An Invariance Principle for Nonlinear Switched Systems

A. Bacciotti and L. Mazzi
Dipartimento di Matematica del Politecnico di Torino
C.so Duca degli Abruzzi, 24 - 10129 Torino - Italy
bacciott@calvino.polito.it — luisa.mazzi@polito.it

Abstract

In this paper we address the problem of extending La Salle Invariance Principle to switched system. We prove an extension of the invariance principle relative to dwell-time switched solutions, and a second one relative to constrained switched systems.

Keywords: switched systems, stability, invariance principle, dwell time

1 Introduction

In spite of their apparent simplicity, switched systems may exhibit a very complicated dynamical behavior. For instance, an example in [8] shows that the LaSalle invariance principle does not apply to switched systems, at least as long as we are interested in asymptotic stability under completely arbitrary switching (other examples will be discussed later). By imposing appropriate restrictions about the admissible trajectories, global asymptotic stability results are proved in [8] for linear switched systems with multiple Liapunov functions, and satisfying a special observability condition. Later on, the results of [8] have been extended in [9] to nonlinear switched systems. Since the decrease conditions are expressed by means of non-strict inequalities, the results of [8], [9] (in particular, Theorem 7 and 9 in [9]) can be interpreted as extensions of the LaSalle invariance principle.

In the present paper, by following a different, more traditional approach, we obtain a different extension of the LaSalle invariance principle. It applies to switched systems defined by finite families of continuous (nonlinear) vector fields and its statement is closer in spirit to the classical one. This paper is organized as follows. The basic material, a motivating example and the statement of the main theorem are presented in Section 2, where we also point out a possible extension to the case of nonsmooth Liapunov functions. The proof is given in Section 3. In Section 4 we extend the result to constrained switched systems, by using multiple Liapunov functions. In Section 5 we compare our results to those obtained by other authors (in particular [9], [17], [14]), and we show the limitations of our results by means of some examples.

We wish to thank the referee, whose remarks led us to notably improve the previous version of Theorem 2.

2 Basic definitions and statement of the theorem

Geometric control theory, widely developed during the last three decades of the past century, relies on the idea that an input system can be interpreted as a family of vector fields (polysystem). A similar formalism can be adopted to describe the main object of study in the recent theory of switched systems (see [13], [12] and references therein). It should be remarked, however, that these two theories are usually focused on the investigation of different properties. Geometric control theory successfully addressed controllability and optimality problems, while switched system theory is basically concerned with stability.

The systems we are dealing with in this paper, are defined by finite families

$$\mathcal{F} = \{f_p(x), \quad p \in P\} \tag{1}$$

where $P = \{1, \dots, N\}$ and each $f_p(x)$ is a continuous vector field of $\mathbf{R}^n$ such that $f_p(0) = 0$. For simplicity, we assume that all the $f_p(x)$'s are forward complete. By a *switched solution* of (1) we mean any continuous curve $\varphi(t) : [0, +\infty) \to \mathbf{R}^n$ enjoying the following property: there exist a positively divergent sequence of times $\tau_0 = 0 < \tau_1 < \tau_2 < \dots$ (called *switching times*), and a sequence of indices $p_0, p_1, p_2, \dots$ (with $p_k \in P$) such that $\varphi(t)$ is an integral curve of the vector field $f_{p_k}(x)$ for $t \in (\tau_k, \tau_{k+1})$. The set of all the switched solutions will be denoted by $\mathcal{S}$. Note that the sequence $\{\tau_k\}$ associated to a switched solution is not unique, since we can add, if convenient, new "artificial" switching times. In particular, our definition includes the case of solutions with finitely many "essential" switches.

Equivalently, a switched system can be described as

$$\dot{x} = \sum_{p=1}^{N} u_p f_p(x)$$

where the set of admissible inputs is given by all the piecewise constant functions $u = (u_1, \dots, u_N) : [0, +\infty) \to U$, and U is the set of the vectors of the canonical basis of $\mathbf{R}^N$. In the literature about switched systems, an input function of this type is often referred to as a *switching signal*; the usual notation for switching signals is $\sigma(t) : [0, +\infty) \to P$. Without loss of generality, we assume that all switching signals are right continuous.

The following definitions are taken from [5] (but see also [13], [12]).

Definition 1 *We say that the origin is* stable under arbitrary switching *for (1) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each $\varphi \in \mathcal{S}$,*

$$|\varphi(0)| < \delta \quad \Longrightarrow \quad |\varphi(t)| < \varepsilon$$

for each $t \geq 0$. We say that the origin is globally attractive under arbitrary switching *if*

$$\lim_{t \to +\infty} \varphi(t) = 0,$$

for each $\varphi \in \mathcal{S}$. We finally say that the origin is globally asymptotically stable under arbitrary switching *for (1) if it is stable and globally attractive.*

Definition 2 *We say that the origin is* uniformly attractive *for (1) if for each* $\eta > 0$ *and for each* $R > 0$ *there exists* $T > 0$ *such that, for each* $\varphi \in \mathcal{S}$,

$$|\varphi(0)| < R, \ t > T \implies |\varphi(t)| < \eta.$$

Consistently with a commonly used terminology, we say that the origin is GUAS *for system (1) if it is stable under arbitrary switching and uniformly attractive.*

Definition 3 *Let* Ω *be an open subset of* $\mathbf{R}^n$ *containing the origin. We say that* $V(x) : \Omega \to [0, +\infty)$ *is a* common weak Liapunov function *for (1) if it is of class* C^1*, positive definite, and the following holds:*

$$\nabla V(x) \cdot f_p(x) \leq 0 \tag{2}$$

for each $x \in \Omega$ *and each* $p \in P$*. We say that* $V(x)$ *is a* common strict Liapunov function *for (1) if in (2) the strict inequality holds for each* $x \in \Omega \setminus \{0\}$.

If all the vector fields $f_p(x)$ are locally Lipschitz continuous, it has been proved that GUAS is equivalent to the existence of a smooth, everywhere defined and radially unbounded, strict common Liapunov function ([15]). The proof of the following proposition is straightforward (see [3], [5]).

Proposition 1 *Assume that there exists a common weak Liapunov function for (1). Then the origin is stable under arbitrary switching.*

Example 1 (see [8]). Consider the system formed by the pair of planar linear differential systems (see Fig. 1)

$$\dot{x} = A_1 x = \begin{pmatrix} -x_2 \\ x_1 - 2x_2 \end{pmatrix} , \qquad \dot{x} = A_2 x = \begin{pmatrix} x_2 \\ -x_1 - 2x_2 \end{pmatrix} .$$

It is easy to check that $V(x_1, x_2) = x_1^2 + x_2^2$ is a common weak Liapunov function. Note that here $\Omega = \mathbf{R}^n$ and $V(x_1, x_2)$ is radially unbounded. This implies in particular that all the switched solutions are bounded in the future. This system does not admit a common strict Liapunov function. Indeed, such a Liapunov function would be a Liapunov function also for the differential inclusion

$$\dot{x} \in \mathrm{co}\{A_1 x, A_2 x\} . \tag{3}$$

This in turn would imply that (3) is asymptotically stable at the origin (with respect to all its solutions i.e., all absolutely continuous curves satisfying (3)). But this is impossible, since any point of the form $(\bar{x}_1, 0)$ is an equilibrium solution for (3).

As a matter of fact, as indicated in [8], for this example it is possible to construct switched solutions starting from points of the form $(\bar{x}_1, 0)$ which do not converge to the origin. In Fig. 2 it is represented the behaviour of such a solution. The sequence of switching times is given by $\tau_1 = 1$, $\tau_k = \tau_{k-1} + \frac{1}{k}$. The upper graph draws the time-evolution of the x_1 axis, the lower one the time evolution of the x_2 axis. We may see that x_1 rapidly stabilizes around a value $\hat{x} > 1/2$.

∎

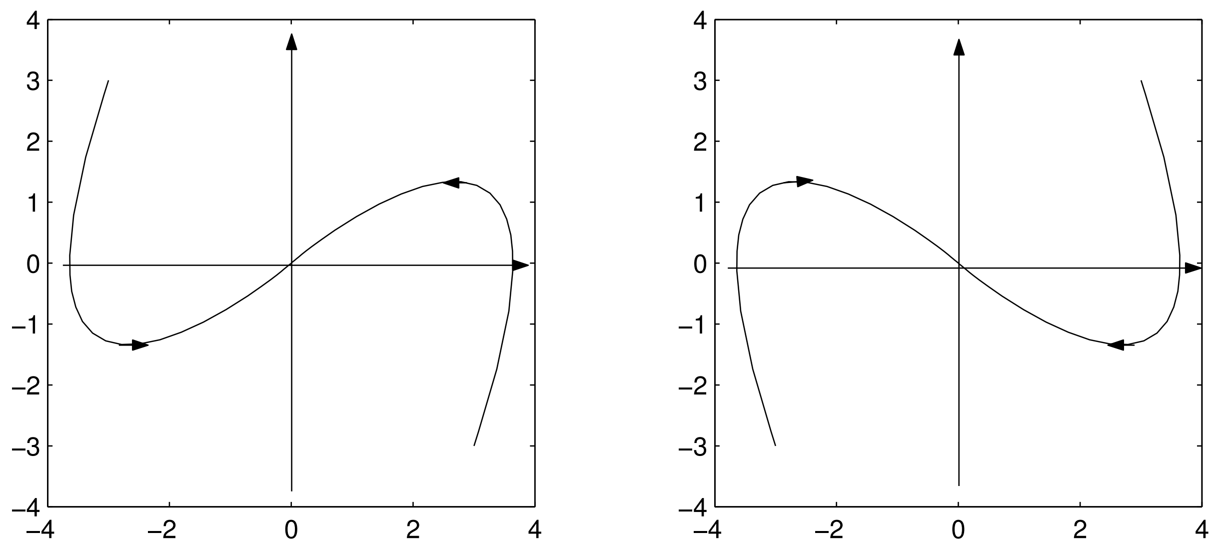

Figure 1: Phase portraits of the two systems of Example 1.

Example 1 suggests that in order to obtain asymptotic stability criteria of the LaSalle invariance principle type by exploiting the knowledge of a weak common Liapunov function, one needs to impose some restrictions on the admissible switched solutions.

Definition 4 *We say that* $\varphi \in \mathcal{S}$ *has a* nonvanishing dwell time *if there exist a sequence* $\{\tau_k\}$ *of switching times associated to* φ *and* $h > 0$ *such that*

$$\inf_k (\tau_{k+1} - \tau_k) \geq h \ . \tag{4}$$

Any value of h *for which (4) holds will be called a* dwell time *for* φ*.*

According to the notation introduced in [8], the set of all the switched solutions possessing a nonvanishing dwell time is denoted by $\mathcal{S}_{\rm dwell} \subset \mathcal{S}$. We emphasize that we do not ask for a universal dwell time, the value of h in Definition 4 being dependent on the specific solution $\varphi \in \mathcal{S}_{\rm dwell}$. The origin is said to be *asymptotically stable* with respect to dwell time solutions if Definition 1 holds with $\mathcal{S}$ replaced by $\mathcal{S}_{\rm dwell}$.

The LaSalle invariance principle is a classical, very useful achievement of dynamical system theory ([10]). It was extended to differential inclusions with upper semicontinuous, compact convex valued righthand side in [7] (see [2], [16] for further improvements). From this point of view, the main difference between the classical dynamical system framework and the differential inclusion one, is the lack of uniqueness of solutions. To face this aspect of the problem, the notion of invariance is replaced in [7] by the notion of weak invariance.

In the context of switched systems, we say that a compact set M is *weakly invariant* with respect to (1) if for each $x \in M$ there exist an index $p \in P$, a solution $\varphi(t)$ of the vector field $f_p(x)$, and a real number $b > 0$ such that $\varphi(0) = x$ and $\varphi(t) \in M$ for either $t \in [-b, 0]$ or $t \in [0, b]$. We also say

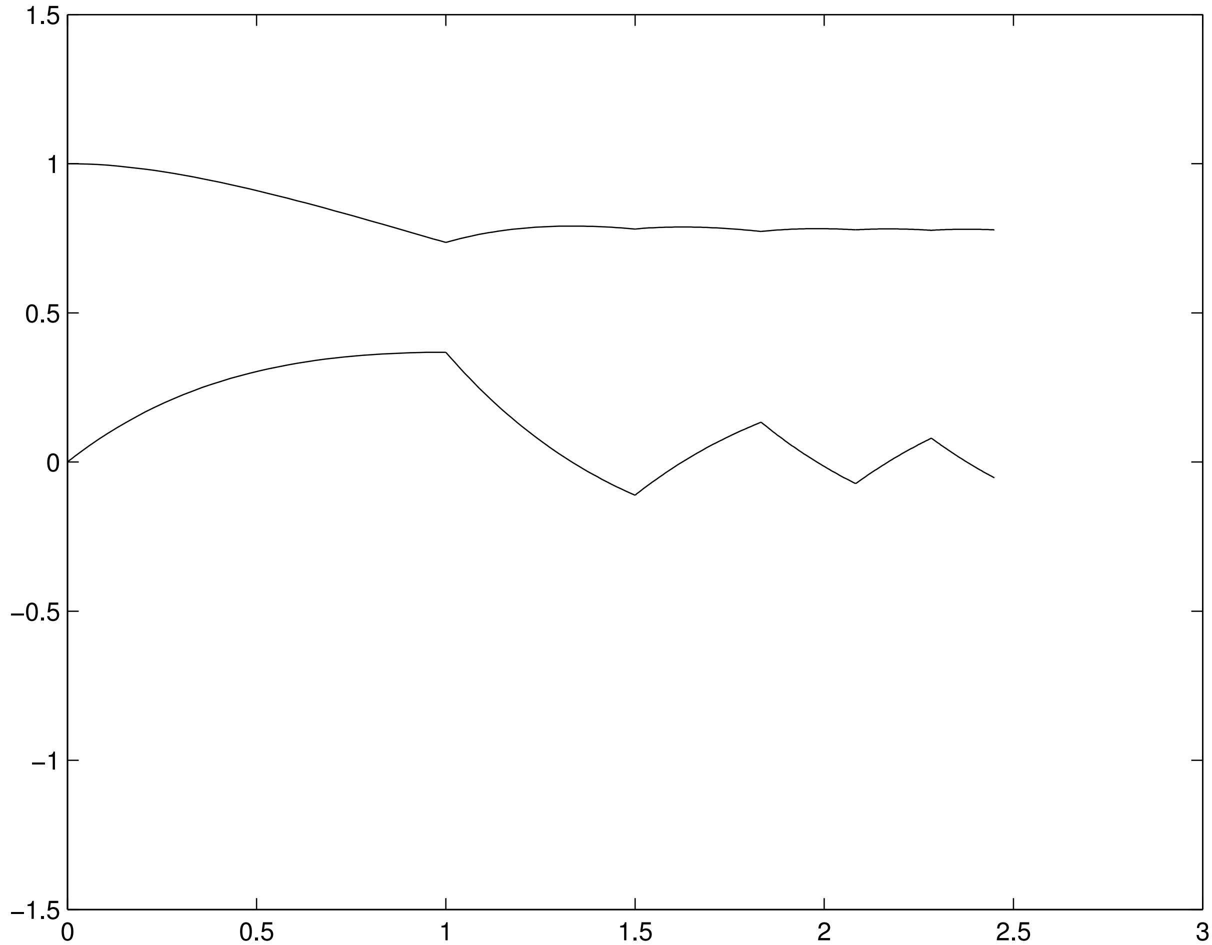

Figure 2: Time evolution of the two coordinates of a trajectory not tending to the origin.

that a switched solution $\varphi(t)$ is *attracted* by a compact set M if for each $\varepsilon > 0$ there exists a time $T > 0$ such that for each $t \geq T$ one has

$$\varphi(t) \in \mathcal{B}(\varepsilon, M)$$

where $\mathcal{B}(\varepsilon, x)$ is the open ball of radius ε centered at x, and $\mathcal{B}(\varepsilon, M) = \cup_{x \in M} \mathcal{B}(\varepsilon, x)$. Clearly, $\varphi(t)$ is attracted by M if and only if $\lim_{t \to +\infty} \operatorname{dist}(\varphi(t), M) = 0$.

We are now ready to state our main result.

Theorem 1 *Let $V(x) : \Omega \to [0, +\infty)$ be a weak common Liapunov function for (1). Let $l > 0$ and let Ω_l be the connected component of the level set $\{x \in \Omega : V(x) < l\}$ such that $0 \in \Omega_l$. Assume that Ω_l is bounded, and let*

$$Z = \{\ x \in \Omega : \exists p \in P \text{ such that } \nabla V(x) \cdot f_p(x) = 0\}\ .$$

Finally, let M be the union of all the compact, weakly invariant sets which are contained in $Z \cap \Omega_l$. Then every solution $\varphi(t) \in \mathcal{S}_{\rm dwell}$ such that $\varphi(0) \in \Omega_l$ is attracted by M.

Theorem 1 can be used to verify that the system of Example 1 is asymptotically stable at the origin with respect to dwell time solutions. Indeed, in that case it is easy to check that the unique compact, weakly invariant set contained in Z is the origin. We present another example, which shows that Theorem 1 may be seen as a complement to the results of [4].

Example 2 Consider the switched system $\{f_1(x), f_2(x)\}$, where

$$f_1(x) = \begin{pmatrix} -x_2 \\ x_1 - x_2^k \end{pmatrix} , \qquad f_2(x) = \begin{pmatrix} x_2 \\ -x_1 - x_2^k \end{pmatrix}$$

and k is any odd integer ($k \geq 1$). For $k = 1$, $f_1(x)$ and $f_2(x)$ are linear vector fields (actually, two stable foci); the origin is not asymptotically stable under arbitrary switching for the given system, since it does not meet the necessary and sufficient condition given in [4]. For $k > 1$, the same conclusion can be established by using an argument similar to that of Example 1. Even in this case, we can take $V(x_1, x_2) = x_1^2 + x_2^2$ as a weak common Liapunov function. Again, the unique compact, weakly invariant set contained in Z is the origin, so that the origin is asymptotically stable with respect to dwell time solutions.

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In some cases, it may be easier to find a common weak Liapunov function which is locally Lipschitz continuous, but not of class C^1. By following the same lines of [2], Theorem 1 can be extended to these situations. Let $V(x) : \Omega \to [0, +\infty)$ be locally Lipschitz continuous, where Ω is an open set containing the origin. Let us denote by $\partial_C V(x)$ the Clarke gradient of $V(x)$, and by

$$\dot{\overline{V}}(x) = \{a \in \mathbf{R} : \exists p \in P \text{ such that } \zeta \cdot f_p(x) = a, \ \forall \zeta \in \partial_C V(x)\} .$$

We modify Definition 3, by saying that $V(x)$ is a *nonsmooth common weak Liapunov function* for (1) if $V(x)$ is locally Lipschitz continuous, positive definite and

$$\dot{\overline{V}}(x) \subset (-\infty, 0] .$$

If we set $Z = \{x \in \mathbf{R}^n : 0 \in \dot{\overline{V}}(x)\}$, we may prove Theorem 1 also in the case of nonsmooth common weak Liapunov functions.

3 Proof of Theorem 1

Let $\varphi : [0, +\infty) \to \mathbf{R}^n$ be a continuous curve. Recall that the *ω-limit set* of φ is defined as

$$\Lambda_\varphi^+ = \{y : \lim_{k \to +\infty} \varphi(t_k) = y\}$$

where t_k is any positively divergent sequence of real numbers. It is clear that if φ is bounded, Λ_φ^+ is nonempty and compact.

If φ is a switched solution, it may happen that no solution of the f_p's lies in Λ_φ^+. The weak invariance property holds if φ is a dwell time switched solution, as we prove in Proposition. 2.

Let $p \in P$. We agree to say that a function $\psi(t)$ is a solution of the differential equation

$$\dot{x} = f_p(x) \tag{5}$$

on a closed interval $[c, d]$ if it is continuous on $[c, d]$, and it is a solution of (5) on (c, d).

Moreover, we say that a function $\psi(t)$ satisfies property (p) on an interval $[c, d]$ if $\psi(t)$ is a solution of (5) on $[c, d]$.

From the general theory on differential equations we derive the following :

Lemma 1 *Let $[c, d]$ be an interval containing the origin, and let $\{\varphi_k(t), t \in [c, d]\}$ be an equi-bounded family of solutions of $\dot{x} = f_p(x)$, such that $\varphi_k(0) \to a$, as $k \to +\infty$.*

Then there exists a subsequence $\{\varphi_{k_i}(t)\}$, uniformly convergent on $[c, d]$ to a solution $\varphi(t)$ of $\dot{x} = f_p(x)$, such that $\varphi(0) = a$.

Proposition 2 *Let $\varphi(t)$ be a switched solution of dwell-time h, bounded in the future. Then Λ_φ^+ is weakly invariant.*

Actually, we shall prove that, if $a \in \Lambda_\varphi^+$, there exist an index p, an interval $[\gamma, \delta]$ ($0 \in [\gamma, \delta]$, $\delta - \gamma \geq h/2$) and a function $\psi(t)$, satisfying property (p) on $[\gamma, \delta]$, such that $\psi(0) = a$ and $\psi(t) \in \Lambda_\varphi^+$, $\forall t \in [\gamma, \delta]$.

Proof. φ is bounded in the future, therefore there exists $R > 0$ such that $\varphi(t) \subset \mathcal{B}(R, 0)$, for all $t \geq 0$.

Let $\{\tau_k\}$ be a positively divergent, increasing sequence of switching times relative to the switched solution φ such that $\tau_{k+1} - \tau_k \geq h$, for all $k \in \mathbf{N}$.

Let $a \in \Lambda_\varphi^+$ and let $\{t_j\}$ be a positively divergent, increasing sequence, such that $\varphi(t_j) \to a$ as $t \to +\infty$.

Since the set of indices P is finite, there exists an index $p \in P$ and subsequences $\{t_{j_i}\}$ and $\{\tau_{k_i}\}$ such that, for all $i \in \mathbf{N}$,

1. $t_{j_i} \in [\tau_{k_i}, \tau_{k_i+1}]$,
2. $\varphi(t)$ satisfies property (p) on $[\tau_{k_i}, \tau_{k_i+1}]$.

We define,

$$\psi_i(t) := \varphi(t + t_{j_i}).$$

These functions satisfy property (p) on $[\tau_{k_i} - t_{j_i}, \tau_{k_i+1} - t_{j_i}]$. Moreover, $\psi_i(0) = \varphi(t_{j_i}) \to a$ as $i \to +\infty$. We set

$$\alpha_i := \tau_{k_i} - t_{j_i}, \qquad \beta_i := \tau_{k_i+1} - t_{j_i}.$$

These numbers satisfy the following properties, for all $i \in \mathbf{N}$,

1. $\beta_i - \alpha_i \geq h$,

2. $\alpha_i \leq 0$, $\beta_i \geq 0$,

3. $\psi_i(t)$ satisfy property (p) in $[\alpha_i, \beta_i]$.

Moreover, $\psi_i(t) \subset \mathcal{B}(R, 0)$, for all $t \in [\alpha_i, \beta_i]$, for all i.

Claim. There exists a subsequence $\{\psi_{i_k}(t)\}$ defined on a common interval $[\gamma, \delta] \subset \cap_{i=1}^{+\infty}[\alpha_i, \beta_i]$, with $\gamma - \delta \geq h/2$.

If the Claim holds, by Lemma 1 there exists a subsequence, still denoted by $\{\psi_{i_k}(t)\}$ for sake of simplicity, uniformly converging to a function $\psi(t)$ with property (p) on $[\gamma, \delta]$. Since $\psi_{i_k}(t) = \varphi(t + t_{j_{i_k}})$, $\psi(t) \in \Lambda_\varphi^+$, for all $t \in [\gamma, \delta]$.

In order to prove the claim, we have to consider several different cases, depending on the structure of the set of intervals $\{[\alpha_i, \beta_i]\}$.

Case 1. There exists a subsequence $\{\alpha_{i_k}\}$ such that $\alpha_{i_k} = 0$, for all k (or a subsequence $\beta_{i_k} = 0$).

In this case, since φ has dwell time h, $[0, h] \subseteq [0, \beta_{i_k})$. Thus, we have found an interval $[\gamma, \delta] = [0, h]$ where all the functions of the subsequence $\psi_{i_k}(t)$ are defined.

If $\beta_{i_k} = 0$, we get an analogous result, with $[\gamma, \delta] = [-h, 0]$.

Case 2. Case 1 does not occur and there exists a subsequence $\{\alpha_{i_k}\}$ such that $\alpha_{i_k} \to 0$ as $k \to +\infty$ (or a subsequence $\beta_{i_k} \to 0$ as $k \to +\infty$).

Without loss of generality, we can suppose that $\alpha_{i_k} < 0$, for all k. The functions $\psi_{i_k}(t)$ satisfy property (p) on $[\alpha_{i_k}, \alpha_{i_k} + h] \subseteq [\alpha_{i_k}, \beta_{i_k}]$.

Since $\alpha_{i_k} \to 0$, there exists $\tilde{k}$ such that, if $k > \tilde{k}$, then

$$-\frac{h}{2} \leq \alpha_{i_k} < 0, \qquad \frac{h}{2} \leq \alpha_{i_k} + h < h.$$

Therefore, all the functions $\psi_{i_k}(t)$ are defined on $[0, \frac{h}{2}] = [\gamma, \delta]$, and they satisfy property (p) on this interval.

If there is a sequence $\beta_{i_k} \to 0$, by a similar procedure we can find a common interval $[-\frac{h}{2}, 0]$.

Case 3. Case 1 and Case 2 do not occur and $\{\beta_i\}$ (or $\{\alpha_i\}$) is unbounded.

If $\{\beta_i\}$is unbounded, there exist a subsequence $\{\beta_{i_k}\}$ and an index $\hat{k}$ such that, if $k > \hat{k}$, $\beta_{i_k} > h$ and $\alpha_{i_k} < 0$. Therefore, all the functions $\psi_{i_k}(t)$ are defined on $[0, h] = [\gamma, \delta] \subset \cap_{i_k = i_{\hat{k}+1}}^{+\infty}[\alpha_{i_k}, \beta_{i_k}]$, and they satisfy property (p) on this interval.

If the set α_i is unbounded, by a similar procedure we can find a common interval $[-h, 0]$.

Case 4. None of the previous cases occur.

Both $\{\alpha_i\}$ and $\{\beta_i\}$ are bounded. By Bolzano-Weierstrass Theorem, we can find two subsequences α_{i_k} and β_{i_k} such that:

1. $\alpha_{i_k} \to \alpha$ as $k \to +\infty$, $\beta_{i_k} \to \beta$ as $k \to +\infty$.

2. $\beta - \alpha \geq h$,

3. $\alpha < 0$, $\beta > 0$.

There exists $\tilde{k}$ such that if $k > \tilde{k}$, then

$$\begin{array}{l} \alpha - \frac{h}{4} \leq \alpha_{i_k} \leq \alpha + \frac{h}{4} \\ \beta - \frac{h}{4} \leq \beta_{i_k} \leq \beta + \frac{h}{4}. \end{array}$$

Let us set

$$\begin{array}{l} \gamma := \min\left\{0, \alpha + \frac{h}{4}\right\} \\ \delta := \max\left\{0, \beta - \frac{h}{4}\right\}, \end{array}$$

We have selected an interval of length $\gamma - \delta \geq h/2$ where all the functions $\psi_{i_k}(t)$ are defined and satisfy property (p).

Thus we have proved that it is always possible to find a subsequence of functions $\psi_k(t)$ such that:

1. There exists an interval $[\gamma, \delta]$, containing 0, of length $\gamma - \delta \geq h/2$, where $\psi_k(t)$ satisfies property (p), for all k.

2. $\psi_k(t) = \varphi(t + t_k)$, for all $t \in [\gamma, \delta]$.

3. $\psi_k(0) \to a$, as $k \to +\infty$.

4. $\psi_k(t) \subset \mathcal{B}(R, 0)$, for all k, for all $t \in [\gamma, \delta]$.

We may apply Lemma 1 to prove that $\psi_k(t)$ uniformly converges to a function $\psi(t)$ satisfying property (p) on $[\gamma, \delta]$. Therefore $\psi(t) \subseteq \Lambda_\varphi^+$, for all $t \in [\gamma, \delta]$.

In the proof of the lemma it is fundamental that all the solutions have a nonvanishing dwell time. Otherwise, the times t_j could belong to intervals of arbitrarily small length, and it could not be possible to obtain the arc of solution contained in the ω-limit set.

Proof of Theorem 1.

First of all, let us remark that $M \neq \emptyset$, since $0 \in M$.

Step 1. Let us consider a point $\overline{x} \in \Omega_l$ and a dwell time switched solution $\varphi(t)$ such that $\varphi(0) = \overline{x}$. Since

$$g(t) := V(\varphi(t))$$

is nonincreasing and $g(0) = V(\overline{x}) = \overline{l} < l$, we have that $\varphi(t) \in \Omega_l$, for all $t \geq 0$. Therefore, $\varphi(t)$ is bounded in the future and Λ_φ^+ is nonempty and compact.

Step 2. By the monotonicity of $g(t)$, we have

$$\lim_{t \to +\infty} g(t) = \lim_{t \to +\infty} V(\varphi(t)) = l_0 \leq V(\overline{x}) = \overline{l} < l.$$

Step 3. Let us consider $a \in \Lambda^+_\varphi$ and a positively divergent sequence $\{t_j\}$ such that $\lim_{j\to+\infty} \varphi(t_j) = a$. Then

$$\lim_{j\to+\infty} V(\varphi(t_j)) = \lim_{j\to+\infty} g(t_j) = \lim_{t\to+\infty} g(t) = l_0.$$

By the continuity of $V(\varphi(t))$,

$$\lim_{j\to+\infty} V(\varphi(t_j)) = V(a).$$

Therefore, $V(a) = l_0$, for all $a \in \Lambda^+_\varphi$.

Step 4. By Proposition 2, there exists an interval $[\gamma, \delta]$ containing the origin and a function $\psi(t)$ such that,

1. $\psi(0) = a$,
2. $\psi(t) \subseteq \Lambda^+_\varphi$, for all $t \in [\gamma, \delta]$,
3. $\psi(t)$ is a solution of $\dot{x} = f_p(x)$, for some p, for all $t \in [\gamma, \delta]$.

Then $V(\psi(t)) = V(a) = l_0$, for all $t \in [\gamma, \delta]$, and

$$\frac{d}{dt} V(\psi(t)) = \nabla V(\psi(t)) \cdot f_p(\psi(t)) = 0 \tag{6}$$

for all $t \in (\gamma, \delta)$. Since $\nabla V(\psi(t)) \cdot f_p(\psi(t))$ is continuous, (6) actually holds also for $t = \gamma$ and $t = \delta$. By setting $t = 0$, we conclude that $a \in Z$.

We have thus proven that $\Lambda^+_\varphi \subset Z \cap \Omega_l$. The conclusion follows, since by definition, φ is attracted by Λ^+_φ.

4 Constrained switches

In many applications, switching among the vector fields of (1) is not completely arbitrary, but it is subject to constraints which may concern the functional nature of the input signal, or be dependent on the state of the system (see [1], [6], [13], [12]). The more natural framework for studying state-dependent switching rules is that of hybrid systems. However, as we shall show later (Example 3), in some simple situations state-dependent switching rules can be described by introducing certain constraints on the set of admissible inputs, and treated in our context. To this purpose, it is convenient to extend the Liapunov function approach by considering multiple Liapunov functions ([3], [13], [12], [8], [9]).

In this section we denote by Σ the set of all switching signals $\sigma(t) : [0, +\infty) \to P$, and by Σ_{dwell} the set of all switching signals generating dwell time solutions. Moreover, let us denote by $\mathcal{P}(\Sigma)$ the power set of Σ.

Definition 5 *By a* constrained switched system *we mean a pair* $(\mathcal{F}, \Upsilon)$ *where* $\mathcal{F}$ *is a finite family of continuous vector fields as in (1), and* Υ *is a map from* $\mathbf{R}^n$ *to* $\mathcal{P}(\Sigma)$.

Some comments about the role played in the previous definition by the set valued map Υ are in order. The map Υ contains the information about the restrictions imposed on the admissible inputs. More precisely, for each $x_0 \in \mathbf{R}^n$, it specifies the set of input signals which are admissible when the system is initialized at x_0. A *solution* of a constrained switched system corresponding to a given initial state x_0 is any solution of the initial value problem

$$\begin{cases} \dot{x} = f_{\sigma(t)}(x) \\ x(0) = x_0 \end{cases} \tag{7}$$

where $\sigma(t) \in \Upsilon(x_0)$. The definitions of stability and asymptotic stability can be easily updated in order to be applied to a constrained system $(\mathcal{F}, \Upsilon)$.

Definition 6 *Let Ω be an open set of $\mathbf{R}^n$ containing the origin. We say that a constrained switched system $(\mathcal{F}, \Upsilon)$ admits* multiple weak Liapunov functions *if there exists a family of smooth functions $V_p(x)$, $p \in P$, such that the following conditions hold.*

1. *For each $p \in P$, $V_p : \Omega \to [0, +\infty)$ is positive definite and of class C^1.*

2. *For all $p \in P$ and for all $x \in \Omega$, $\nabla V_p(x) \cdot f_p(x) \leq 0$.*

3. *For each $x_0 \in \mathbf{R}^n$ and each $\sigma \in \Upsilon(x_0)$, let $\{\tau_k\}$ be a sequence of switching times associated to σ. For every pair of switching times $\tau_h < \tau_j$ such that $\sigma(\tau_h) = \sigma(\tau_j) = p$, and for every solution $\varphi(t)$ of (7) we have,*
$$V_p(\varphi(\tau_{h+1})) \geq V_p(\varphi(\tau_j)) \ . \tag{8}$$

The meaning of condition (8) can be explained in the following way: for each switching signal, if there is a pair of intervals on which the same vector field f_p is active, then the value assumed by V_p along the trajectory at the left endpoint of the second interval, must be not greater than the value assumed by V_p along the trajectory at the right end point of the first interval. This last condition, paired with the second one, ensures that the functions $V_p(\varphi(t))$ are non increasing on the union of the intervals where f_p is active.

Actually, conditions of this kind appear in several non-equivalent forms in the literature (see [3], [8], [9]). The form adopted here is the same as it appear in [9] (Assumption 4).

Theorem 2 *Consider a constrained system $(\mathcal{F}, \Upsilon)$ and assume that $\Upsilon(\overline{x}) \subseteq \Sigma_{\text{dwell}}$ for each $\overline{x} \in \mathbf{R}^n$. Let $\{V_p(x) : \Omega \to [0, +\infty)\}$ be a family of weak Liapunov functions for $(\mathcal{F}, \Upsilon)$.*

Assume that, for all $l > 0$ and for all $p \in P$, the set $\Omega_l^p = \{x \in \Omega \ : \ V_p(x) < l\}$ is connected and bounded. Let us consider the sets:

$$\begin{array}{rcl} \Omega_l & = & \cap_{p \in P} \Omega_l^p \\ Z & = & \{ \ x \in \Omega : \exists p \in P \text{ such that } \nabla V_p(x) \cdot f_p(x) = 0\}, \end{array}$$

Moreover, let M be the intersection of all the weakly invariant sets contained in Z.

Then, for all $l > 0$, for all $\overline{x} \in \Omega_l$ and for all solutions $\varphi(t)$ of $(\mathcal{F}, \Upsilon(\overline{x}))$, $\varphi(t)$ is attracted by M.

Before proving Theorem 2, we need to introduce some additional notation and to prove a Lemma, to guarantee that any solution starting in Ω_l is bounded in the future.

We consider a switching signal $\sigma(t) \in \Upsilon(\overline{x})$ and a solution $\varphi(t)$ of $(\mathcal{F}, \Upsilon(\overline{x}))$. Let $\{\tau_i\}$ be a switching sequence associated to σ. We denote by I_p the union of the intervals $[\tau_i, \tau_{i+1})$ where $\sigma(\tau_i) = p$, that is the union of the intervals where φ satisfies property (p).

Moreover, let $U_l = \cup_{p\in P}\Omega_l^p$.

Lemma 2 *Under the same hypotheses as in Theorem 2, for all $l > 0$, there exists $m \geq l$ such that for all $\overline{x} \in \Omega_l$ and for all solutions $\varphi(t)$ of $(\mathcal{F}, \Upsilon(\overline{x}))$, $\varphi(t) \in \overline{U}_m$ for all $t \geq 0$, and hence $\varphi(t)$ is bounded in the future.*

Proof of Lemma 2. Let $P = \{1, \dots, N\}$. We shall proceed by induction on the number N of the vector fields contained in $\mathcal{F}$.

Step 1: $N = 1$. In this case, we see that the statement is true, with $m = l$. Indeed, if $\overline{x} = \varphi(0) \in \Omega_l = \Omega_l^1$, then, by the monotonicity hypothesis, $V_1(\varphi(t)) \leq V_1(\varphi(0)) < l$, for all $t \geq 0$, and $\varphi(t) \in \Omega_l = U_l \subseteq \overline{U}_l$.

Step N. We suppose that the statement is true for any subfamily of $\mathcal{F}$ formed by $N-1$ vector fields.

Let $\overline{x} \in \Omega_l$. If φ is any solution of $(\mathcal{F}, \Upsilon(\overline{x}))$ such that there exists $p \in P$ and a switching signal $\sigma(t)$ associated to φ such that $\sigma(t) \neq p$, for all $t \geq 0$, then by induction there exists $\kappa_p \geq l$ such that $\varphi(t) \in \overline{U}_{\kappa_p}$, for all $t \geq 0$.

Let $C = \cup_{p\in P}\overline{U}_{\kappa_p}$, and $m_p = \max_{x\in C} V_p(x)$. We claim that the number we are looking for is

$$m = \max\{m_p,\ p \in P\}.$$

To prove the claim, first of all let us observe that, for all $p \in P$, the following chain of inclusions holds:

$$\Omega_l \subseteq \overline{U}_{\kappa_p} \subseteq C \subseteq \overline{\Omega}_m \subseteq \overline{\Omega}_m^p \subseteq \overline{U}_m.$$

Actually, since $\kappa_p \geq l$, the inclusion $\Omega_l \subseteq \overline{U}_{\kappa_p}$ is true, because $\Omega_l \subseteq \Omega_l^q \subseteq \Omega_{\kappa_p}^q \subseteq U_{\kappa_p}$, for any q.

Moreover, the inclusion $C \subseteq \overline{\Omega}_m$ is true since if $x \in C$, then $V_p(x) \leq m_p \leq m$; therefore $x \in \overline{\Omega}_m^p$. The other inclusions are trivially verified.

Now, let us consider $\overline{x} \in \Omega_l$, a switching signal $\sigma(t) \in \Upsilon(\overline{x})$ and a solution $\varphi(t)$ of $(\mathcal{F}, \Upsilon(\overline{x}))$. Let $\{\tau_i\}$ be a switching sequence associated to σ, and let I_p defined as above. Let us suppose by contradiction that there exists $T > 0$ such that $\varphi(T) \notin \overline{U}_m$, that is, $V_p(\varphi(T)) > m$, for all $p \in P$.

Let $q \in P$ be such that $\sigma(T) = q$, so that $T \in I_q$. By Definition 6, we have that $V_q(\varphi(t)) \geq V_q(\varphi(T)) > m$, for all $t \in I_q \cap [0, T]$. By hypothesis, $V_q(\varphi(0)) < l \leq m$, hence $0 \notin I_q$ and $\tau_{i_q} = \min I_q \neq 0$, and $\varphi(\tau_{i_q}) \notin \overline{\Omega}_m^q$.

For all $t \in [0, \tau_{i_q})$, $\sigma(t) \neq q$, so that $\varphi(t)$ switches among $N-1$ vector fields. By induction, we have that $\varphi(t) \in \overline{U}_{\kappa_q} \subseteq \overline{\Omega}_m^q$ for all $t \in [0, \tau_{i_q})$. Then

$$\lim_{t\to\tau_{i_q}^-} \varphi(t) = \varphi(\tau_{i_q}) \in \overline{U}_{\kappa_p} \subseteq \overline{\Omega}_m^q,$$

a contradiction, since $\varphi(\tau_{i_q}) \notin \overline{\Omega}^q_m$.

∎

Proof of Theorem 2. Given $l > 0$, let $\overline{x} \in \Omega_l$, $\sigma(t) \in \Upsilon(\overline{x})$, $\varphi(t)$ a solution of $(\mathcal{F}, \Upsilon(\overline{x}))$. By Definition 6, for any $p \in P$, the function $\tilde{g}_p(t) = V_p(\varphi(t))$ is continuous on $[0, +\infty)$ and nonincreasing on I_p. We may modify $\tilde{g}_p(t)$ in order to obtain functions $g_p(t)$, continuous, nonincreasing on $[0, +\infty)$ and such that $g_p(t) = \tilde{g}_p(t)$ on I_p. Since for any $p \in P$, $g_p(0) = V_p(\varphi(0)) = l_p < l$, then $\lim_{t\to+\infty} g_p(t) = \bar{l}_p \leq l_p < l$.

By Lemma 2, $\varphi(t)$ is bounded in the future, so that $\Lambda^+_\varphi \neq \emptyset$. Given $a \in \Lambda^+_\varphi$, let us consider a divergent sequence $\{t_j\}$ such that $\varphi(t_j) \to a$ as $j \to +\infty$. Possibly by passing to a subsequence, we may find, as in the proof of Proposition 2, an index $p \in P$ and a sequence of switching times $\{\tau_{k_j}\}$ such that:

- $t_j \in [\tau_{k_j}, \tau_{k_j+1})$,
- $\varphi(t)$ satisfies property (p) on $[\tau_{k_j}, \tau_{k_j+1})$, that is $\sigma(t_j) = \sigma(\tau_{k_j}) = p$.

As in the proof of Proposition 2, we may prove that there exist an interval $[\gamma, \delta]$, $(0 \in [\gamma, \delta])$ and functions $\psi_j(t) = \varphi(t + t_j)$ defined on $[\gamma, \delta]$, satisfying the following properties:

- $\psi_j(t)$ uniformly converges to $\psi(t)$ on $[\gamma, \delta]$,
- $\psi(t) \subseteq \Lambda^+_\varphi$, for all $t \in [\gamma, \delta]$,
- $\psi(t)$ satisfies property (p) on $[\gamma, \delta]$,
- $\psi(0) = a$.

By construction, for all $t \in [\gamma, \delta]$:

$$\begin{aligned} V_p(\psi(t)) &= \lim_{j\to+\infty} V_p(\psi_j(t)) = \lim_{j\to+\infty} V_p(\varphi(t + t_j)) \\ &= \lim_{j\to+\infty} g_p(t + t_j) = \lim_{s\to+\infty} g_p(s) = \bar{l}_p < l. \end{aligned}$$

We conclude that

$$V_p(\psi(t)) = \bar{l}_p = g_p(a) = V_p(a), \quad \forall t \in [\gamma, \delta].$$

Hence

$$\nabla V_p(a) \cdot f_p(a) = \lim_{t\to 0} \frac{V_p(\psi(t)) - V_p(a)}{t} = 0.$$

The remaining part of the proof follows the same line as in Theorem 1.

∎

Example 3 Consider the family of planar linear systems

$$\dot{x} = A_1 x = \begin{pmatrix} -2x_2 - x_1 \\ x_1 \end{pmatrix}, \qquad \dot{x} = A_2 x = \begin{pmatrix} -x_2 - x_1 \\ 2x_1 \end{pmatrix}.$$

It is easy to check that $V_1(x_1, x_2) = x_1^2 + 2x_2^2$ is a weak Liapunov function for the first system, while $V_2(x_1, x_2) = 2x_1^2 + x_2^2$ is a weak Liapunov function for the second one. The origin is a stable focus for both systems, and in both cases the trajectories are run counterclockwise. We assume that switching is determined according to the following rule:

- the system defined by matrix A_1 is active for $x_1 x_2 > 0$;

• the system defined by matrix A_2 is active for $x_1 x_2 < 0$.

Since the time needed to go from one side to the other side of each quadrant is constant for every system, this state space-dependent switching rule can be easily reinterpreted as a time-dependent switching rule, by assigning the length of the time intervals between switching times. However, it is clear that the length of the first time interval depends on the initial state. Thus, in this way we have selected a subset $\Upsilon(x_0) \subset \Sigma_{\text{dwell}}$.

It is easy to check that the assumptions of Theorem 2 are fulfilled, with $Z = \{x \in \mathbf{R}^2 : \ x_1 = 0\}$. Since only the trivial solution lies in Z, the origin is asymptotically stable.

∎

5 Further remarks

First of all, we wish to compare our results to those appeared recently in the literature. In particular, Theorems 7 and 9 of [9] represent criteria for asymptotic stability of switched systems, which make use of multiple weak Liapunov functions. Let us remark that Theorem 1 in this paper provides information about the asymptotic behavior of trajectories even if the system is not asymptotically stable. Of course, under additional conditions, Theorem 1 can be used to check asymptotic stability. From this point of view, it seems to be alternative to the results of [8] and [9]. Indeed, on one hand, the dwell time condition imposed in Theorem 1 is more restrictive than Assumption 3 in [9]. On the other hand, the small-time observability condition (Assumption 2 of [9]) is not always easy to check. For instance, in Example 2 of this paper (see Section 2) asymptotic stability can be easily ascertained by means of Theorem 1 (with a common Liapunov function), while it is not clear how to verify the small-time observability condition, since there are no indications on how to construct the function γ used there.

The observability condition of [9] can be used in junction with our invariance principle, as well: it actually implies that the attractive invariant set M reduces to the origin.

The extension to the LaSalle invariance principle to hybrid automata is one of the main concern of [17], (see [14] as well). Such an extension is performed under the assumption that the automaton is deterministic and continuous, thus ensuring uniqueness of solutions and continuous dependence with respect to the initial data. Hence, is does not apply to the systems considered in this paper, which undergo switching in an almost arbitrary way. Finally, we point out that a "controlled" invariance principle for discrete time systems in given in [11].

We wish now to remark some limitations of our result. First of all, the extension of the invariance principle to switched systems requires some adjustments, regarding the type of switched solutions we deal with. Moreover, we need a generalized notion of the "bad set" Z and a weakened notion of invariance. As a consequence, there are cases where our theorem is not applicable, although the switched system is actually asymptotically stable under arbitrary switching, as it is illustrated by the following examples.

Example 4 Let us consider the linear stable focus and the linear stable node

$$\dot{x} = A_1 x = \begin{pmatrix} -x_1 - x_2 \\ x_1 \end{pmatrix} , \qquad \dot{x} = A_2 x = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} .$$

The quadratic function $V(x) = x_1^2 + x_2^2$ is a common weak Liapunov function. Moreover, $Z = \{x_1 = 0\}$. Since $x_1 = 0$ is union of orbits of the second system, according to Theorem 1 we may at most conclude that all the switched solutions tend to the x_2-axis.

Anyway, in this case the function $W(x) = x_1^2 + x_1x_2 + x_2^2$ is a common strict Liapunov function, therefore the origin is asymptotically stable under arbitrary switching.

∎

Example 5 Once again, we consider two systems; the first one is the same stable focus introduced in Example 4 (see Fig. 3, on the left). As a second system, we consider the continuous differential system:

$$\dot{x} = f(x) = \begin{cases} A_1(x) = \begin{pmatrix} -x_1 - x_2 \\ x_1 \end{pmatrix}, & \text{if } x_1 < 0 \\ A_3(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, & \text{if } x \geq 0. \end{cases}$$

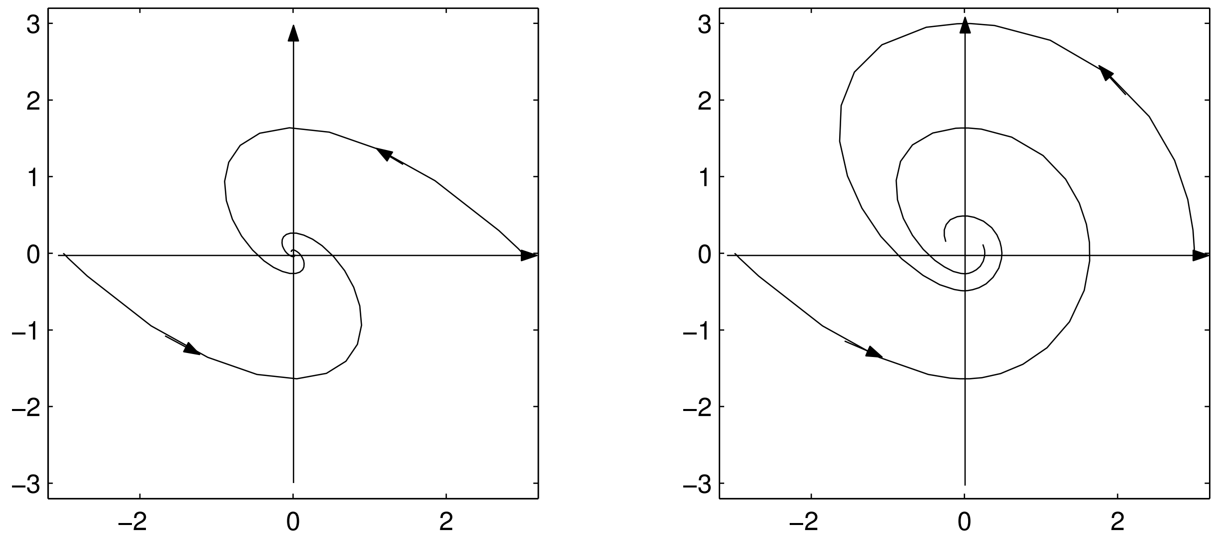

Figure 3: Phase portraits of the two systems of Example 5.

On the right side of Fig. 3 we may see how the second system is asymptotically stable at the origin, even if the phase portrait in the half-plane $x_1 \geq 0$ is that of a linear center.

The quadratic function $V(x) = x_1^2 + x_2^2$ is once again a common weak Liapunov function; in this case $Z = \{(x_1, x_2) : \ x_1 \geq 0\}$. The largest weakly invariant (but not invariant) set contained in Z is the same Z, because the half-circles centered at the origin are arcs of orbits of the second system.

Hence, we do not obtain any useful information via Theorem 1. We may conclude just the same that the origin is asymptotically stable under arbitrary switching using the method of the "worse case" proposed by Boscain in [4] (see also [5]).

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