Phi^{n/2}_{n/2}(i/2, \cdot)$ is (up to a constant) the unique τ-radial function which is harmonic and in L^2 (see the end of Section 4).

Finally we give the Plancherel theorem in the generic case.

THEOREM 5.5 ([6], Thm. 5.8). (I) *Let* $1 \leqslant j \leqslant n-1$, $j \neq n/2$. *For all* $F \in C_0^\infty(G, \tau_j, \tau_j)$, *we have the Plancherel formula*

$$\|F\|^2_{L^2(G,\tau_j,\tau_j)} = \int_0^\infty \left\{\mathrm{d}\nu_{j-1}(\lambda)\,|\mathcal{H}^j_{j-1}(F)(\lambda)|^2 + \mathrm{d}\nu_j(\lambda)\,|\mathcal{H}^j_j(F)(\lambda)|^2\right\}.$$

Moreover, the map $F \to \mathcal{H}(F)$ *extends to an isometry of* $L^2(G, \tau_j, \tau_j)$ *onto the space*

$$L^2(\mathbb{R}^+, \mathrm{d}\nu_{j-1}) \overset{\perp}{\oplus} L^2(\mathbb{R}^+, \mathrm{d}\nu_j).$$

(II) *Let* $\tau = \tau_{n/2}$. *For all* $F \in C_0^\infty(G, \tau, \tau)$, *we have the Plancherel formula*

$$\begin{aligned}\|F\|^2_{L^2(G,\tau,\tau)} &= \int_0^\infty \mathrm{d}\nu(\lambda)\left\{|\mathcal{H}^{n/2}_{n/2-1}(F)(\lambda)|^2 + |\mathcal{H}^{n/2}_{n/2}(F)(\lambda)|^2\right\} + \\ &\quad + d_\tau\, n\, 2^{1-4n}|\mathcal{H}^{n/2}_{n/2}(F)(i/2)|^2,\end{aligned}$$

where $\mathrm{d}\nu = \mathrm{d}\nu_{n/2-1} = \mathrm{d}\nu_{n/2}$ *and* $d_\tau = \binom{n}{n/2}$. *Moreover, the map* $F \to \mathcal{H}(F)$ *extends to an isometry of* $L^2(G, \tau, \tau)$ *onto the space*

$$L^2(\mathbb{R}^+, \mathrm{d}\nu) \overset{\perp}{\oplus} L^2(\mathbb{R}^+, \mathrm{d}\nu) \overset{\perp}{\oplus} \mathbb{C} \cdot \Phi^{n/2}_{n/2}\left(\frac{i}{2}, \cdot\right).$$

The surjectivity statement is proved as follows. First one can show ([6], Prop. 7.6) that for all $j = 1, \ldots, n-1$,

$$\{F \in C_0^\infty(G, \tau_j, \tau_j) : \mathrm{d}F = 0\} = \mathrm{d}(C_0^\infty(G, \tau_j, \tau_{j-1})),$$

and

$$\{F \in C_0^\infty(G, \tau_j, \tau_j) : \delta F = 0\} = \delta(C_0^\infty(G, \tau_j, \tau_{j+1})),$$

where for two K-types δ_1, δ_2, we denote by $C_0^\infty(G, \delta_1, \delta_2)$ the space of smooth compactly supported maps α: $G \to \mathrm{Hom}(V_{\delta_1}, V_{\delta_2})$ satisfying $\alpha(k_1 x k_2) = \delta_2(k_2^{-1})\alpha(x)\delta_1(k_1^{-1})$, $\forall x \in G$, $\forall k_1, k_2 \in K$.

Then, as in the usual Hodge–Kodaira decomposition theorem for p-forms on noncompact Riemannian manifolds, one shows that for all $j \neq n/2$

$$\overline{\mathrm{d}(C_0^\infty(G, \tau_j, \tau_{j-1}))} \overset{\perp}{\oplus} \overline{\delta(C_0^\infty(G, \tau_j, \tau_{j+1}))} = L^2(G, \tau_j, \tau_j),$$

whereas for $j = n/2$

$$\overline{\mathrm{d}(C_0^\infty(G, \tau_{n/2}, \tau_{n/2-1}))} \overset{\perp}{\oplus} \overline{\delta(C_0^\infty(G, \tau_{n/2}, \tau_{n/2+1}))} \overset{\perp}{\oplus} \mathbb{C} \cdot \Phi_{n/2}^{n/2}\left(\frac{i}{2}, \cdot\right)$$
$$= L^2(G, \tau_{n/2}, \tau_{n/2}).$$

Finally, since on $\mathrm{d}(C_0^\infty(G, \tau_j, \tau_{j-1}))$ and on $\delta(C_0^\infty(G, \tau_j, \tau_{j+1}))$ the spherical transform reduces to a single Jacobi transform (cf. (5.2) and (5.3)), one concludes using the Plancherel theorem for the Jacobi transform ([13], Theorem 2.4).

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