Well quasi-orders, better quasi-orders, and classification problems in descriptive set theory
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**Somehow vague questions:**

- Are there any interesting relations — from the point of view of DST — that turn out to be wqo?
- Where do they come out?
- What can be said about their complexity?
Scope of the talk

I will mainly concentrate on the second question: I will present a new (surprisingly easy) example of a bqo that arose studying relations from a DST perspective, and discuss some problem that stem from it.
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$B(X)$ is the $\sigma$-algebra of Borel subsets of $X$.

A function $f: X \to Y$ between topological spaces is Borel if $\forall B \in B(X) f^{-1}(B) \in B(Y)$ or, equivalently, $\forall B \in \Sigma^0_1(Y) f^{-1}(B) \in B(X)$. 
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Polish and standard Borel spaces

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If $X$ is standard Borel and $A \in \mathcal{B}(X)$, then $A$, with the induced $\sigma$-algebra, is standard Borel as well.
A subset $A$ of a standard Borel space $X$ is *analytic* (or $\Sigma^1_1$) if there are a standard Borel space $Y$ and a Borel function $f : Y \to X$ such that $A = im f$. 
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A motivating example

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$$ x \in \prod_{i \in I} 2^{A^{ar(i)}} \times \prod_{j \in J} A^{A^{ar(j)}} \times A^K = X_L $$
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\( X_L \) is a Polish space under the product topology.
A motivating example

The Polish group $\text{Sym}(A)$ acts on $X_L$ by the logic action: for $g \in \text{Sym}(A), x \in X_L$, the model $gx$ is obtained from model $x$ by permutating the elements of $A$ by $g$. 

Theorem. (Lopez-Escobar) The Borel subsets of $X_L$ that are invariant under the logic action are exactly the sets of the form $B_\phi = \{ x \in X_L | x | = \phi \}$ for some $L_{\omega_1^{\omega}}$-sentence $\phi$.

Example. Let $L = \{ \leq \}$, where $\leq$ is a binary relation symbol, so that $X_L = 2^{2^\mathbb{N}}$. Let $\phi$ be an axiom for linear orders. Then $\text{LO} = \{ x \in X_L | x | = \phi \}$ is a Borel (actually closed) subset of $X_L$. 
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Let $E, F$ be $n$-ary relations on standard Borel spaces $X, Y$, respectively.
Borel reducibility

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if there is a Borel function $f : X \to Y$ such that

$$\forall x_1, \ldots, x_n \in X \ (E(x_1, \ldots, x_n) \Leftrightarrow F(f(x_1), \ldots, f(x_n)))$$

Main examples for classes of countable structures:

Analytic equivalence relations, like isomorphism or bi-embeddability in some classes of countable structures. Analytic quasi-orders, like embeddability.
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If $\mathcal{R}$ is a class of relations on standard Borel spaces, then a relation $F$ is $\mathcal{R}$-complete if

1. $F \in \mathcal{R}$
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- Does \( \mathcal{R} \) admit complete elements? Who are them?
- What is the complexity, both in the Borel hierarchy and w.r.t. \( \leq_B \), of the elements of \( \mathcal{R} \)?
Several classes of relations have been proven to admit complete elements.

**Examples:**

- **Borel equivalence relations with at most countable classes.** A complete element is the orbit equivalence relation of $F_2$ on $2^{F_2}$.
- **Isomorphism relations for countable structures.** A complete element is isomorphism on countable graphs.
- **Analytic quasi-orders.** A complete element is embeddability on countable graphs.
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**Remark.** An $R$-complete relation must be sufficiently complex both w.r.t. the projective hierarchy and combinatorially, since it Borel embeds every element of $R$. In particular, a wqo cannot be $R$-complete, unless all elements of $R$ are wqo's.
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2. **Recursive isomorphism**

   **Theorem.** (C., 2002) The relation of recursive isomorphism on countable linear orders is complete for Borel equivalence relations with at most countable classes.
3. Embeddability

Theorem. (Laver, 1971) The relation of embeddability on countable linear orders (in fact on $\sigma$-scattered ones) is a bqo.

The results proved by Laver and by van Engelen, Miller, Steel, 1987) are actually stronger.

Denote by $\leq_c$ the relation of continuous order preserving embeddability between linear orders.

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Preserving bqo’s

**Definition.** (Louveau, Saint Raymond, 1990) Let $C$ be a class of structures with morphisms between them, such that the identities are $C$-morphisms, and $C$-morphisms are closed under composition.
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quasi-ordered by

$$f_0 \leq f_1 \iff \exists g : \text{dom} f_0 \to \text{dom} f_1 \text{ a } \mathcal{C} \text{-morphism}$$

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Class $C$ preserves bqo’s if whenever $Q$ is a bqo, then $Q^C$ is bqo as well.
Preserving bqo’s

**Theorem.** (Laver, 1971) The class of \(\sigma\)-scattered linear orders under \(\leq_i\) preserves bqo’s.

**Theorem.** (van Engelen, Miller, Steel, 1987) The class of countable linear orders under \(\leq_c\) preserve bqo’s.
5. Epimorphisms
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The restriction of $\leq_s$ to $LO$ is an analytic quasi-order.
Some easy remarks:

- If \( f: L \to K \) witnesses \( K \leq_s L \), then \( f \) has a right inverse \( g: K \to L \) witnessing \( K \leq_i L \). So \( \leq_s \subseteq \leq_i \).

- The converse does not hold, e.g., \( \omega, \omega + 1 \).

- \( K \leq_s L \) if and only if \( L = \sum_{i \in K} L_i \).

- If \( L \) has a minimum (or a maximum), while \( K \) does not, then \( K \not\leq L \).

- If \( K, L \) do not have maximum and \( K \leq_s L \), then \( \text{cof}(K) = \text{cof}(L) \).

Similarly for orders without minimum.

As a first consequence, an analogous of Laver's result cannot hold for \( \leq_s \): if \( \kappa, \lambda \) are distinct infinite cardinals, they are incomparable under \( \leq_s \).
Some easy remarks:

- If $f : L \to K$ witnesses $K \leq_s L$, then $f$ has a right inverse $g : K \to L$ witnessing $K \leq_i L$. So $\leq_s \subseteq \leq_i$.

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- $K \leq_s L$ if and only if $L = \sum_{i \in K} L^i$.

- If $L$ has a minimum (or a maximum), while $K$ does not, then $K \not\leq L$.

- If $K, L$ do not have maximum and $K \leq_s L$, then $\text{cof}(K) = \text{cof}(L)$.

Similarly for orders without minimum.

As a first consequence, an analogous of Laver's result cannot hold for $\leq_s$: if $\kappa, \lambda$ are distinct infinite cardinals, they are incomparable under $\leq_s$. 
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Epimorphisms on linear orders

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As a first consequence, an analogous of Laver’s result cannot hold for $\leq_s$: if $\kappa, \lambda$ are distinct infinite cardinals, they are incomparable under $\leq_s$. 

Lemma
If $K$ has minimum, maximum, and it is complete, then for any order $L$,

$$K \leq_i L \Rightarrow K \leq_s L$$

Proof.
The bqo $\leq_s$ on countable orders

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If $f : K \leq_i L$ and $a = \min K$, define $g : L \to K$ by

$$g(y) = \begin{cases} 
  a & \text{if } y < f(a) \\
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\end{cases}$$

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The bqo $\leq_s$ on countable orders

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**Corollary**

$\leq_s$ is a bqo on countable complete linear orders with maximum and minimum.
The bqo $\leq_s$ on countable orders

Let $LIN_3$ be the class of all linear orders coloured in three colours: an element of $LIN_3$ is a linear order $L$ together with a function $c : L \to 3$. 
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$LIN_3$ can be quasi-ordered as follows: given

$$K = (K, c), L = (L, c') \in LIN_3$$

set

$$(K, c) \leq_{col} (L, c') \iff \exists f : K \rightarrow L \text{ injective, order preserving, continuous, and such that } \forall x \in K c(x) = c' f(x)$$

So $(LIN_3, \leq_{col})$ is a bqo on the subclass of $LIN_3$ consisting of countable orders.
The bqo $\leq_s$ on countable orders

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The \( \leq_s \) on countable orders

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By the theorem of van Engelen, Miller, Steel, \( \leq_{col} \) is a bqo on the subclass of \( LIN_3 \) consisting of countable orders.
Given a linear order $L$ let its closure $\bar{L}$ be defined by completing $L$ and then possibly adding a first or a last element, in case $L$ does not have them.
The b.qo $\leq_s$ on countable orders

Given a linear order $L$ let its closure $\bar{L}$ be defined by completing $L$ and then possibly adding a first or a last element, in case $L$ does not have them. The complete colouring of $L$ is the map $c_L : \bar{L} \rightarrow 3$ defined by

$$c_L(x) = \begin{cases} 
2 & \text{if } x \in L \\
1 & \text{if } x \in \{\min \bar{L}, \max \bar{L}\} \text{ and } x \notin L \\
0 & \text{otherwise}
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So $(\bar{L}, c_L) \in LIN_3$.

Notice that if $L$ is countable, then $L$ is scattered if and only if $\bar{L}$ is countable.
The bqo $\leq_s$ on countable orders

**Lemma**

Given linear orders $K, L$, if $c_K \leq_{col} c_L$, then $K \leq_s L$. 
The bqo $\leq_s$ on countable orders

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**Theorem**

$\leq_s$ is a bqo on scattered countable linear orders.

**Proof.**

If $L$ is countable scattered, then $\bar{L}$ is countable. By the above lemma, the map $\Phi : K \mapsto c_K$ satisfies

$$\Phi(K) \leq_{col} \Phi(L) \Rightarrow K \leq_s L$$
The bqo $\leq_s$ on countable orders

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Given linear orders $K, L$, if $c_K \leq_{col} c_L$, then $K \leq_s L$.

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Since $\leq_{col}$ is a bqo on countable orders, $\leq_s$ is a bqo on scattered countable orders.
It remains to consider non-scattered countable orders.
It remains to consider non-scattered countable orders. If $L$ is a countable, non-scattered linear ordering, there are four mutually disjoint possibilities, depending on the existence of *scattered initial or final tails*: 

1. $\eta \leq s_L$
2. $L = L_0 + \hat{L}$, for some unique $L_0, \hat{L}$, with $L_0$ scattered and $\eta \leq s_{\hat{L}}$
3. $L = \hat{L} + L_1$, for some unique $L_1, \hat{L}$, with $L_1$ scattered and $\eta \leq s_{\hat{L}}$
4. $L = L_0 + \hat{L} + L_1$, for some unique $L_0, L_1, \hat{L}$, with $L_0, L_1$ scattered and $\eta \leq s_{\hat{L}}$

It remains to show that $\leq_s$ is a bqo on each of these four classes. The orders in case 1 are all $\leq_s$-equivalent.
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Since $\leq_s$ is already being proved to be a bqo on countable scattered linear orders (previous theorem), the assignment $L \mapsto L_0$ proves that $\leq_s$ is also a bqo on this class.
Theorem

- If $\alpha$ is a successor ordinal and $\beta$ is any ordinal, then
  $$\alpha \leq_{s} \beta \iff \alpha \leq \beta$$
\( \leq_s \) on ordinals

**Theorem**

- If \( \alpha \) is a successor ordinal and \( \beta \) is any ordinal, then
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- If \( \alpha \) is a limit ordinal and \( \beta \) is a successor ordinal, then \( \alpha \nleq_s \beta \)

**Corollary**

Let \( \beta \) be a non-null ordinal. Then \( \alpha \leq_s \beta \) for every non-null \( \alpha \leq \beta \) if and only if \( \beta \) is countable and a finite multiple of an indecomposable ordinal: \( \beta = \omega \delta m \).
Theorem

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- If $\alpha = \omega^{\gamma_0} n_0 + \ldots + \omega^{\gamma_k} n_k$, $\beta = \omega^{\delta_0} m_0 + \ldots + \omega^{\delta_h} m_h$ are limit ordinals (i.e., $\gamma_k, \delta_h > 0$), then
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Let $\beta$ be a non-null ordinal. Then $\alpha \leq_s \beta$ for every non-null $\alpha \leq \beta$ if and only if $\beta$ is countable and a finite multiple of an indecomposable ordinal: $\beta = \omega^\delta m$. 
Definition

A linear order $L$ is strongly surjective if it surjects order-preservingly onto any of its sub-orders, i.e., for any order $K$, $K \leq L \Rightarrow K \leq sL$.

So, the strongly surjective ordinals are those of the form $\omega \delta m$ for some at most countable $\delta$ and $m > 0$. 
Strongly surjective orders

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Some properties of strongly surjective orders

Strongly surjective orders are not closed under sums (e.g., $\omega \gamma_0 n_0 + \omega \gamma_1 n_1$).

However:

\[ \text{Let } I \text{ be any order and, for each } i \in I, \text{ let } L_i \text{ be a strongly surjective order. Then } \sum_{i \in I} L_i \text{ is strongly surjective if and only if, for every non-empty } J \subseteq I, \sum_{j \in J} L_j \leq s \sum_{i \in I} L_i \]

\[ \text{If } L, M \text{ are strongly surjective, then } LM \text{ is strongly surjective.} \]

\[ \text{If } L \text{ is scattered and } LM \text{ is strongly surjective, then } M \text{ is strongly surjective.} \]

\[ \text{If } L \text{ is strongly surjective, then for any ordinal } \alpha, \text{ the } \alpha \text{-th Hausdorff condensation } L(\alpha) \text{ is strongly surjective (and similarly for several other condensations).} \]
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- If $L$ is strongly surjective, then for any ordinal $\alpha$, the $\alpha$-th Hausdorff condensation $L^{(\alpha)}$ is strongly surjective (and similarly for several other condensations).
Examples of strongly surjective orders

The order $\eta$ is strongly surjective.

The following are all the strongly surjective complete orders:

1. $(\omega \gamma n) \ast \omega \delta m$
2. $(\omega \gamma n) \ast \omega \delta m$, for $\gamma, \delta$ countable ordinals and $n, m > 0$
3. $\zeta \alpha m$, for $\alpha$ a countable ordinal and $m > 0$
4. $(\omega \alpha 0) \ast m$ and its reversal $\omega \alpha 0 \ast$,
5. $(\omega \alpha 0) \ast m + (\omega \gamma) \ast \omega$ and its reversal $\omega \gamma \ast m + (\omega \alpha 0) \ast$, m
6. $(\omega \alpha 0) \ast m + \sum_{i \in \omega} (\omega \alpha ji) \ast$ and its reversal $\sum_{i \in \omega} \ast \omega \alpha ji + \omega \alpha 0 m$
Examples of strongly surjective orders

The order $\eta$ is strongly surjective.
The following are all the strongly surjective complete orders:

- $(\omega^\gamma n)^*, \omega^\delta m, (\omega^\gamma n)^* + \omega^\delta m$, for $\gamma, \delta$ countable ordinals and $n, m > 0$
- $\zeta^\alpha m$, for $\alpha$ a countable ordinal and $m > 0$
- $(\omega^{\alpha_0})^* m$ and its reversal $\omega^{\alpha_0} \omega^*$
- $(\omega^{\alpha_0})^* m + (\omega^\gamma)^* \omega$ and its reversal $\omega^\gamma \omega^* + \omega^{\alpha_0} m$
- $(\omega^{\alpha_0})^* m + \sum_{i \in \omega}(\omega^{\alpha_j})^* \omega$ and its reversal $\sum_{i \in \omega} \omega^{\alpha_j} + \omega^{\alpha_0} m$
A descriptive set theoretic result

Finding all strongly surjective orders with one gap is doable but requires rather long and tedious computation by cases.
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A descriptive set theoretic result

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A reason for this difficulty may be suggested by the following

**Theorem**

The class of strongly surjective orders is a $\Pi^1_2$ subset of LO. Moreover it is both $\Sigma^1_1$-hard and $\Pi^1_1$-hard.
A descriptive set theoretic result

Finding all strongly surjective orders with one gap is doable but requires rather long and tedious computation by cases. For those with two or more gaps the computation seems to explode. No clue how to identify those with infinitely many gaps.

A reason for this difficulty may be suggested by the following

**Theorem**

*The class of strongly surjective orders is a $\Pi^1_2$ subset of LO. Moreover it is both $\Sigma^1_1$-hard and $\Pi^1_1$-hard*

We do not know any sharper classification.
Uncountable strongly surjective orders

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Note that  

**Proposition.** Any strongly surjective order is *short*, i.e., neither $\omega_1$ nor $\omega_1^*$ embed into it.
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**Question.** Is there an uncountable strongly surjective orders?

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**Proposition.** Any strongly surjective order is short, i.e., neither $\omega_1$ nor $\omega_1^*$ embed into it.

Trying to play around with the most familiar (short) uncountable orders does not produce any result. Indeed

**Theorem**

*The only linear order of the form $\rho_0 \cdot \ldots \cdot \rho_n$, where each $\rho_i$ is one of $\eta, \theta, \lambda$, to be strongly surjective is $\eta$.***
However, recall the following

**Definition.** If \( \kappa \) is an infinite cardinal, a linear order \( L \) is \( \kappa \)-dense if it has no end points and any open interval of \( L \) has cardinality \( \kappa \).
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Consider the following statement:

$\varphi_\kappa$: Up to isomorphism there is a unique $\kappa$-dense suborder of $\mathbb{R}$. 

So $\varphi_{\aleph_0}$ holds in ZFC, while the consistency of $\varphi_{\aleph_1}$ with ZFC was proven by Baumgartner. Moreover $\varphi_{\aleph_1}$ follows from PFA.

The question about the consistency of $\varphi_{\aleph_2}$ seems to be open.
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Uncountable strongly surjective orders

**Theorem**

*Assume* $\varphi_\kappa$. 

Questions.

▶ What about the existence of an uncountable strongly surjective order in ZFC? In ZFC + CH? In L?
Uncountable strongly surjective orders

Theorem
Assume $\varphi_\kappa$. Then there is a strongly surjective order of cardinality $\kappa$. Actually there are infinitely many of them, pairwise non-$\leq_i$-equivalent.
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