Structural instability of nonlinear plates modelling suspension bridges

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E. Berchio, F. Gazzola, C. Zanini, *Which residual mode captures the energy of the dominating mode in second order Hamiltonian systems?*, preprint

A general Hamiltonian system

Let \( U = U(y_1, \ldots, y_N) \) be a potential energy.

\[
\begin{align*}
\ddot{y}_1(t) + \frac{\partial U}{\partial y_1}(y_1(t), \ldots, y_N(t)) &= 0, \\
\ddots & \quad \ddots \\
\ddot{y}_N(t) + \frac{\partial U}{\partial y_N}(y_1(t), \ldots, y_N(t)) &= 0.
\end{align*}
\]

The Hamiltonian is given by the total energy

\[
E(y_1, \ldots, y_N, \dot{y}_1, \ldots, \dot{y}_N) := \sum_{i=1}^{N} \frac{\dot{y}_i^2}{2} + U(y_1, \ldots, y_N).
\]

We may think to the function \( y_i = y_i(t) \) as the amplitude (fourier coefficient) of each of the a so-called oscillation modes.
A general Hamiltonian system (transfer of energy)

Our main purpose is to study the transfer of energy from one mode to one or more of the other ones.

For example suppose that for some $k \in \{1, \ldots, N\}$ at least one among $y_k(0)$ and $\dot{y}_k(0)$ is different from zero and that $y_i(0) = \dot{y}_i(0) = 0$ for any $i \in \{1, \ldots, N\}, i \neq k$.

Figure: The first longitudinal mode
A general Hamiltonian system (transfer of energy)

A transfer of energy to other oscillation modes may occur.

Figure: The second longitudinal mode
A general Hamiltonian system (transfer of energy)

Figure: The third longitudinal mode
A general Hamiltonian system (transfer of energy)

The transfer of energy from one oscillation mode to the other ones is simply due to the fact that the system does not admit any stationary wave solution corresponding to that mode.

Suppose now that the system admits a solution
\[ Y_0(t) = (y_{0,1}(t), \ldots, y_{0,N}(t)) \]
satisfying \[ y_{0,N}(t) = 0 \] for any \( t \geq 0 \).

Figure: The first torsional mode
Let $Y_0 = Y_0(t)$ be the solution considered before. Consider now the solution $Y(t) = (y_1(t), \ldots, y_N(t))$ of the system satisfying

\[
y_1(0) = y_{0,1}(0), \ldots, y_{N-1}(0) = y_{0,N-1}(0), y_N(0) = A,
\]
\[
y_1'(0) = y_{0,1}'(0), \ldots, y_{N-1}'(0) = y_{0,N-1}'(0), y_N'(0) = B.
\]

The main question is to understand if the solution $Y_0$ is stable or not with respect to perturbations of $y_N$.

We can say that $Y_0$ is a stable solution if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $|A| + |B| < \delta$

\[
|Y(t) - Y_0(t)| + |Y'(t) - Y_0'(t)| < \varepsilon \quad \text{for any } t \in \mathbb{R}.
\]
Figure: The second torsional mode
Oscillation modes

\[
\begin{aligned}
\begin{cases}
u_{tt} + Lu = f(u) & \text{in } \Omega \times (0, \infty), \\
\text{Boundary Conditions on } \partial\Omega.
\end{cases}
\end{aligned}
\]

Here by oscillation mode we mean “linear oscillation modes”. We may choose the eigenfunctions of the following eigenvalue problem

\[
\begin{aligned}
\begin{cases}
Lu = \lambda u & \text{in } \Omega, \\
\text{Boundary Conditions on } \partial\Omega.
\end{cases}
\end{aligned}
\] (EP)

Denote by \(w_1, \ldots, w_n, \ldots\) a complete orthonormal system of eigenfunctions of (EP) and by \(\lambda_1 \leq \cdots \leq \lambda_n \leq \cdots\) the corresponding eigenvalues.

\[
u(\cdot, t) = \sum_{n=1}^{+\infty} y_n(t)w_n(\cdot)
\]
A model with interacting oscillators

\[
\begin{align*}
\frac{m\ell^2}{3} \theta'' &= \ell \cos \theta \left[ f(y - \ell \sin \theta) - f(y + \ell \sin \theta) \right] \\
my'' &= -\left[ f(y - \ell \sin \theta) + f(y + \ell \sin \theta) \right]
\end{align*}
\]
A model with interacting oscillators

\begin{align*}
\theta_i''(t) + 3 \frac{\partial U}{\partial \theta_i}(\Theta(t), Y(t)) &= 0 \\
y_i''(t) + \frac{\partial U}{\partial y_i}(\Theta(t), Y(t)) &= 0.
\end{align*}

where \((\Theta, Y) = (\theta_1, \ldots, \theta_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}\) and

\[U(\Theta, Y) = \sum_{i=1}^{n} [F(y_i + \sin \theta_i) + F(y_i - \sin \theta_i)] + \frac{1}{2} \sum_{i=0}^{n} [K_y(y_i - y_{i+1})^2 + K_\theta(\theta_i - \theta_{i+1})^2]\]

with the notation \(y_0 = y_{n+1} = \theta_0 = \theta_{n+1} = 0\).
A possible choice for $f$ is

$$f(s) = k_1 s + k_2 s^2 + k_3 s^3.$$ 

Some models of suspension bridges

$y$: deflection in the vertical direction;

$\theta$: angle of torsion.

\[
\begin{cases}
M y_{tt} + E l y_{xxxx} + f(y + \ell \sin \theta) + f(y - \ell \sin \theta) = 0 \\
\frac{M \ell^2}{3} \theta_{tt} - \mu \ell^2 \theta_{xx} + \ell \cos \theta (f(y + \ell \sin \theta) + f(y - \ell \sin \theta)) = 0.
\end{cases}
\]

$x \in (0, L), \ t > 0$.

The Kirchhoff-Love model for the plate

\[ \Omega := (0, L) \times (-\ell, \ell), \quad 2\ell \cong \frac{L}{150}. \]

\[ \mathbb{E}_B(u) = \frac{E d^3}{12(1 - \sigma^2)} \int_\Omega \left( \frac{1}{2} (\Delta u)^2 + (\sigma - 1) \det(D^2 u) \right) dxdy \]

- \( u \): deflection in the vertical direction;
- \( E > 0 \): Young modulus;
- \( 0 < \sigma < \frac{1}{2} \): Poisson ratio;
- \( d \): thickness;

\[ \mathbb{E}_T(u) = \mathbb{E}_B(u) - \int_\Omega fu dxdy. \]

\{0\} \times (-\ell, \ell) and \{L\} \times (\ell, \ell): hinged edges.

(0, L) \times \{-\ell\} and (0, L) \times \{\ell\}: free edges.

Some comments on the plate model

- In the model of the plate only vertical displacements are allowed and in our case, when torsional oscillations appear, horizontal displacements are not negligible.
- The deck of the bridge is not actually a plate.
- Rocard writes: «The plate as a model is perfectly correct and corresponds mechanically to a vibrating suspension bridge».
- Due to the fact that the width of the deck is quite small with respect to its length, the deformation of any cross section is negligible.

The plate model

The plate satisfies the Euler-Bernoulli plate model.

\[ \frac{E \alpha^3}{12(1-\sigma^2)} \Delta^2 u = f \]

in \( \Omega \)

\[ u(0, y) = u_{xx}(0, y) = u(L, y) = u_{xx}(L, y) = 0 \]

for \( y \in (-\ell, \ell) \)

\[ u_{yy}(x, \pm \ell) + \sigma u_{xx}(x, \pm \ell) = 0 \]

for \( x \in (0, L) \)

\[ u_{yyy}(x, \pm \ell) + (2 - \sigma)u_{xxy}(x, \pm \ell) = 0 \]

for \( x \in (0, L) \)
The plate model

- $u_0$ equilibrium position if, the plate had no weight, there were no loads acting on the plate.
- $u_w$ equilibrium position of the plate if it was only subject to its own weight $w$.
- $u_h$ equilibrium position if the plate had no weight and if was subject to the restoring force of the cables-hangers.
- If both the weight and the action of the cables-hangers are considered, the two effects cancel and the equilibrium position $u_0 \equiv 0$ is recovered.

The plate model

Figure: The plate $\Omega$ and its subset $\omega$ (dark grey) where the hangers act.

- $\omega := (0, L) \times \left[(-\ell, -\ell + \varepsilon) \cup (\ell - \varepsilon, \ell)\right]$;
- $\Upsilon = \Upsilon(y)$: characteristic function of the set $\omega$;
- $h(y, u) = \Upsilon(y)(k_1u + k_2u^3)$: restoring force due to the cables-hangers system.

The plate model

\[ H(y, u) := \gamma(y) \int_0^u h(y, t) \, dt = \gamma(y) \left( \frac{k_1}{2} u^2 + \frac{k_2}{4} u^4 \right). \]

\[ \mathcal{E}_T(u) = \mathcal{E}_B(u) + \int_{\Omega} (H(y, u) - fu) \, dx \, dy. \]

\[
\begin{cases}
\frac{E d^3}{12(1-\sigma^2)} \Delta^2 u + h(y, u) = f & \text{in } \Omega \\
u(0, y) = u_{xx}(0, y) = u(L, y) = u_{xx}(L, y) = 0 & \text{for } y \in (-\ell, \ell) \\
u_{yy}(x, \pm\ell) + \sigma u_{xx}(x, \pm\ell) = 0 & \text{for } x \in (0, L) \\
u_{yyy}(x, \pm\ell) + (2 - \sigma) u_{xxy}(x, \pm\ell) = 0 & \text{for } x \in (0, L).
\end{cases}
\]

Problem (PS) admits a unique solution which is also the unique minimizer for the total energy \( \mathcal{E}_T \).
The plate model

\[ \mathcal{E}_u(t) := \frac{m}{2|\Omega|} \int_{\Omega} u_t^2 \, dx \, dy + \mathbb{E}_T(u). \]

\[ \mathcal{A}(u) := \int_{0}^{T} \left( \frac{m}{2|\Omega|} \int_{\Omega} u_t^2 \, dx \, dy \right) \, dt - \int_{0}^{T} \mathbb{E}_T(u) \, dt. \]

\[
\begin{aligned}
\frac{m}{|\Omega|} u_{tt} + \frac{E d^3}{12(1-\sigma^2)} \Delta^2 u + h(y, u) &= f \\
u(0, y, t) &= u_{xx}(0, y, t) = u(L, y, t) = u_{xx}(L, y, t) = 0 \\
u_{yy}(x, \pm \ell, t) + \sigma u_{xx}(x, \pm \ell, t) &= 0 \\
u_{yyy}(x, \pm \ell, t) + (2 - \sigma) u_{xxy}(x, \pm \ell, t) &= 0 \\
u(x, y, 0) &= u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y)
\end{aligned}
\]

Problem \((PE)\) is well posed and every solution \(u\) of \((PE)\) is globally defined in time.
The evolution problem with a damping term

\[
\begin{aligned}
\frac{m}{|\Omega|} u_{tt} + \delta u_t + \frac{E d^3}{12(1-\sigma^2)} \Delta^2 u + h(y, u) &= f \\
for \quad (y, t) \in (-\ell, \ell) \times (0, T) \\
\end{aligned}
\] 

\[
\begin{aligned}
u(0, y, t) &= u_{xx}(0, y, t) = u(L, y, t) = u_{xx}(L, y, t) = 0 \\
for \quad (x, t) \in (0, L) \times (0, T) \\
\end{aligned}
\] 

\[
\begin{aligned}
u_{yy}(x, \pm \ell, t) + \sigma u_{xx}(x, \pm \ell, t) &= 0 \\
for \quad (x, t) \in (0, L) \times (0, T) \\
\end{aligned}
\] 

\[
\begin{aligned}
u_{yyy}(x, \pm \ell, t) + (2 - \sigma) u_{xxy}(x, \pm \ell, t) &= 0 \\
for \quad (x, t) \in (0, L) \times (0, T) \\
\end{aligned}
\] 

In presence of a damping term the solution converges as \( t \to +\infty \) to the unique solution of the stationary problem.

Three fundamental questions

(Q1) why do longitudinal oscillations suddenly transform into torsional oscillations?

From Smith-Vincent we read: « the only torsional mode which developed under wind action on the bridge or on the model is that with a single node at the center of the main span ». This raises the following question:

(Q2) why do torsional oscillations appear with a node at midspan?

Three fundamental questions

From the report by O. H. Ammann, T. von Kármán, G. B. Woodruff we read: «the motions, which a moment before had involved a number of waves (nine or ten) had shifted almost instantly to two». This raises a third natural question

(Q3) are there longitudinal oscillations which are more prone to generate torsional oscillations?

The eigenvalue problem

\[
\begin{aligned}
    \Delta^2 w &= \lambda w & \text{in } \Omega \\
    w(0, y) &= w_{xx}(0, y) = w(\pi, y) = w_{xx}(\pi, y) = 0 & \text{for } y \in (-\ell, \ell) \\
    w_{yy}(x, \pm \ell) + \sigma w_{xx}(x, \pm \ell) &= 0 & \text{for } x \in (0, \pi) \\
    w_{yyyy}(x, \pm \ell) + (2 - \sigma)w_{xxy}(x, \pm \ell) &= 0 & \text{for } x \in (0, \pi).
\end{aligned}
\]

The set of eigenvalues may be ordered in an increasing sequence \( \{\lambda_k\} \) of strictly positive numbers diverging to \( +\infty \) and any eigenfunction belongs to \( C^\infty(\overline{\Omega}) \).

(i) For any $m \geq 1$ there exists a sequence of eigenvalues $\lambda_{k,m} \uparrow +\infty$ such that $\lambda_{k,m} > m^4$ for all $k \geq 1$; the corresponding eigenfunctions are of the kind

$$
\left[ a \cosh \left( y \sqrt{\lambda_{k,m}^{1/2} + m^2} \right) + b \sinh \left( y \sqrt{\lambda_{k,m}^{1/2} + m^2} \right) \\
+c \cos \left( y \sqrt{\lambda_{k,m}^{1/2} - m^2} \right) + d \sin \left( y \sqrt{\lambda_{k,m}^{1/2} - m^2} \right) \right] \sin(mx)
$$

for suitable constants $a, b, c, d$, depending on $m$ and $k$;

(ii) if the unique positive solution $m$ of $\tanh(\sqrt{2}m\ell) = \left( \frac{\sigma}{2 - \sigma} \right)^2 \sqrt{2}m\ell$ is an integer $m_* \in \mathbb{N}$, then $\lambda = m_*^4$ is an eigenvalue with corresponding eigenfunction

$$
\left[ \sigma \ell \sinh(\sqrt{2}m_* y) + (2 - \sigma) \sinh(\sqrt{2}m_* \ell) y \right] \sin(m_* x) ;
$$
The eigenvalue problem

(iii) for any $m \geq 1$, there exists an eigenvalue $\lambda_m \in ((1 - \sigma)^2 m^4, m^4)$ with corresponding eigenfunction

$$
\begin{bmatrix}
(\sqrt{\lambda_m} - (1 - \sigma)m^2) & \frac{\cosh(y\sqrt{m^2 - \sqrt{\lambda_m}})}{\cosh(\ell\sqrt{m^2 - \sqrt{\lambda_m}})} \\
(\sqrt{\lambda_m} + (1 - \sigma)m^2) & \frac{\cosh(y\sqrt{m^2 - \sqrt{\lambda_m}})}{\cosh(\ell\sqrt{m^2 - \sqrt{\lambda_m}})}
\end{bmatrix}
\sin(mx);
$$

(iv) for any $m \geq 1$, satisfying $\ell m\sqrt{2} \coth(\ell m\sqrt{2}) > \left(\frac{2-\sigma}{\sigma}\right)^2$, there exists an eigenvalue $\lambda_m \in (\lambda_m, m^4)$ with corresponding eigenfunction

$$
\begin{bmatrix}
(\sqrt{\lambda_m} - (1 - \sigma)m^2) & \frac{\sinh(y\sqrt{m^2 + \sqrt{\lambda_m}})}{\sinh(\ell\sqrt{m^2 + \sqrt{\lambda_m}})} \\
(\sqrt{\lambda_m} + (1 - \sigma)m^2) & \frac{\sinh(y\sqrt{m^2 + \sqrt{\lambda_m}})}{\sinh(\ell\sqrt{m^2 + \sqrt{\lambda_m}})}
\end{bmatrix}
\sin(mx);
$$

(v) There are no eigenvalues other than the ones characterized in (i) – (iv).
The eigenvalue problem

Figure: Eigenfunction of the first eigenvalue.

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Structural instability of nonlinear plates modelling suspension bridges
The eigenvalue problem

Figure: Eigenfunction of the second eigenvalue.

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The eigenvalue problem

Figure: Eigenfunction of the third eigenvalue.

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The eigenvalue problem

Figure: Eigenfunction of the first torsional eigenvalue.

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The eigenvalue problem

Figure: Eigenfunction of the second torsional eigenvalue.

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Structural instability of nonlinear plates modelling suspension bridges
Finite dimensional reduction

In our analysis we consider the first 14 longitudinal eigenvalues/eigenfunctions and the first 2 torsional eigenvalue/eigenfunctions.

In increasing order the first 10 eigenvalues are longitudinal; then we have the first torsional eigenvalue, then other 4 longitudinal eigenvalues and finally the second torsional eigenvalue.

<table>
<thead>
<tr>
<th>eigenvalue</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
<th>$\lambda_5$</th>
<th>$\lambda_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>kind</td>
<td>$\mu_{1,1}$</td>
<td>$\mu_{2,1}$</td>
<td>$\mu_{3,1}$</td>
<td>$\mu_{4,1}$</td>
<td>$\mu_{5,1}$</td>
<td>$\mu_{6,1}$</td>
</tr>
<tr>
<td>$\sqrt{\text{eigenvalue}} \approx$</td>
<td>0.98</td>
<td>3.92</td>
<td>8.82</td>
<td>15.68</td>
<td>24.5</td>
<td>35.28</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>eigenvalue</th>
<th>$\lambda_7$</th>
<th>$\lambda_8$</th>
<th>$\lambda_9$</th>
<th>$\lambda_{10}$</th>
<th>$\lambda_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>kind</td>
<td>$\mu_{7,1}$</td>
<td>$\mu_{8,1}$</td>
<td>$\mu_{9,1}$</td>
<td>$\mu_{10,1}$</td>
<td>$\nu_{1,2}$</td>
</tr>
<tr>
<td>$\sqrt{\text{eigenvalue}} \approx$</td>
<td>48.02</td>
<td>62.73</td>
<td>79.39</td>
<td>98.03</td>
<td>104.61</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>eigenvalue</th>
<th>$\lambda_{12}$</th>
<th>$\lambda_{13}$</th>
<th>$\lambda_{14}$</th>
<th>$\lambda_{15}$</th>
<th>$\lambda_{16}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>kind</td>
<td>$\mu_{11,1}$</td>
<td>$\mu_{12,1}$</td>
<td>$\mu_{13,1}$</td>
<td>$\mu_{14,1}$</td>
<td>$\nu_{2,2}$</td>
</tr>
<tr>
<td>$\sqrt{\text{eigenvalue}} \approx$</td>
<td>118.62</td>
<td>141.19</td>
<td>165.72</td>
<td>192.21</td>
<td>209.25</td>
</tr>
</tbody>
</table>

**Table:** Approximate value of the least 16 eigenvalues of for $\sigma = 0.2$ and $\ell = \frac{\pi}{150}$. 
We write $u(x, y, t) \approx \sum_{k=1}^{16} y_k(t)w_k(x, y)$ where $u$ solves

$$u_{tt} + \gamma \Delta^2 u + \gamma(y)(u + u^3) = 0 \quad \text{in } (0, \pi) \times \left(-\frac{\pi}{150}, \frac{\pi}{150}\right) \times [0, +\infty).$$

We also write

$\varphi_k(t) = y_k(t)$ for $k = 1, \ldots, 10,$

$\tau_1(t) = y_{11}(t),$

$\varphi_k(t) = y_{k+1}(t)$ for $k = 11, \ldots, 14,$

$\tau_2(t) = y_{16}(t).$
Finite dimensional reduction

\[
\begin{aligned}
\varphi_k''(t) + \gamma \mu_{k,1} \varphi_k(t) + \Phi_k(\varphi_1(t), \ldots, \varphi_{14}(t), \tau_1(t), \tau_2(t)) &= 0, \quad k \in 1, \ldots, 14, \\
\tau_k''(t) + \gamma \nu_{k,2} \tau_k(t) + \Gamma_k(\varphi_1(t), \ldots, \varphi_{14}(t), \tau_1(t), \tau_2(t)) &= 0, \quad k = 1, 2.
\end{aligned}
\]

We study analytically the interaction between each longitudinal mode \(\varphi_k, k = 1, \ldots, 14\), and one of the torsional modes \(\tau_l, l = 1, 2\).

We give a suitable definition of (linearized) stability for the \(k\)-th longitudinal mode \(k = 1, \ldots, 14\) with respect to the \(l\)-th torsional mode.

We prove that for small values of the total energy, the \(k\)-th mode, \(k = 1, \ldots, 14\), is stable with respect to the \(l\)-th torsional mode, \(l = 1, 2\).
Numerical results for the complete system

\[
\begin{align*}
\varphi_j''(t) + \gamma\mu_j,1\varphi_j(t) + \Phi_j(\varphi_1(t), \ldots, \varphi_{14}(t), \tau_1(t), \tau_2(t)) &= 0, \quad j \in \{1, \ldots, 14\}, \\
\tau_l''(t) + \gamma\nu_{l,2}\tau_l(t) + \Gamma_l(\varphi_1(t), \ldots, \varphi_{14}(t), \tau_1(t), \tau_2(t)) &= 0, \quad l = 1, 2, \\
\varphi_k(0) &= A, \quad \varphi_k'(0) = 0, \\
\varphi_j(0) &= \varphi_j'(0) = 0, \quad j \in \{1, \ldots, 14\}, \quad j \neq k, \\
\tau_l(0) &= \tau_l'(0) = 0, \quad l = 1, 2.
\end{align*}
\]

\((P_{0,k})\)
Numerical results for the complete system

\[
\begin{aligned}
\varphi_j''(t) + \gamma \mu_{j,1} \varphi_j(t) + \Phi_j(\varphi_1(t), \ldots, \varphi_{14}(t), \tau_1(t), \tau_2(t)) &= 0, \quad j \in \{1, \ldots, 14\}, \\
\tau_m''(t) + \gamma \nu_{m,2} \tau_m(t) + \Gamma_m(\varphi_1(t), \ldots, \varphi_{14}(t), \tau_1(t), \tau_2(t)) &= 0, \quad m = 1, 2, \\
\varphi_k(0) &= A, \quad \varphi_k'(0) = 0, \\
\varphi_j(0) &= \varphi_j'(0) = 0, \quad j \in \{1, \ldots, 14\}, j \neq k, \\
\tau_m(0) &= \tau_m'(0) = \delta \quad \text{if} \quad m = l, \tau_m(0) = \tau_m'(0) = 0 \quad \text{otherwise}.
\end{aligned}
\]

\((P_{\delta,k,l})\)
Numerical results for the complete system

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
<tr>
<td>$A_1(k)$</td>
<td>2.38</td>
<td>1.89</td>
<td>3.83</td>
<td>2.27</td>
<td>1.92</td>
<td>1.66</td>
<td>1.02</td>
</tr>
<tr>
<td>$A_2(k)$</td>
<td>5.17</td>
<td>4.38</td>
<td>4.94</td>
<td>8.08</td>
<td>4.18</td>
<td>4.05</td>
<td>3.87</td>
</tr>
</tbody>
</table>

Table: Numerical values of $A_1(k)$ and $A_2(k)$ when $\gamma = 5.17 \cdot 10^{-4}$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1(k)$</td>
<td>$&gt; 10$</td>
<td>$&gt; 10$</td>
<td>$&gt; 10$</td>
<td>0.62</td>
<td>1.08</td>
<td>1.46</td>
<td>1.83</td>
</tr>
<tr>
<td>$A_2(k)$</td>
<td>3.59</td>
<td>3.10</td>
<td>1.87</td>
<td>$&gt; 10$</td>
<td>$&gt; 10$</td>
<td>$&gt; 10$</td>
<td>$&gt; 10$</td>
</tr>
</tbody>
</table>

Table: Numerical values of $A_1(k)$ and $A_2(k)$ when $\gamma = 5.17 \cdot 10^{-4}$. 
Instability of longitudinal modes

Figure: Problem $P_{\delta,k,l}$ with $k = 10$, $l = 2$, $A = 2$ and $\delta = 5 \cdot 10^{-4}$
Instability of longitudinal modes

Figure: Problem $P_{\delta,k,l}$ with $k = 10$, $l = 1$, $A = 2$ and $\delta = 5 \cdot 10^{-4}$
(Q1) why do longitudinal oscillations suddenly transform into torsional oscillations?
Answer: presence of stability thresholds.

(Q2) why do torsional oscillations appear with a node at midspan?
Answer: for \( k = 8, 9, 10 \) we have \( A_1(k) > A_2(k) \).

(Q3) are there longitudinal oscillations which are more prone to generate torsional oscillations?
Answer: the critical amplitude of oscillation of a longitudinal mode depends on the ratio between the torsional and longitudinal frequencies. This ratio reaches its minimum for the 10th longitudinal mode.