Closed Loop Stabilization of Switched Systems

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Abstract

In this paper we address the problem of closed loop stabilization of switched control systems. First we propose a general definition of closed loop switched solution of a control system. Then we use it in order to get a stabilizability result for planar bilinear systems. This result is finally compared with existing literature.

Keywords: Switched systems, bilinear systems, stabilization, discontinuous feedback laws.

1 Introduction

This paper addresses the closed loop switched stabilization problem for finite dimensional nonlinear systems. The main results concern planar systems defined by pairs of linear vector fields and are strictly related to the recent literature about hybrid and switched systems. This paper also aims to propose a set of definitions and a suitable terminology, which may help when the stability analysis is performed in a non conventional setting.

By an *input system* we mean a system of equations of the form

$$\dot{x} = f(x, u) \tag{1}$$

where $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ is the *state variable*, $u \in \mathbf{R}^m$ is the *input variable*, and $f : \mathbf{R}^{n+m} \to \mathbf{R}^n$ is measurable and locally bounded. Given a nonempty subset U of $\mathbf{R}^m$, let us denote by $\mathcal{M}$ the set of all the measurable, locally bounded functions $u(t) : [0, +\infty) \to U$. An *open loop system* can be described as a pair $(f, \mathcal{U})$ where f defines the system equations (1), and $\mathcal{U} \subseteq \mathcal{M}$ is called the set of *admissible inputs*. For any $u(\cdot) \in \mathcal{U}$, (1) becomes a system of time dependent equations

$$\dot{x} = f(x, u(t)) \ . \tag{2}$$

We emphasize that f is not required to be continuous, so that existence of solutions (in the usual sense) of (2) is not guaranteed, in general.

In this paper we deal with a variety of notions of stability, that we now attempt to introduce in a unified manner. We find convenient to refer to an open loop system $(f, \mathcal{U})$. Assume that to each $\bar{x} \in \mathbf{R}^n$ and to each $u(\cdot) \in \mathcal{U}$, a set $\mathcal{S}_{\bar{x},u(\cdot)}$ has been assigned. The elements of $\mathcal{S}_{\bar{x},u(\cdot)}$ are absolutely continuous curves of $\mathbf{R}^n$ that we agree to call "solutions" corresponding to the initial state $\bar{x}$ and the admissible input $u(\cdot)$. We say that the open loop system $(f, \mathcal{U})$ is *(globally) asymptotically stable at the origin* (with respect to the assigned $\mathcal{S}_{\bar{x},u(\cdot)}$) if the following conditions hold:

(P0) for all $\bar{x} \in \mathbf{R}^n$ and all $u(\cdot) \in \mathcal{U}$, $\mathcal{S}_{\bar{x},u(\cdot)} \neq \emptyset$;

(P1) for all $\bar{x} \in \mathbf{R}^n$, all $u(\cdot) \in \mathcal{U}$ and all $\varphi(\cdot) \in \mathcal{S}_{\bar{x},u(\cdot)}$, $\varphi(t)$ is defined for each $t \geq 0$ and $\varphi(0) = \bar{x}$;

(P2) for all $\bar{x} \in \mathbf{R}^n$, all $u(\cdot) \in \mathcal{U}$ and all $\varphi(\cdot) \in \mathcal{S}_{\bar{x},u(\cdot)}$, $\lim_{t\to+\infty} \varphi(t) = 0$;

(P3) for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $\bar{x} \in \mathbf{R}^n$, all $u(\cdot) \in \mathcal{U}$ and all $\varphi(\cdot) \in \mathcal{S}_{\bar{x},u(\cdot)}$,

$$|\bar{x}| < \eta \Longrightarrow |\varphi(t)| < \varepsilon, \ \forall t \geq 0 \ .$$

Remark 1 Note that if U is a singleton, then (1) becomes a system of time invariant ordinary differential equations, and conditions (P0), ..., (P3) reduce to the classical definition of asymptotic stability.

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Remark 2 Note that if f is continuous, then the term "solution" is unambiguously intended in Carathéodory sense (the definition of Carathéodory solution is recalled in Appendix A), and condition (P0) is automatically fulfilled.

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By a *state-static-memoryless feedback law* we mean a function

$$u = k(x) : \mathbf{R}^n \to \mathbf{R}^m \ . \tag{3}$$

Replacing (3) in (1) we obtain the so-called *closed-loop system*

$$\dot{x} = f(x, k(x)) \ . \tag{4}$$

By virtue of Remark 1, our definition of asymptotic stability applies to (4), as well. Roughly speaking, the input system (1) is said to be *stabilizable at the origin* if there exists a feedback law (3) such that the closed-loop system (4) is asymptotically stable at the origin. Of course, by virtue of Remark 2, this notion is well founded if f is continuous, and if we limit ourselves to continuous feedback laws. On the contrary, when discontinuous feedback laws are allowed, the right-hand-side of (4) may be discontinuous even if f is smooth. As already recalled, in this case Carathéodory solutions need not to exist and conditions (P0), ... (P3) have to be referred to some notion of generalized solution[1] as it will be explained in the Section 2. Now, it is well known that the question whether a given discontinuous feedback law $u = k(x)$ stabilizes (1) at the origin, may have different answers if we adopt different notions of solutions (see [9]; some examples are also reported in Section 2).

Discontinuous feedback laws are unavoidable when the input is constrained to take values in a finite set U. This is a typical requirement, for instance, in geometric control theory and in the more recent theory of switched systems. The more natural definition of switched system is usually formulated from an "open loop" point of view. From now on, let $U \subset \mathbf{R}^m$ be a finite set (not reduced to a singleton)[2]. Let $\mathcal{U}_{pc} \subset \mathcal{M}$ be the set of all functions which are piecewise constant[3]. We say that an open loop system $(f, \mathcal{U})$ is a *switched system* if $\mathcal{U} = \mathcal{U}_{pc}$, and we say that a solution of a switched system (i.e., any Carathéodory solution of (1) corresponding to an input $u(\cdot) \in \mathcal{U}_{pc}$) is an *open loop switched solution.* Note that under the assumption that f is continuous, open loop switched solutions exist for any initial state; moreover, they are continuous and piecewise of class C^1. In particular, the set of points where a switched solution is not differentiable cannot have accumulation points (accumulation of switching points is also called Zeno's phenomenon[4]; it is usually considered strongly undesirable for practical purposes).

We say that the input system (1) is *asymptotically stable under arbitrary switching* if $(f, \mathcal{U}_{pc})$ is asymptotically stable with respect to all its open loop switched solutions. Asymptotic stability under arbitrary switching has been investigated in some early papers (see for instance [6]) and, more recently, in [7] and [11] (but see also the good surveys [12], [18] and the references therein).

In this paper we are rather interested in stability under appropriate switching, which can be viewed as a switched systems version of the well known asymptotic controllability notion ([10]). Consider a switched system $(f, \mathcal{U}_{pc})$, and let $\mathcal{S}_{\bar{x}, u(\cdot)}$ be the set of all the open loop switched solutions corresponding to the initial state $\bar{x}$ and the input $u(\cdot)$. We say that the system is *asymptotically stable under appropriate switching* if the following properties hold:

1. for all $\bar{x} \in \mathbf{R}^n$, there exists $u(\cdot) \in \mathcal{U}_{pc}$ such that
   (a) $\mathcal{S}_{\bar{x}, u(\cdot)} \neq \emptyset$;
   (b) for all $\varphi(\cdot) \in \mathcal{S}_{\bar{x}, u(\cdot)}$, $\varphi(t)$ is defined for all $t \geq 0$ and $\lim_{t \to +\infty} \varphi(t) = 0$;
2. for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $\bar{x} \in \mathbf{R}^n$ with $|\bar{x}| < \eta$, there exists $u(\cdot) \in \mathcal{U}_{pc}$ such that (a), (b) hold, and $|\varphi(t)| < \varepsilon$ for all $t \geq 0$ and all $\varphi(\cdot) \in \mathcal{S}_{\bar{x}, u(\cdot)}$.

[1] In our opinion, property (P0) must be considered an essential component of the stability concept; indeed, in absence of such an existence requirement, some important and useful results, such as the invariance LaSalle principle, may fail to be true.

[2] In principle, U might be any subset of $\mathbf{R}^n$, but for many practical applications only the case where U is finite is of interest.

[3] This means that on every compact interval there are at most finitely many points where the function is discontinuous; moreover, between two consecutive points where it is discontinuous, the function is constant. The value of the function at the discontinuity points does not matter.

[4] Some authors distinguish left and right accumulation [21], [17]

Most of the known results about asymptotic stability under appropriate switching concern systems which have a linear or a bilinear structure ([15], [21], [27]). For instance in [27] the authors consider systems defined by pairs of real matrices (A, B). Such systems can be written in the form

$$\dot{x} = uAx + (1-u)Bx \tag{5}$$

where u denotes a scalar input taking values in $U = \{0, 1\}$. Of course, the stability properties of (5) depends on the dynamic behavior of the two "subsystems"

$$\dot{x} = Ax\ , \qquad \dot{x} = Bx\ . \tag{6}$$

It is well known that appropriate switching between the trajectories of the subsystems (6) may result in a stabilizing effect, even if both matrices A and B are not Hurwitz ([25]). The investigation carried out in [27] concerns pairs of 2×2 matrices (A, B), and covers the following cases: 1) both A and B have complex eigenvalues with positive real part; 2) both A and B have real eigenvalues of opposite sign; 3) both A and B have real positive eigenvalues. Using a similar approach, and exploiting the existence of a common Lyapunov function, the case of a pair of oscillators (i.e., the case where both A and B have conjugate imaginary eigenvalues) is considered in [15], [16]. However, we notice that the switching rules proposed in the mentioned papers, although depending on the position in the state space, exhibit in some cases a hysteretic behavior. Hence, they are hybrid in nature and cannot be realized under the action of a pure state-static-memoryless feedback law.

These considerations enlight the need of introducing a new notion of stabilization, that we propose to call closed loop switched stabilization or, in short, s-stabilization. It relies on the construction of a state-static-memoryless feedback, and enables us to preserve at the same time the switching nature of the system. This will be done in Section 2, where we also present some illustrative examples.

We are in a position to state the main question of interest in this paper: *to find out conditions and/or classes of systems for which asymptotic stability under appropriate switching implies s-stabilizability.* We address this question in Section 3, and we give a positive answer for certain types of systems of the form (5): typically, pairs of harmonic oscillators, but our approach can be applied to other cases, as well. Let us emphasize that, since U is a finite set, a feedback law is expected to be discontinuous and hence, the closed loop system has a discontinuous righthand side. One remarkable feature of the feedback law designed in this paper is that it stabilizes the system, regardless the most usual notions of solution. The main result is proved in Section 4. Section 5 contains some examples and Section 6 points out some possible extentions.

We finally remark that the feedback law proposed in this paper cannot be deduced from the results of [10], [1]. Indeed, in [10] the authors make use of sampling solutions: as we remark in Section 2, they are different form closed loop switched solutions and can be better interpreted in the context of hybrid systems. In [1], the authors refer to Carathéodory solutions in general, and do not consider the problem of avoiding Zeno's phenomenon. Moreover we remark that feedback law proposed in this paper is not a "patchy feedback" in the sense of [1].

2 General definitions

The main purpose of this section is to set up a precise version of the feedback stabilization problem, which may represent a natural closed-loop counterpart of the problem of asymptotic stability under appropriate switching. For the moment, we refer to the general form (1). As explained in the introduction, the main feature of our problem is that the set $U \subset \mathbf{R}^m$ where the admissible inputs take values, is finite. Hence, the stabilizing feedback law $u = k(x)$ to be constructed, is expected to be discontinuous, in general. The resulting closed-loop system (4) will have a discontinuous righthand side, and many different notions of solution can be adopted (see for instance [14], [13], [4]). Here, we consider three possibilities: Krasowski solutions, Carathéodory solutions and closed loop switched solutions. Krasowski and Carathéodory solutions are classical; they are recalled in Appendix A for reader's convenience. As far as closed loop switched solutions are concerned, we propose the following definition.

Definition 1 *Let us consider system (1), and let $u = k(x)$ be a given feedback law. We say that an absolutely continuous curve $\varphi(t) : I \to \mathbf{R}^n$, where I is an interval of real numbers, is a* closed loop switched solution *if*

(a) there exists $u(t) \in \mathcal{U}_{pc}$ such that $k(\varphi(t)) = u(t)$ for all $t \in I$;

(b) $\dot{\varphi}(t) = f(\varphi(t), k(\varphi(t)))$ for a.e. $t \in I$.

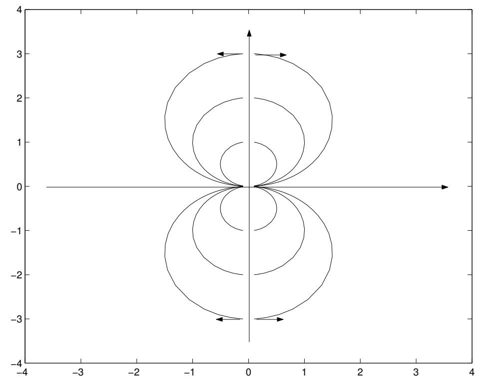

Figure 1: A C- and s- (but not K-) asymptotically stable system.

In other words, a closed loop switched solution is a Carathéodory solution of system (4) which can be reproduced as a switched open-loop solution. However, in general there may be Carathéodory solutions which are not of closed loop switched type (see Example 3 below).

Closed loop switched solutions should be compared with sampling solutions, as defined for instance in [10], [19]. It turns out that any closed loop switched solution is sampling, but there are sampling solutions which are not closed loop switched. We note in addition that sampling solutions always exist for any initial condition, while closed loop switched solutions may fail to exist.

We are now ready to propose our definitions of stabilizability.

Definition 2 *We say that the input system (1) is* K-stabilizable *(respectively,* C-stabilizable*,* s-stabilizable*) at the origin if there exists a state-static-memoryless feedback* $u = k(x) : \mathbf{R}^n \to U$ *such that properties (P0), (P1), (P2), (P3) hold with respect to the set of all the Krasowski (respectively, Carathéodory, closed loop switched) solutions* $\varphi(t)$ *of (4).*

Note that the existence condition (P0) is automatically fulfilled for Krasowski solutions, but not for Carathéodory and closed loop switched solutions. Note also that closed loop switched and Carathéodory solutions, when they exist, are particular cases of Krasowski solutions. However, because of the existence requirement, K-stabilizability does not imply in general C- and s-stabilizability.

Of course, one can formulate a further definition of stabilizability with respect to Filippov solutions. The notion of Filippov solution is very popular and has been already applied in the context of switched systems (see for instance [24], [21], [23]). However, as far as the results of this paper are concerned, what can be said for Krasowski solutions holds for Filippov solutions, as well. As explained in Appendix A, Krasowski solutions are more general than Filippov solutions. Thus, we can limit ourselves to consider the latter ones.

The following examples may help to clear up the situation.

Example 1 The two-dimensional system

$$\begin{cases} \dot{x}_1 = (x_1^2 - x_2^2)u \\ \dot{x}_2 = 2x_1x_2u \end{cases}$$

is known as Artstein's circles example. It is known that the feedback law

$$u = k(x) = \begin{cases} 1 & \text{if } x_1 \geq 0 \\ -1 & \text{if } x_1 < 0 \end{cases}$$

C-stabilizes the system at the origin (see Figure 1). In fact, the same feedback law provides s-stabilization, as well. However, this feedback law is not K-stabilizing: indeed, each point of the form $(0, \bar{x}_2)$ represents an equilibrium solution, in the sense of Krasowski.

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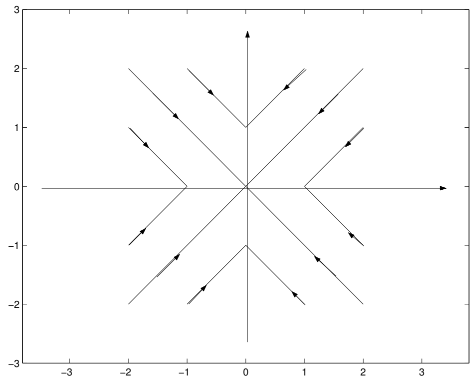

Figure 2: A K- (not C-, not s-) asymptotically stable system.

Example 2 Consider the two-dimensional system

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \end{cases}$$

where the control (u_1, u_2) is subject to take values on the set

$$\{(0,0),(1,1),(1,-1),(-1,-1),(-1,1)\} \ .$$

It is clear that the feedback law

$$u = k(x) = \begin{cases} u_1 = u_2 = 0 & \text{if } x_1 = x_2 = 0 \\ u_1 = u_2 = -1 & \text{if } x_1 \geq 0, x_2 > 0 \\ u_1 = -u_2 = -1 & \text{if } x_1 > 0, x_2 \leq 0 \\ u_1 = u_2 = 1 & \text{if } x_1 \leq 0, x_2 < 0 \\ u_1 = -u_2 = 1 & \text{if } x_1 < 0, x_2 \geq 0 \end{cases}$$

K-stabilizes the system at the origin (see Figure 2). However, it stabilizes the system neither in Carathéodory sense nor in the closed loop switched sense. Indeed, for initial states of the form $(\bar{x}_1, 0)$ or $(0, \bar{x}_2)$ there exist no Carathéodory and closed loop switched solutions. Moreover, for each other initial state $(\bar{x}_1, \bar{x}_2)$ with $\bar{x}_1 \neq \bar{x}_2$, the (unique) Carathéodory solution is not continuable on $[0, +\infty)$.

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Example 3 Consider again a two-dimensional system

$$\begin{cases} \dot{x}_1 = -u_1 + 2u_2 \\ \dot{x}_2 = -2u_1 - u_2 \end{cases}$$

where the input variables are subject to the constraints $u_i \in \{-1, 0, 1\}$ for $i = 1, 2$. This system is K-stabilized and C-stabilized at the origin by means of the feedback law

$$u_1 = \operatorname{sgn} x_1 \ , \qquad u_2 = \operatorname{sgn} x_2$$

(see Figure 3). However, we note that all nontrivial Carathéodory solutions of the closed-loop system reach the origin in finite time, and present a converging sequence of switches. Hence, for each initial state $(\bar{x}_1, \bar{x}_2) \neq (0, 0)$ there exist no closed loop switched solutions and, as a consequence, the feedback is not s-stabilizing. This example is basically due to Filippov (see [13], see also [25] and [23]).

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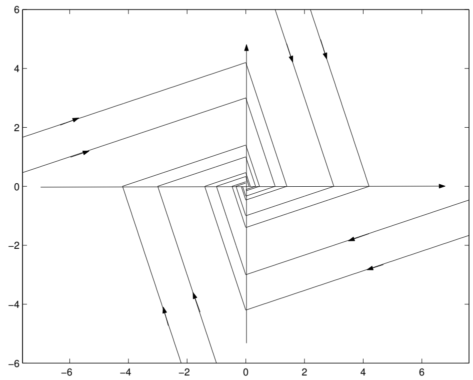

Figure 3: A K-, C- (not s-) asymptotically stable system.

3 Planar bilinear systems

In this section we consider systems of the form (5), with $x \in \mathbf{R}^2$. For certain pairs (A, B) we show how to construct state-static-memoryless feedback laws which provide at the same time, K-stabilizability, C-stabilizability and s-stabilizability. As already pointed out in the Introduction, systems of the form (5) with $x \in \mathbf{R}^2$ are considered also in [27], [15]. However, we emphasize once more that the switching rules determined in [27], [15] do not correspond, in general, to state-static-memoryless feedback laws. In addition, our result covers a different class of systems: indeed, we prescribe the form of the matrix A, but we do not need any special form for the matrix B.

Systems of the form (5) are considered also in [3], with $x \in \mathbf{R}^n$. The main contribution of [3] is a sufficient condition (actually, a generalization of [26]) for the existence of a discontinuous feedback which provides stabilization with respect to Krasowski solutions and hence, a fortiori, in the sense of Filippov solutions. However, the feedback law constructed in [3] gives rise to a closed-loop system whose Krasowski or Filippov solutions cannot be reproduced, in general, as open loop switched solutions. Moreover, the result of [3] does not apply to pairs of oscillators with opposite directions, a case covered by the results of the present paper.

In a different context, the hybrid stabilization problem for a pairs of oscillators is solved in [2] by the construction of a pair of piecewise linear feedback laws and a timed automaton.

Now we state the main result of this paper, whose proof will be exposed in the next section.

Theorem 1 *Consider system (5), and assume that*

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \tag{7}$$

where $\omega \neq 0$. Then, there exists a discontinuous feedback law taking values in $\{0, 1\}$ and providing at the same time K-stabilizability, C-stabilizability and s-stabilizability if and only if the following condition holds:

(H) there exists $x \in \mathbf{R}^2$ such that $\omega \cdot \det(Ax \vdots Bx) > 0$.

Remark 3 Theorem 1 can be actually applied under the more general assumption that A has a pair of conjugate imaginary eigenvalues $\pm i\omega$. Indeed, the form (7) can be always recovered by a preliminary change of coordinates associated to a matrix P with $\det P > 0$ (i.e., preserving the orientation of the plane), which transforms the pair of matrices A, B into the pair $P^{-1}AP$, $P^{-1}BP$. This can be done without affecting condition (H). Indeed, we point out that if $x = Py$, then

$$\omega \cdot \det(P^{-1}APy \vdots P^{-1}BPy) = \omega \cdot (\det P^{-1})(\det(APy \vdots BPy)) = \omega \cdot (\det P^{-1})(\det(Ax \vdots Bx)) .$$

Note that the phase portrait of the vector field Ax is an isochronous center: the trajectories are circumferences which are run counterclockwise if $\omega > 0$, and clockwise if $\omega < 0$.

Remark 4 We report here some comments about Theorem 1.

• In the proof of the theorem we make use of a Lyapunov function. According to the terminology of [15], it can be reviewed as a *common weak Lyapunov-like function*.

• Condition (H) covers in particular the case $\omega \cdot \det(Ax \vdots Bx) > 0$ for each $x \neq 0$. However, in this case the matrix B is Hurwitz and the solution of the problem is trivial. The most interesting case arises when the quadratic form $\det(Ax \vdots Bx)$ is indefinite. However, we may have also the case where $\omega \cdot \det(Ax \vdots Bx) \geq 0$ for each $x \neq 0$, but it does not vanish (see Example **??** in Section 5).

• The feedback law provided by the previous theorem may give rise, in some cases, to Krasowski solutions which are not Carathéodory solutions.

• By the same method it is possible to address certain cases where the vector field Ax has an unstable equilibrium point (see Section 6).

• Theorem 1 may be seen as a constructive proof of the fact that asymptotic controllability implies asymptotic stabilizability ([10, 1]) in the particular case of bilinear systems. In particular, in [1] it is proved that an asymptotically controllable system can be asymptotically stabilized with respect to Carathéodory solutions by means of a "patchy" feedback law. We remark that the feedbacks we construct, though stabilizing the system with respect to Carathéodory solutions, are not patchy.

4 Proof of Theorem 1

Proof of the necessary part.

We need the following lemma. It concerns Krasowski solutions of discontinuous differential equations defined by pairs of continuous vector fields.

Lemma 1 *Let $f_1(x), f_2(x)$ be two continuous vector fields of $\mathbf{R}^n$. Let $f(x)$ be defined in such a way that for each $x \in \mathbf{R}^n$ either $f(x) = f_1(x)$ or $f(x) = f_2(x)$. Then,*

$$\mathbf{K}f(x) = \begin{cases} \{f(x)\} & \textit{if } f \textit{ is continuous at } x \\ \mathrm{co}\{f_1(x), f_2(x)\} & \textit{otherwise.} \end{cases}$$

The proof of this lemma is easy and it is not reported. Now assume that (H) is false; in other words, assume that $\omega \cdot \det(Ax \vdots Bx)) \leq 0$ for each $x \in \mathbf{R}^2$. Let us consider the function

$$V(x_1, x_2) = \frac{x_1^2 + x_2^2}{2} \ . \tag{8}$$

It is clear that

$$\nabla V(x_1, x_2) \cdot Ax = 0 \quad \text{and} \quad V(x_1, x_2) \cdot Bx = -\frac{1}{\omega}\det(Ax \vdots Bx) \geq 0$$

for each $x \in \mathbf{R}^2$. According to Lemma 1, for any Krasowski solution of the closed loop system we therefore have

$$\nabla V(\varphi(t)) \cdot \dot{\varphi}(t) \geq 0 \tag{9}$$

for a.e. $t \geq 0$. Assume that the closed loop system is K-stabilized by the feedback law $u = k(x)$, and let $\varphi(t)$ a solution of (4) such that $\|\varphi(0)\| > 2\sqrt{\omega}$, which implies $V(\varphi(0)) > 2$. Then, for some $T > 0$ we should have $\|\varphi(T)\| < \sqrt{2\omega}$, that is $V(\varphi(T)) < 1$, which is impossible because of (9). The proof of the necessary part is complete, since C-solutions and S-solutions, when they exist, are also K-solutions.

Proof of the sufficient part.

The most intriguing aspect of this part of the proof consists in the construction of the feedback law. A crucial role is played by the identification of certain conic or semiconic regions of $\mathbf{R}^2$.

Let $v \neq 0$ be a given vector of $\mathbf{R}^2$. By a *ray* we mean the set of points of the form λv, where $\lambda \geq 0$. Given a pair of linearly independent vectors $v, w \in \mathbf{R}^2$, by a *semiconic region* (engendered by v, w) we mean the set of points of the form

$$\lambda v + \mu w \tag{10}$$

where λ, μ are arbitrary nonnegative numbers. By a *conic region* we finally means the set of points of the form (10) where λ, μ are both nonnegative or both nonpositive.

In other word, a ray can be viewed as a half straight line issuing from the origin, and a semiconic region as the convex region bounded by a pair of distinct rays. We need to distinguish between these two rays, in a way which depends on the vector field Ax. We agree to call the *right side* of a semiconic region of $\mathbf{R}^2$, the one which can be superposed to the other by a counterclockwise rotation of an angle less that π if $\omega > 0$, or by a clockwise rotation of an angle less that π if $\omega < 0$. The other ray will be called the *left side*. Note that according to these definitions, semiconic and conic regions are closed and have a nonempty interior.

We are interested in the following lemma, relating linear vector fields and semiconic regions.

Lemma 2 *Let D be a semiconic region of $\mathbf{R}^2$, and let E be a real 2×2 matrix. Assume that*

$$(Ex) \cdot (Ax) > 0 \tag{11}$$

for each $x \in D$ $(x \neq 0)$. Let R and L be respectively the right and left side of D. Then, there exists $T > 0$ such that for each $\bar{x} \in R$ $(\bar{x} \neq 0)$ the corresponding solution of the system

$$\dot{x} = Ex \tag{12}$$

reaches for the first time some point $\bar{y} \in L$ $(y \neq 0)$ exactly at the time T. A similar result holds if the quadratic form $(Ex) \cdot (Ax)$ is strictly negative on $D \setminus \{0\}$, with $\bar{x} \in L$ and $\bar{y} \in \mathbf{R}$.

Proof Let us denote by (ρ, θ) the polar coordinates of the plane and consider the equation

$$\dot{\theta} = \frac{1}{\rho^2}(Ex) \cdot \begin{pmatrix} -\omega x_2 \\ \omega x_1 \end{pmatrix} = \omega \left(E \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right) \tag{13}$$

which is independent on ρ and describes the projection of the trajectories of system (12) on $S = \{x \in \mathbf{R}^2 : |x| = 1\}$. Condition (11) guarantees that (13) has no singular points on the compact arc $D \cap S$. Let p and q be respectively the intersection points of R and L with S. Then T is defined as the time needed to reach q starting from p along the solution of (13).

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According to condition (H), the open set

$$\mathcal{O} = \{x \in \mathbf{R}^2 : \omega \cdot \det(Ax \vdots Bx) > 0\}$$

is nonempty. Of course, if $x \in \mathcal{O}$ $(x \neq 0)$ then $\lambda x \in \mathcal{O}$ for each $\lambda \neq 0$. Moreover, by continuity, if $x \in \mathcal{O}$ then $\mathcal{O}$ contains a whole neighborhood of x. Hence, there exists a semicone $C \subset \mathcal{O} \cup \{0\}$. Let R and L be respectively the right and left side of C. Note that according to our agreements, if $x \in R$ $(x \neq 0)$ the vector Ax points inward C. Moreover, we must have $Bx \neq 0$ for each $x \in \mathcal{O}$ and, hence, for each $x \in C$ $(x \neq 0)$. Let us also remark that the reciprocal orientation of Ax and Bx is the same for $x \in C$. This follows by the simple observation that $\det(Ax \vdots Bx)$ represents the third component of the wedge product $W_1(x) \wedge W_2(x)$, where $W_1(x), W_2(x)$ are the vectors of $\mathbf{R}^3$ obtained by adding to Ax, Bx a third zero component. Let us distinguish two cases.

Case 1. For each $x \in R$ $(x \neq 0)$, $(Ax) \cdot (Bx) > 0$ (Bx points inward C, as well as Ax).

Case 2. For each $x \in R$ $(x \neq 0)$, $(Ax) \cdot (Bx) < 0$ (Bx points outward C).

Note that in force of our assumptions, only these two cases are actually possible.

Continuation of the proof for Case 1.

Clearly, we can take a ray $M \subset C$ $(M \neq R, M \neq L)$ such that, denoting by $G \subset C$ the semicone delimited by R and M, we still have $(Ax) \cdot (Bx) > 0$ for any $x \in G$ $(x \neq 0)$ (see Figure 4). Let $\tilde{G} = R \cup \operatorname{Int} G$. Let us define the feedback law

$$u = k(x) = \begin{cases} 1 & \text{if } x \in \mathbf{R}^2 \setminus \tilde{G} \\ 0 & \text{if } x \in \tilde{G} \end{cases}. \tag{14}$$

The closed loop system is

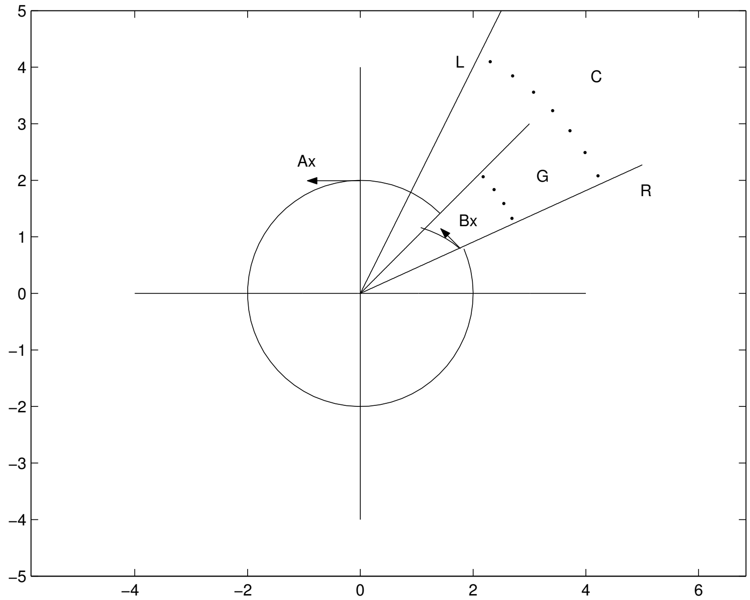

Figure 4: The semicones C and G (case $\omega > 0$)

$$\dot{x} = \begin{cases} Ax & \text{if } x \in \mathbf{R}^2 \setminus \tilde{G} \\ Bx & \text{if } x \in \tilde{G} \quad . \end{cases} \tag{15}$$

The first goal is to show that for each initial state $p_0 \in \mathbf{R}^2$ there exists at least one switched solution issuing from p_0.

If $p_0 \in \mathbf{R}^2 \setminus \tilde{G}$ ($p_0 \neq 0$), then the trajectory issuing from p_0 is an arc of circumference (centered at the origin), which intersects the ray R at some point $p_1 \neq 0$ after a finite duration T_0. Re-starting from p_1, the trajectory can be continued as an integral curve of the vector field Bx. Lemma 2 guarantees that after a finite duration T_B this trajectory intersects the ray M at some point $p_2 \neq 0$. Assuming p_2 as a new starting point, the trajectory is further continued according to the vector field Ax, and the ray R is reached again at some point $p_3 \neq 0$, after a finite duration T_A. The reasoning can be iterated. We have so constructed for each initial state $p_0 \in \mathbf{R}^2 \setminus \tilde{G}$ ($p_0 \neq 0$) a solution of the closed loop system. Such a solution can be also described in open loop terms. To this purpose, we can use a partition of the time axis determined by the points

$$0, \tau_0 = T_0, \tau_1 = T_0 + T_B, \tau_2 = T_0 + T_B + T_A, \tau_3 = T_0 + 2T_B + T_A, \ldots.$$

In order to complete our task, we have only to remark that the sequence $\{\tau_i\}$ does not have a finite limit (i.e., the sequence $\{\tau_i\}$ actually diverges and "zenoness" cannot occur). The argument is similar if $p_0 \in \tilde{G}$. Existence of switched solutions also implies the existence of Carathéodory solutions. To finish the proof for the Case 1, it remains to show that the closed loop system (15) is asymptotically stable with respect to Krasowski solutions. We use the criterion recalled in Appendix B, with the Lyapunov function (8).

First of all, we point out that the Krasowski operator associated to (15) gives rise to the differential inclusion

$$\dot{x} \in \begin{cases} \{Ax\} & \text{if } x \in \mathbf{R}^2 \setminus G \\ \{\alpha Ax + (1-\alpha)Bx : \alpha \in [0,1]\} & \text{if } x \in R \cup M \\ \{Bx\} & \text{if } x \in \operatorname{Int} G \; . \end{cases}$$

Now, for any $x \in \mathbf{R}^2$ we have

$$\nabla V(x) \cdot Ax = 0$$

while for $x \in \operatorname{Int} G$ we have

$$\nabla V(x) \cdot Bx = -\frac{1}{\omega}\det(Ax \vdots Bx) < 0 \ .$$

Hence, we easily infer that

$$\dot{\overline{V}}(x) \subseteq (-\infty, 0] \qquad \forall x \in \mathbf{R}^2 \tag{16}$$

and

$$Z = \{x : 0 \in \dot{\overline{V}}(x)\} = \mathbf{R}^2 \setminus \mathrm{Int}\, G \ .$$

From (16) we deduce that the origin is stable, and that each Krasowski solution is attracted by the maximal invariant set contained in Z. As already remarked, each trajectory issuing from a nonzero initial state $p_0 \in Z$ leaves Z in finite time. Hence, the maximal weakly invariant set contained in Z reduces to the origin. This completes the proof for the first case. By the way, we remark that in this case for each initial state there is a unique Krasowski solution, which is in fact a switched solution, as well.

Continuation of the proof for Case 2.

As in the previous case, we can take a ray $M \subset C$ ($M \neq R, M \neq L$) such that for each $x \in G$ ($x \neq 0$) (G being the semicone delimited by R and M), the inequality $Ax \cdot Bx < 0$ is still valid. Take a further ray $N \subset G$ ($N \neq R, N \neq M$); denote by K the semicone whose right and left sides are respectively given by R and N, and by H the semicone whose right and left sides are respectively given by N and M (see Figure 5).

Next we construct two sequences of points $\{p_i\}$, $\{q_i\}$ ($i \in \mathbf{Z}$). Let p_0 be the intersection point between S and R, and let $\varphi_0(t)$ be the solution of the vector field Ax issuing from p_0. According to Lemma 2, there exists a time $T_A > 0$ such that $\varphi_0(T_A)$ coincides with the intersection point between S and N, while $\varphi_0(t) \in K$ for each $t \in [0, T_A]$. Let $q_0 = \varphi_0(T_A)$ and $\Gamma_0 = \varphi_0((0, T_A])$. Then let $\psi_0(t)$ be the solution of the vector field Bx issuing from q_0. Using again Lemma 2, we find a time $T_B > 0$ such that $\psi_0(T_B)$ coincides with the intersection point between S and R, while $\psi_0(t) \in K$ for each $t \in [0, T_B]$. Let $p_1 = \psi_0(T_B)$ and $\Delta_1 = \psi_0((0, T_B])$. The procedure can be iterated for any positive integer i, by considering at each step the solution $\varphi_i(t)$ of the vector field Ax issuing from p_i, defining $q_i = \varphi_i(T_A)$ and $\Gamma_i = \varphi_i((0, T_A])$, continuing with the solution $\psi_i(t)$ of the vector field Bx issuing from q_i and defining $p_{i+1} = \psi_i(T_B)$, $\Delta_i = \psi_i((0, T_B])$. The procedure can be repeated also for any negative integer i, following the trajectories of Ax and Bx backward in time.

Now we define a sequence

$$\ldots, \tau_{-3} = -T_A - 2T_B, \tau_{-2} = -T_A - T_B, \tau_{-1} = -T_B, \tau_0 = 0, \tau_1 = T_A, \tau_2 = T_A + T_B, \tau_3 = 2T_A + T_B, \tau_4 = 2T_A + 2T_B, \ldots$$

and a curve $\sigma(t)$ in the following way:

$$\sigma(t) = \begin{cases} \ldots & \\ \psi_0(t - \tau_{-1}) & t \in [\tau_{-1}, \tau_0) \\ \varphi_0(t) & t \in [\tau_0, \tau_1) \\ \psi_1(t - \tau_1) & t \in [\tau_1, \tau_2) \\ \varphi_1(t - \tau_2) & t \in [\tau_2, \tau_3) \\ \ldots \ . & \end{cases}$$

Note that the image of $\sigma(t)$, denoted by Σ, is formed by the pairwise distinct union of the arcs Γ_i, Δ_i. We distinguish the sets

$$\Sigma_A = \cup \Gamma_i \quad \text{and} \quad \Sigma_B = \cup \Delta_i \ .$$

Claim The curve $\sigma(t)$ is unbounded for $t \to -\infty$; in addition, $\lim_{t \to +\infty} \sigma(t) = 0$.

We prove the second statement of the claim, the proof of the first one being similar. As already established in the proof of the first case, moving inside G along the solutions $\psi_i(t)$ of the vector field Bx the values of the function $V(x)$ decrease. As a consequence, the sequence of real numbers $\{|p_i|\}$ is decreasing. Assume that $\lim_{i \to +\infty} |p_i| = l > 0$. Then, the solutions $\psi_1(t), \psi_2(t), \ldots$ uniformly converges for $t \in [0, T_B]$ to some function $\tilde{\varphi}(t)$. The image of this function must be contained in the level curve $\{x : V(x) = l^2/2\}$, which is of course an arc of a circumference. On the other hand, according to well known theorems about ordinary differential equations, $\tilde{\varphi}(t)$ must be also a solution of the vector filed Bx. But this is impossible, since by virtue of (H), Bx is not tangent to a circumference.

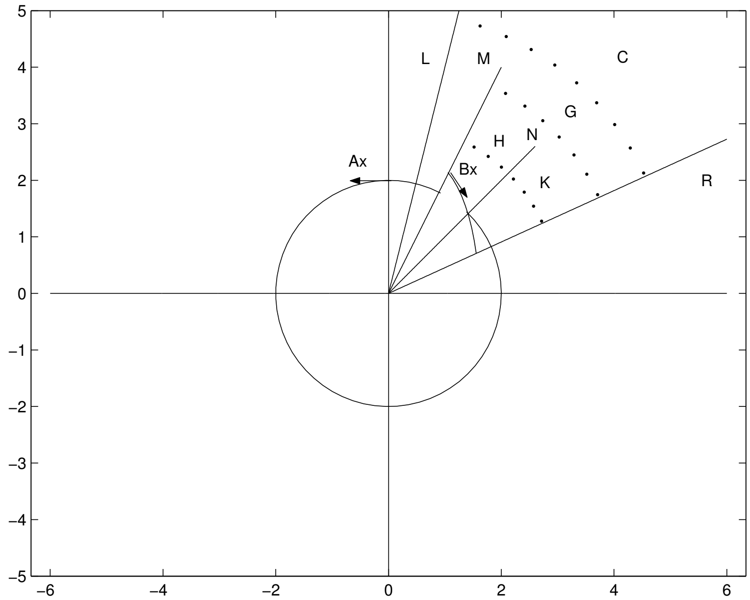

Figure 5: The semicones C, G, H and K (case $\omega > 0$)

We are now ready to define our feedback law. The set Σ divides K into two part. Let K_0 be the set of points of K bounded by Σ and the ray N. Let moreover $\Omega = \mathrm{Int}\, K_0 \cup H \cup \Sigma_B$, $\Omega^c = \mathbf{R}^2 \setminus \Omega$, and

$$u = k(x) = \begin{cases} 1 & \text{if } x \in \Omega^c \\ 0 & \text{if } x \in \Omega \end{cases} . \tag{17}$$

The corresponding closed loop system is

$$\dot{x} = \begin{cases} Ax & \text{if } x \in \Omega^c \\ Bx & \text{if } x \in \Omega \end{cases} . \tag{18}$$

As in the proof of Case 1, we first show that for each $p_0 \in \mathbf{R}^2$ there exists at least one closed loop switched solution. Assume first that $p_0 \in \Omega^c$; then the system evolves according to the vector field Ax until the set Σ_B is reached in finite time. Therefore, the evolution is continued following the curve $\sigma(t)$. Instead, if $p_0 \in \Omega$, the system evolves according to the vector field Bx, until the set Σ_A is reached in finite time. Even in this case, the evolution is continued following the curve $\sigma(t)$. It is clear that in both cases the trajectory satisfies the required conditions.

Finally, we prove that the closed loop system is asymptotically stable with respect to Krasowski solutions. Again, we make use of the Lyapunov function (8). The Krasowski operator gives rise to the differential inclusion

$$\dot{x} \in \begin{cases} \{Ax\} & \text{if } x \in \mathrm{Int}\, \Omega^c \\ \{\alpha Ax + (1-\alpha)Bx : \alpha \in [0,1]\} & \text{if } x \in \Sigma \cup M \\ \{Bx\} & \text{if } x \in \mathrm{Int}\, \Omega\ . \end{cases}$$

This time we have $Z = \overline{\Omega^c}$. To conclude the proof, we have to check that for all $l > 0$, the origin is the maximal weakly invariant set contained in $Z \cap L_l$. Let us show first that there exists no bounded Krasowski solutions contained in $\Sigma \cup M$.

Recall that the vectors Ax and Bx are not parallel for $x \in \Sigma \cup M$ ($x \neq 0$). Hence, the closed loop system has not equilibrium solutions in the sense of Krasowski. If $p \in \Sigma$, the unique Krasowski solution issuing from p coincides with $\sigma(t)$ and we already know that this solution is unbounded. If $p \in M$, then we have uniqueness of backward solutions but not of forward solutions. Indeed, there exist three Krasowski (forward in time) solutions issuing from p. The first enters immediately into $\mathrm{Int}\, \Omega$ and hence it is not contained in Z. The second enters immediately into $\mathrm{Int}\, \Omega^c$ and the third one slides along M. However, the last one represents the unique Krasowski solution issuing from p backward in time. Note that the first two forward solutions are both closed loop switched solutions as well, while the third one is neither a Carathèodory solution nor a closed loop switched solution. In any case, no complete Krasowski solution issuing from $p \in M$ is contained in $Z \cap L_l$. Finally, we remark that by continuing backward in

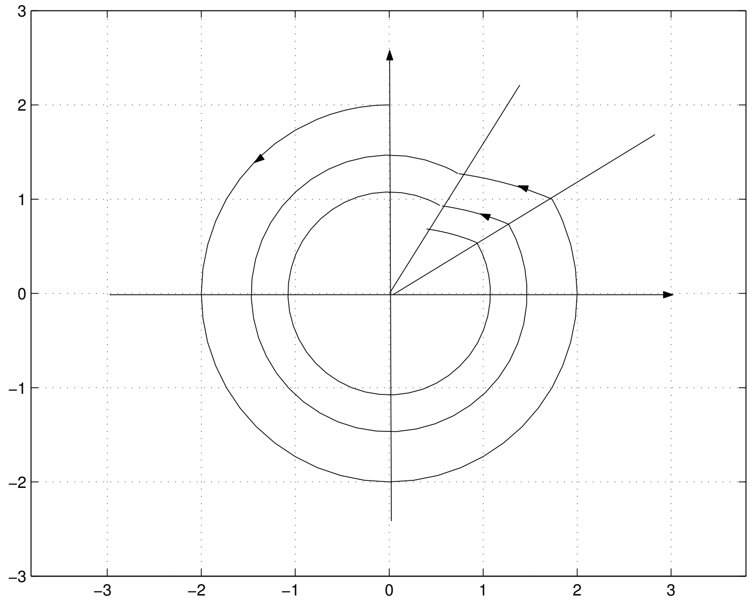

Figure 6: A trajectory of system (5) with both matrices A and B with purely imaginary eigenvalues.

time any Krasowski solution of the closed loop system issuing from a point $p \in \operatorname{Int}\Omega^c$, we hit the ray M in finite time. The proof is complete.

5 Examples

In this section we present a number of particular cases covered by Theorem 1.

We denote by λ_1^B, λ_2^B the eigenvalues of the matrix B. The trivial computation of the quantity $\det(Ax \vdots Bx)$ shows that system (5) is always K-C-s-stabilizable in the following cases:

- λ_1^B, λ_2^B complex conjugate with null real part, $B \neq \alpha A$ with $\alpha \in \mathbf{R}$;
- $\lambda_1^B, \lambda_2^B \in \mathbf{R}$, $\lambda_1^B \lambda_2^B < 0$.

Other particular situations covered by Theorem 1 are illustrated by the following examples.

Example 4 Let us consider system (5) with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & \alpha \\ -1 & 1 \end{pmatrix}.$$

If $\alpha > 0$ matrix B gives rise to an unstable focus. If $\alpha = 4$ condition (H1) is satisfied then system (5) is K-C-s-stabilizable; if $\alpha = 1$ condition (H1) is not satisfied then system (5) is not K-C-s-stabilizable.

Example 5 Let us consider system (5) with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & \alpha \\ 0 & 2 \end{pmatrix}.$$

Matrix B gives rise to an unstable node. If $\alpha = 3$ condition (H1) is satisfied then system (5) is K-C-s-stabilizable; if $\alpha = 0$ condition (H1) is not satisfied then system (5) is not K-C-s-stabilizable.

Example 6 Let us consider system (5) with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

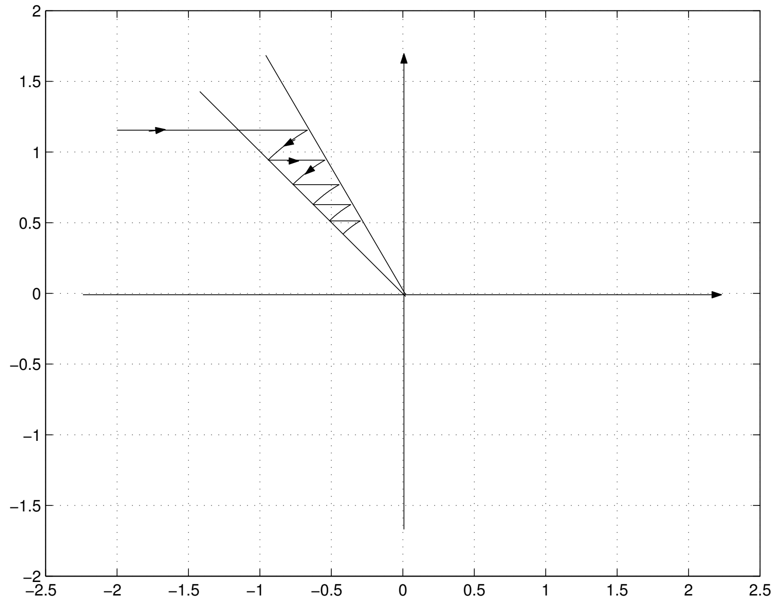

Figure 7: A trajectory of Example **??**.

Matrix B has one null eigenvalues. Condition (H1) is satisfied then system (5) is K-C-s-stabilizable. Note that in this case $\det(Ax \vdots Bx) \geq 0$ for all $x \in \mathbf{R}^2$.

6 Extensions

We now list some situations which have been already considered in the literature. We refer in particular to the articles by Xu and Antsaklis [27] and by Boscain [7]. By using a procedure analogue to the one developed in the proof of Theorem 1, it is possible to modify the control laws suggested by the authors in order to construct state-static-memoryless feedback laws.

In the following we denote by λ_1^A, λ_2^A and λ_1^B, λ_2^B the eigenvalues of the matrices A and B respectively. We remember that interesting cases are the ones in which both A and B have at least one eigenvalue with nonnegative real part.

In [27] necessary and sufficient conditions for system (5) to be stable under appropriate switching are found in the following cases:

- $\lambda_1^k, \lambda_2^k \in \mathbf{R}$, $\lambda_1^k \lambda_2^k < 0$, $k = A, B$;
- $\lambda_1^k, \lambda_2^k \in \mathbf{R}$, $\lambda_1^k \lambda_2^k > 0$, $k = A, B$;
- λ_1^k, λ_2^k complex conjugates, $k = A, B$.

In these cases the same necessary and sufficient conditions hold for system (5) being stabilizable.

In [7] systems of the form

$$\dot{x} = u\tilde{A} + (1-u)\tilde{B} \tag{19}$$

are considered in connection with the problem of stabilizability under arbitrary switching (see the Introduction). Admissible feedback laws are measurable functions $u : [0, \infty) \to [0, 1]$. In the paper necessary and sufficient conditions in order to have stability under arbitrary switching are found. Of course a necessary condition for the system being asymptotically stable under arbitrary switching is that both matrices $\tilde{A}$ and $\tilde{B}$ have strictly negative eigenvalues. The necessary and sufficient conditions for stability under arbitrary switching of (**??**) are expressed in terms of the eigenvalues of the matrices $\tilde{A}$ and $\tilde{B}$. Let us denote by $\tilde{\varphi}_{x,u}(t)$ the solution of (**??**) corresponding to the control law

$u(t)$ and the initial condition x. Assume that the eigenvalues of $\tilde{A}$ are complex conjugates with negative real parts and the eigenvalues of $\tilde{B}$ are real and negative. It is proved in [7] that a set of conditions doesn't hold if and only if there exists $x \in \mathbf{R}^2\backslash\{0\}$ and $u(t)$ such that $\tilde{\varphi}_{x,u}(t) \to \infty$ as $t \to \infty$. Actually Boscain's conditions do not hold if and only if for any x there exists $u(t)$ such that $\tilde{\varphi}_{x,u}(t) \to \infty$ as $t \to \infty$ and moreover $u(t)$ can be chosen such that $u(t) \in \{0,1\}$ and $u(t)$ has discontinuities only on a couple of lines through the origin.
Let us now consider our original system (5) with λ_1^A, λ_2^A complex conjugates with positive real parts and $\lambda_1^B, \lambda_2^B \in \mathbf{R}$, $\lambda_1^B\lambda_2^B > 0$ and system (**??**) with $\tilde{A} = -A$ and $\tilde{B} = -B$. Note that the eigenvalues of $\tilde{A}$ and $\tilde{B}$ are the opposites of the eigenvalues of A and B and that for any fixed $u(t)$ the trajectories of the two systems are the same except for their orientation. Denote by $\varphi_{x,u}(t)$ the solution of (5) corresponding to the initial condition x and the control law $u(t)$. We have that Boscain's conditions do not hold if and only if for all x there exists $u(t)$ such that $\tilde{\varphi}_{x,u}(t) \to \infty$ as $t \to \infty$, i.e. if and only if for all x there exists $u(t)$ such that $\varphi_{x,u}(t) \to 0$ as $t \to \infty$.

Finally in case

- λ_1^A, λ_2^A complex conjugates and $\lambda_1^B, \lambda_2^B \in \mathbf{R}$ with $\lambda_1^B\lambda_2^B > 0$,

if Boscain's conditions do not hold we get a time dependent switching law which can be modified in a K-C-s-stabilizing feedback by using a procedure analogue to the one used in the proof of Theorem 1.

Our Theorem 1 gives necessary and sufficient conditions for K-C-s stabilizability in the cases

- λ_1^A, λ_2^A complex conjugates with null real part and any λ_1^B, λ_2^B.

Cases excluded by the previous enumeration are

- $\lambda_1^A, \lambda_2^A \in \mathbf{R}$, $\lambda_1^A\lambda_2^A < 0$ and $\lambda_1^B, \lambda_2^B \in \mathbf{R}$, $\lambda_1^B\lambda_2^B > 0$;
- $\lambda_1^A, \lambda_2^A \in \mathbf{R}$, $\lambda_1^A\lambda_2^A < 0$ and λ_1^B, λ_2^B complex conjugates;

and all cases in which either A or B have one or two null eigenvalues. These cases can be treated in an analogous way. In particular, in the case $\lambda_1^A, \lambda_2^A \in \mathbf{R}$, $\lambda_1^A\lambda_2^A < 0$ and λ_1^B, λ_2^B complex conjugates, K-C-s stabililization can be always achieved by means of a state-static-memoryless feedback, in an analogous way to the one shown in the following example.

Example 7 Let us consider system (5) with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

We have that $\lambda_1^A = 1, \lambda_2^A = -1$, i.e. matrix A gives rise to a saddle point and $\lambda_1^B = 1+i, \lambda_2^B = 1-i$. i.e. matrix B gives rise to an unstable focus. Let $\mathcal{O} = \{x \in \mathbf{R}^2 : -\det(Ax \vdots Bx) > 0\} = \{(x_1,x_2) \in \mathbf{R}^2 : \frac{x_2}{x_1} < 1-\sqrt{2}, \frac{x_2}{x_1} > 1+\sqrt{2}\}$. Let us consider the three rays $m : x_2 = 5x_1$, $r : x_2 = 4x_1$ and $n : x_2 = 3x_1$, $x_1 > 0$. Let $H \subset \mathcal{O}$ the semiconic region delimited by the rays m and r and $K \subset \mathcal{O}$ the semiconic region delimited by the rays r and n. Note that $Ax|_r \cdot Bx|_r < 0$ and $Ax|_n \cdot Bx|_n < 0$. By using a technique completely analogous to the one used in the proof of Theorem 1, Case 2, we can construct a curve $\Sigma \subset K$ by patching together pieces of trajectories of the systems associated to the matrices A and B respectively, and define the set Ω as the union of the cone H, the interior of the set of points of K bounded by the curve Σ and the ray r and the arcs of Σ which are trajectories associated to the matrix A. Finally the feedback law

$$u = k(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c \end{cases} \tag{20}$$

is a K-C-s-stabilizing feedback.

7 Conclusions

As already remarked in the Introduction, the stabilizability problem for switched systems in the plane has been studied by several authors. In all papers that deal with this problem it is evident the central role played by "conic switching laws". We have not given a precise definition of "conic switching law" here: very roughly speaking they

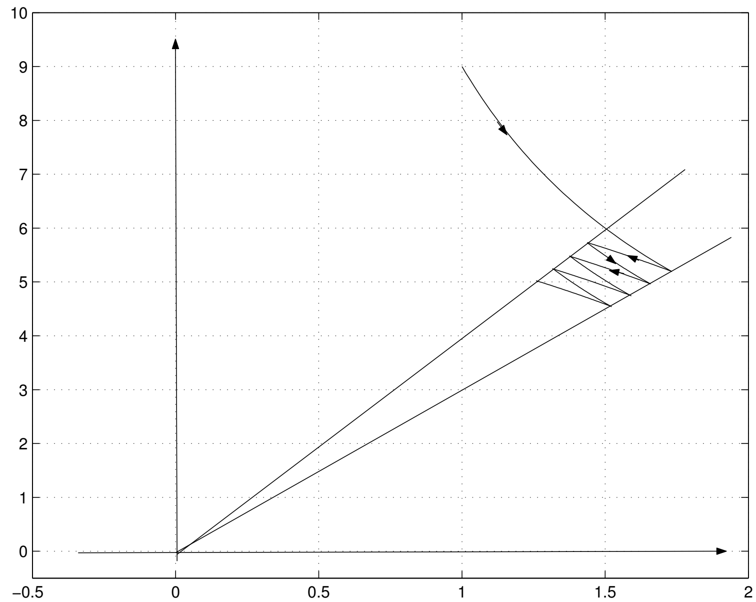

Figure 8: A trajectory of Example **??**.

can be seen as hybrid control laws whose discrete state space is the set $\{0, 1\}$ and whose discrete dynamics is determined by a couple of lines through the origin (see [27]). Such "conic switching laws" are not always expressed as state-static-memoryless feedbacks. In this paper we have considered the particular case of system (5) with the matrix A with imaginary eigenvalues and we have showed how a "conic switching law" can modified into a state-static-memoryless feedback which preserves the switching nature of the system. Moreover we have proved rigorously that these feedbacks actually stabilize. In order to construct state-static-memoryless feedback laws, a procedure analogue to the one developed in the proof of Theorem 1 may be applied to different situations already considered in the literature.

8 Appendix A

In this appendix we recall some facts about autonomous systems of ordinary differential equations

$$\dot{x} = f(x) \ , \quad x \in \mathbf{R}^n \tag{21}$$

where $f(x)$ is in general Lebesgue measurable and locally bounded. Let I denote some interval of real numbers. A *Carathéodory solution* of (19) is a curve $\varphi(t) : I \to \mathbf{R}^n$ such that $\varphi(t)$ is absolutely continuous, and $\dot{\varphi}(t) = f(\varphi(t))$ for a.e. $t \in I$. A *Krasowski solution* of (19) is a curve $\varphi(t) : I \to \mathbf{R}^n$ such that $\varphi(t)$ is absolutely continuous, and

$$\dot{\varphi}(t) \in (F_K f)(\varphi(t))$$

for a.e. $t \in I$, where the "operator" F_K is defined by $(F_K f)(x) = \bigcap_{\delta>0} \overline{\text{co}}\,\{f(\mathcal{B}(x, \delta))\}$.

Under our assumption that $f(x)$ is Lebesgue measurable and locally bounded, the operators F_K associates to f a set valued map which is upper semicontinuous, compact, convex valued. In particular, for each initial state $\bar{x}$ there exists at least one Krasowski solution of (19). Existence of Carathéodory solutions in general requires more specific and restrictive assumptions ([8], [22], [1]).

The most popular notion of generalized solution for differential equations with a discontinuous righthand side is due to Filippov ([13]). Krasowski solutions are more general, in the sense that every Filippov solution is a Krasowski one, but the contrary is not true. Carathéodory solutions, when they exist, are Krasowski solutions, as well. Instead, Carathéodory solutions and Filippov solutions are independent notions.

9 Appendix B

The following invariance principle is proven in [5]. It applies to differential inclusions with an upper semicontinuous, convex, compact valued righthand side of the form

$$\dot{x} \in F(x) \tag{22}$$

and hence, it can be used to derive stability results for Filippov or Krasowski solutions of discontinuous differential equations, provided that a locally Lipschitz continuous, positive definite Lyapunov function $V(x) : \mathbf{R}^n \to \mathbf{R}$ is known. In fact, in this paper we do not need such a generality, since smooth Lyapunov functions suffice. Hence, we can limit ourselves to state the result in a simplified form. Given $V(x) \in C^1$, let

$$\dot{\overline{V}}(x) = \{a \in \mathbf{R} : \exists v \in F(x) \text{ such that } \nabla V(x) \cdot v = a\}.$$

Theorem 2 *Let $F(x)$ be upper semicontinuous, with convex, compact values, and let $V : \mathbf{R}^n \to \mathbf{R}$ be a positive definite function of class C^1. Let us assume that there exists $l > 0$ be such that the set $L_l = \{x \in \mathbf{R}^n : V(x) \le l\}$ is connected and compact. Let $\bar{x} \in L_l$, and let $\varphi(t)$ be any solution of (20) issuing from $\bar{x}$. Suppose that $\dot{\overline{V}}(x) \subseteq (-\infty, 0]$ for each $x \in \mathbf{R}^n$. Let*

$$Z = \{x \in \mathbf{R}^n : 0 \in \dot{\overline{V}}(x)\}$$

and let M be the largest weakly invariant subset of $Z \cap L_l$. Then $\mathrm{dist}(\varphi(t), M) \to 0$ *as $t \to +\infty$.*

Note that under the assumptions of Theorem 2 the set Z is closed.

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