Standard universal dendrites as small Polish structures
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In particular:

- prove counterparts of some results from stability theory;
- find, in this wider context, counterexamples to open problems;
- provide a (yet another) tool to measure complexity of dinamycal systems.
A **Polish structure** is a pair \((X, G)\) where:

- \(G\) is a Polish group acting faithfully on a set \(X\)
  i.e.  \(\forall g, g' \in G \ (g \neq g' \implies \exists a \in X \ ga \neq g'a)\)
- the stabilisers of all singletons are closed

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Notation:

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Independence

A notion of independence in Polish structures is introduced.

Let \( \bar{a} \in X^{<\omega} \) and \( A \subseteq B \subseteq X \) finite.

The idea is to say that \( \bar{a} \) is independent from \( B \) over \( A \) if, once \( A \) has been fixed, asking to fix \( B \) does not add too much constraint on \( \bar{a} \), i.e.

\[
G_B \bar{a} \text{ is big in } G_A \bar{a}.
\]
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However:
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Thus the relations of independence are defined via a pull back to the group $G$. 
Let $\pi_A : G_A \to G_A\bar{a}$ and check whether $g \mapsto g\bar{a}$

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is big in $\pi_A^{-1}(G_A\bar{a}) = G_A$. 
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**Definitions.**
Let \( \bar{a} \in X^{<\omega} \) and \( A, B \subseteq \text{fin} \ X \) (most often \( A \subseteq B \)).

\( \bar{a} \downarrow_{A} B \): \( \bar{a} \) is **o-independent** from \( B \) over \( A \) if \( \pi_A^{-1}(G_{A\cup B}\bar{a}) \) is open in \( G_A \) (written \( \pi_A^{-1}(G_{A\cup B}\bar{a}) \subseteq_{o} G_A \)).

\( \bar{a} \downarrow_{A}^{nm} B \): \( \bar{a} \) is **nm-independent** from \( B \) over \( A \) if \( \pi_A^{-1}(G_{A\cup B}\bar{a}) \) is non-meagre in \( G_A \) (written \( \pi_A^{-1}(G_{A\cup B}\bar{a}) \subseteq_{nm} G_A \)).
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**Remark.** If \( X \) is separable metrisable, the action \( G \times X \to X \) is continuous and \( G_A\bar{a} \) is not meagre in itself, then
\[ \bar{a}_A^\star B \iff G_{A\cup B}\bar{a} \subseteq \star G_A\bar{a} \]

(for \( \star = nm \) it is enough \( X \) being Hausdorff)
Example: $A = \emptyset$.

\[
\bar{a} \downarrow^* B \text{} \iff \{g \in G \mid \exists h \in G_B \ s.t. \ g\bar{a} = h\bar{a}\} \subseteq_G G
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$\overline{\mathcal{a}} \downarrow^* \mathcal{B}$ iff $\{g \in G \mid \exists h \in G_B \ g\overline{\mathcal{a}} = h\overline{\mathcal{a}}\} \subseteq^* G$

The opposite situation: small orbits. Let $A \subseteq_{\text{fin}} X$.

- $dcl(A) = \{a \in X \mid G_A a = \{a\}\}$: definable closure of $A$
- $acl(A) = \{a \in X \mid G_A a \text{ is finite}\}$: strong algebraic closure of $A$
- $Acl(A) = \{a \in X \mid G_A a \text{ is at most countable}\}$: algebraic closure of $A$

For any $A \subseteq X$, define

$$dcl(A) = \bigcup_{A_0 \subseteq_{\text{fin}} X} dcl(A_0),$$

etc.
Basic properties of independence

To develop a counterpart of basic geometric stability theory, five properties of the independence relation are needed:

- **Invariance**: \( \tilde{a} \downarrow_A^* B \iff g\tilde{a} \downarrow_{gA}^* gB \)
- **Symmetry**: \( \tilde{a} \downarrow_A^* \tilde{b} \iff \tilde{b} \downarrow_A^* \tilde{a} \)
- **Transitivity**: \( \tilde{a} \downarrow_A^* B \land \tilde{a} \downarrow_B^* C \iff \tilde{a} \downarrow_A^* C \)
- **Existence of independent extensions**: \( a \in Acl(A) \iff \forall B \ \tilde{a} \downarrow_A^* B \)
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(iff \(\forall a_1, \ldots, a_n \in X, \ G_{a_1, \ldots, a_n} \times X \to X \text{ has at most countably many orbits}\))
Existence of independent extensions

**Theorem.** Let $(X, G)$ be a small Polish structure. Then

\[ \forall \bar{a}, \forall A \subseteq B \subseteq_{\text{fin}} X, \]

**Remark.** The same is not true for $o$-independent extensions.
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- $X^{eq} = \bigcup \{X^n/E \mid E \text{ invariant eq. rel. on } X^n, \text{ s.t } \text{Stab}([a]_E) \leq_c G\}$, the \textit{imaginary extension} of $X$
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Proposition. Every definable set in $X^{eq}$ has a name in $X^{eq}$. 
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Assume \((X, G)\) is a small Polish structure (but in most situations it is enough to ask for the existence of \(nm\)-independent extensions)
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**Definition.** \(NM\) is the function from the collection of orbits over finite sets (in \(X\) or \(X^\text{eq}\)) to \(Ord \cup \{\infty\}\),

\[
NM : (a, A) \mapsto NM(a, A) \in Ord \cup \{\infty\}
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satisfying

\[
NM(a, A) \geq \alpha + 1 \iff \exists B \supseteq \text{fin} A \ (NM(a, B) \geq \alpha \land \neg a \downarrow^A B)
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**Example.** \(NM(a, A) = 0 \iff a \in Acl^{eq}(A)\).
Definition. $(X, G)$ is **nm-stable** if every 1-orbit has ordinal rank, i.e. there is no infinite sequence $A_0 \subseteq A_1 \subseteq \ldots \subseteq_{\text{fin}} X$ and $a \in X$ such that $a$ is nm-dependent from $A_{i+1}$ over $A_i$. 
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**Definition.** If \(D\) is definable over \(A\) in \(X^{eq}\), the \(\mathcal{NM}\)-rank of \(D\) is

\[
\mathcal{NM}(D) = \sup\{\mathcal{NM}(d, A) \mid d \in D\}
\]
Examples (Krupiński)

- $(S^n, \text{Homeo}(S^n))$ has rank 1
- $((S^1)^n, \text{Homeo}((S^1)^n))$ has rank 1
- $([0, 1]^N, \text{Homeo}([0, 1]^N))$ has rank 1
- if $(X, G)$ has rank 1, then $(X^n, G)$ has rank $n$
Definitions.

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- A non-degenerate continuum is hereditarily *(in)decomposable* if all its subcontinua are (in)decomposable.
**The pseudo-arc**

**Definition.** The pseudo-arc is the unique continuum that is hereditarily indecomposable and arc-like:

\[ \forall \varepsilon, \exists f : P \rightarrow [0, 1] \text{ continuous }, \forall y, \ diam(f^{-1}(y)) < \varepsilon \]

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A construction of the pseudoarc:
Fix distinct point \( p, q \in \mathbb{R}^2 \).

Step 0 Draw a simple chain \( U_0 = \{ U_{00}, \ldots, U_{0r_0} \} \) from \( p \) to \( q \) of connected open sets of diameter less than 1. Being a simple chain from \( p \) to \( q \) means:

- \( U_i \cap U_j \iff |i - j| \leq 1 \)
- \( p \in U_{0i} \iff i = 0 \)
- \( q \in U_{0i} \iff i = r_0 \)
**Step k+1** Draw a simple chain $\mathcal{U}_{k+1} = \{U_{k+1,0}, \ldots, U_{k+1,r_{k+1}}\}$ from $p$ to $q$ of connected open sets of diameter less than $\frac{1}{k+2}$ such that

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This last condition means that for all $i, j, m, n$, if

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then there are $s, t$ with $i < s < t < j$ or $i > t > l > j$ such that

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Final step $P = \bigcap_{k \in \mathbb{N}} \bigcup \mathcal{U}_k$ is the pseudoarc.
The pseudo-arc is a quite complicated continuum. Nevertheless it is the generic continuum: the class of pseudo-arcs is dense $G_\delta$ in the space of all continua.

Theorem. (Krupiński) Let $P$ be the pseudo-arc. Then $(P, \text{Homeo}(P))$ is a small, not nm-stable, Polish structure. In particular, the NM-rank of $P$ is $\infty$. Moreover $P$ is an example of a small Polish structure not admitting $o$-independent extensions.
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Dendrites

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**Definition.** Given a point \( x \) in a continuum \( X \), its order \( \text{ord}(x, X) \) is the smallest cardinal \( \beta \) such that \( x \) has a basis of open neighbourhoods whose boundaries have cardinality \( \leq \beta \).
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**Definition.** Given a point $x$ in a continuum $X$, its order $\text{ord}(x, X)$ is the smallest cardinal $\beta$ such that $x$ has a basis of open neighbourhoods whose boundaries have cardinality $\leq \beta$.

All points of a dendrite have order $\leq \aleph_0$. Points of order 1 are called end points; points of order $\geq 3$ are branching points.
The following property might help to visualise a dendrite:

If $X$ is a non-degenerate dendrite, then

$$X = \bigcup_{i \in \mathbb{N}} A_i \cup E(X)$$

where:

- each $A_i$ is an arc, with end points $p_i, q_i$
- $A_{i+1} \cap \bigcup_{j=0}^{i} A_j = \{p_{i+1}\}$
- $\text{diam}(A_i) \to 0$
- $E(X)$ is the set of end points of $X$
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- enumerate \(\{q_n\}_{n \in \mathbb{N}} = (0,1] \cap \mathbb{Q} \times \{0\}\)
- at step \(n\) add \(n\) arcs of diameter \(\leq \frac{1}{2^n}\) intersecting each other and the already achieved construction only in \(q_n\)
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Then all point of \([0, 1] \times \{0\}\) are in distinct orbits.
Let $\emptyset \neq J \subseteq \{3, 4 \ldots, \omega\}$.
There is a unique dendrite $D_J$ such that

- each branching point of $D_J$ has order in $J$
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**Universality property**

$D_J$ is universal for the class of dendrites whose branching points have order in $J$: any such dendrite embeds in $D_J$. 
Theorem. Each \((D_J, \text{Homeo}(D_J))\) is a small Polish structure of \(\mathcal{NM}\)-rank 1.
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Conjectures.

- Each dendrite admits \(nm\)-independent extensions
- If \(D\) is a dendrite and \((D, \text{Homeo}(D))\) is small, then \(\mathcal{NM}(D) = 1\)
Some questions.

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- Find examples of continua $C$ with $1 < \mathcal{N}\mathcal{M}(C) < \infty$. 
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- Find examples of continua $C$ with $1 < \mathcal{N}\mathcal{M}(C) < \infty$.
- The $\mathcal{N}\mathcal{M}$-gap conjecture.
An example of a small Polish structure $(X, G)$ with $\mathcal{N}\mathcal{M}(X) = \omega$ can be obtained as a disjoint sum of small Polish structures of increasing natural rank.
E.g., take $(Y, G)$ of rank 1 and let $X = \bigcup_{n \geq 1} X^n$. 

However, in this example there is no single orbit over finite sets with rank $\geq \omega$ (and $\neq \infty$).

The $\mathcal{N}\mathcal{M}$-gap conjecture.
Let $(X, G)$ be a small Polish structure. Then, for any orbit $o$ of a finite set $A \subseteq X$, one has $\mathcal{N}\mathcal{M}(o) \in \omega \cup \{\infty\}$.
This conjecture is open in the class of small profinite structures; it has been proved for small $m$-stable profinite groups. In this wider context it might be easier to find a counterexample.
An example of a small Polish structure \((X, G)\) with \(NM(X) = \omega\) can be obtained as a disjoint sum of small Polish structures of increasing natural rank. 
E.g., take \((Y, G)\) of rank 1 an let \(X = \bigcup_{n \geq 1} X^n\).

However, in this example there is no single orbit over finite sets with rank \(\geq \omega\) (and \(\neq \infty\)).
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