NONLINEAR STABILIZATION AND OPTIMAL REGULATION: THE SMOOTH CASE

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Abstract

In this paper we investigate the continuous feedback stabilization of nonlinear affine systems, on the basis of an associate optimal regulation problem. We point out several analogies between the present setting and the classical linear theory.

1 Introduction

The relationships between the quadratic regulator problem on the infinite horizon and the stabilization problem, are well understood in the case of linear systems. Analogous results for nonlinear systems have been pointed out occasionally and in a partial way in the literature, but a systematic study seems to have never been carried out. This paper is an attempt in this direction.

In Section 2 we recall some well known definitions and introduce the notation. In Section 3 we focus on the linear case. We briefly recall some typical achievements. This provides an useful model and a logical track for the following nonlinear developments. The proof of the results reviewed in this section are available on many textbooks and will not be reported here.

In Section 4 we consider nonlinear affine systems. By means of suitable reinterpretations and extentions of certain well known results ([3], [7], [10], [14]) and the development of a unified treatment, we show that the linear scheme can be generalized in a satisfactory way. The proofs of the main results are given in Section 5. We note that completing the framework also requires the achievement of some original results and lemmas.

Section 6 contains a new result about radial unboundedness of the value function for the optimal regulation problem: it turns out to be useful in order to read the main theorem as a stabilization result. In Section 7 we report in extended and more precise form a result due to W. Kang ([9]). It shows that under appropriate assumptions, the existence of a smooth stabilizer implies the existence of a stabilizer in damping form (damping control is defined in Sect. 4).

In this paper we will be still confined in a "smooth" setting. This means that we limit ourselves to stabilizing feedback laws which are at least continuous, and to an optimization problem whose value function is at least C^1. A very exciting question is to what extent these "smoothness" assumptions can be relaxed. Partial results in this direction can be reviewed as further contributions to discontinuous Jurdjevic-Quinn theory (see [2]) and will be the subject of a forthcoming paper.

2 Preliminary and notation

For reader's convenience, we recall some basic definitions. In what follows $|v|$ and $||A||$ denote respectively the (euclidean) norm of a vector v and of a matrix A. The transposition of a matrix is denoted by $A^{\mathbf{t}}$. Let

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n \tag{1}$$

be a given system of ordinary differential equations, with f continuous and $f(0) = 0$. The origin is said to be *globally asymptotically stable* if the following two properties hold:

(Stability) $\forall \varepsilon\ \exists \delta$ such that for each solution $\varphi(t)$ of (1),

$$|\varphi(0)| < \delta \Longrightarrow |\varphi(t)| < \varepsilon \quad \forall t \geq 0$$

(Global attraction) for each solution of (1) one has

$$\lim_{t \to +\infty} \varphi(t) = 0 \ .$$

Note that if f is only continuous, uniqueness of solutions is not guaranteed.

A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$. Moreover, V is said to be radially unbounded if $\lim_{|x| \to +\infty} V(x) = +\infty$.

It is well known that (1) is globally asymptotically stable at the origin if and only if there exists a *smooth global strict Liapunov function*, that is a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- $V(x)$ is positive definite and radially unbounded
- $\nabla V(x) f(x)$ is negative definite (i.e., $-\nabla V(x) f(x)$ is positive definite)

(in fact, such a function V can be taken of class C^∞). If in addition we require that $-\nabla V(x) f(x)$ is radially unbounded, then we shall say that $V(x)$ is a *smooth global strong Liapunov function*. As remarked in [12] p. 440, the existence of a strict Liapunov function is actually equivalent to the existence of a (possibly modified) strong Liapunov function.

A function $\alpha : [0, +\infty) \to \mathbb{R}$ is said to be of class $\mathcal{K}_0$ if it is continuous, strictly increasing and such that $\alpha(0) = 0$. If $\alpha \in \mathcal{K}_0$ and in addition $\lim_{r\to+\infty} \alpha(r) = +\infty$, then we say that $\alpha \in \mathcal{K}_0^\infty$.

It is clear that a continuous function V is both positive definite and radially unbounded if and only if there exists a map $\alpha \in \mathcal{K}_0^\infty$ such that $\alpha(|x|) \le V(x)$ for each $x \in \mathbb{R}^n$.

Now, let us consider a time-invariant input system

$$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is at least continuous. Throughout this paper, an *admissible input* is a piecewise continuous and right continuous function $u(t) : [0, +\infty) \to \mathbb{R}^m$. For each admissible input $u(t)$ and each initial state x_0, any corresponding (Carathéodory) solution of (2) will be denoted by $x = \varphi(t; x_0, u(\cdot))$.

A continuous function $k : \mathbb{R}^m \to \mathbb{R}^m$ is called a *feedback*. Replacing u by a feedback $k(x)$ in (2), gives rise to the so-called closed-loop system

$$\dot{x} = f(x, k(x)) \ . \tag{3}$$

The symbol $\varphi_{k(\cdot)}(t; x_0)$ denotes any solution of (3) corresponding to the initial state x_0.

3 Linear systems: summary of results

Consider a linear, time-invariant input system of the form

$$\dot{x} = Ax + Bu \tag{4}$$

where, as before, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. A and B are matrices of appropriate dimensions.

Definition 1 *We say that (4) is* stabilizable *if there exists a continuous feedback $u = k(x)$ such that the origin is a globally asymptotically stable equilibrium position for the closed-loop system. In this case, we also say that $u = k(x)$ is a* stabilizing feedback *or a* stabilizer *for (4).*

There are many interesting characterizations of stabilizable linear systems. In particular, we recall the following one, due to M.J.L. Hautus ([6]).

Theorem 1 *The following properties are equivalent.*

(1.i) System (4) is stabilizable.

(1.ii) rank $(A - \lambda I, B) = n$ *for each $\lambda \in \mathbb{C}$ with $Re\,\lambda \ge 0$.*

(1.iii) There exists a matrix F such that all the eigenvalues of the matrix $A+BF$ lie on the open left half complex plane.

Property (1.iii) actually states that (4) is stabilizable by means of the linear feedback $u = Fx$. Thus, we arrive in particular at the following nice conclusion: if (4) is stabilizable by means of a continuous feedback, then it is also stabilizable by means of a linear feedback.

Now, let us associate to (4) the cost functional

$$\int_0^{+\infty} (|\varphi(t; x_0, u(\cdot))|^2 + |u(t)|^2)\, dt \ . \tag{5}$$

Intuitively, in order to minimize (5), one needs to keep as near to zero as possible both the input $u(t)$ and the solution $\varphi(t; x_0, u(\cdot))$. The following theorem is essentially due to R. Kalman; we refer to [4] for a nice exposition.

Theorem 2 *The following statements are equivalent.*

(2.i) There exists a linear stabilizer $u = Fx$ for (4).

(2.ii) There exists a symmetric, positive definite matrix P which solves the matrix equation

$$PA + A^{\mathbf{t}}P - PBB^{\mathbf{t}}P = -I \tag{6}$$

where I is the identity matrix of $\mathbb{R}^n$.

(2.iii) For each x_0, there exists an admissible input $u^*_{x_0}(t) : [0, +\infty) \to \mathbb{R}^m$ which minimizes (5).

The conclusions of this theorem can be made more precise.

Remark 1 A. If (6) can be solved in the class of the symmetric, positive definite matrices, then in this class the solution is unique.

B. If (6) has a (unique) symmetric, positive definite solution P, then $u = -B^{\mathbf{t}}Px$ is a stabilizing feedback. More precisely, (4) can be stabilized by setting

$$u = -\alpha B^{\mathbf{t}}Px \tag{7}$$

for every $\alpha \geq \frac{1}{2}$. By combining Theorems 1 and 2, we see therefore that if (4) is stabilizable by a continuous feedback, then it is also stabilizable by a feedback of the form (7), for a suitable choice of P and a sufficiently large α.

C. If (6) has a (unique) symmetric, positive definite solution P, then the minimizer $u^*_{x_0}(t)$ of (5) can be expressed as

$$u^*_{x_0}(t) = -B^{\mathbf{t}}P\varphi(t)$$

where $\varphi(t)$ is the unique solution of the Cauchy problem

$$\begin{cases} \dot{x} = Ax - BB^{\mathbf{t}} Px \\ x(0) = x_0 \ . \end{cases}$$

This means that the solution of the optimization problem (4), (5) can be put in feedback form. Note that $\varphi(t)$ coincides with the (optimal) solution $\varphi(t; x_0, u^*_{x_0}(\cdot))$ of (4) corresponding to the initial state x_0 and the optimal control $u^*_{x_0}(t)$.
D. Given x_0, we associate to (5) the value function

$$V(x_0) = \inf_{u(\cdot)} \left(\int_0^{+\infty} (|\varphi(t; x_0, u(\cdot))|^2 + |u(t)|^2) \, dt \right) \tag{8}$$

where the "inf" is taken over all the admissible inputs. If (6) has a (unique) symmetric, positive definite solution P, then $V(x_0)$ is a quadratic form. More precisely, we have $V(x_0) = x_0^{\mathbf{t}} P x_0$.

∎

4 Affine systems: the main theorem

Definition 1 applies also to nonlinear systems, of course. In particular, we are interested in affine, time-invariant systems of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{9}$$

where $x \in \mathbb{R}^n$, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$. The vector fields $f, g_1, \ldots, g_m$ are required to be of class C^1, so that uniqueness of solutions is guaranteed for any admissible input. However, it should be clear that uniqueness of solutions could be lost if we close the loop by means of a (merely) continuous feedback $u = k(x)$. We denote by $G(x)$ the matrix whose columns are $g_1(x), \ldots, g_m(x)$. We also assume that $f(0) = 0$.

We are especially interested in the extention of the properties listed in Theorem 2. Let a continuous, positive definite function $h(x)$ and a number $\gamma > 0$ be given. We associate to (9) the following cost functional

$$J(x_0, u(\cdot)) = \frac{1}{2} \int_0^{+\infty} \left(h(\varphi(t)) + \frac{|u(t)|^2}{\gamma} \right) \, dt \tag{10}$$

where we set for simplicity $\varphi(t) = \varphi(t; x_0, u(\cdot))$. For a given initial state x_0, we say that the minimization problem defined by (10) is *solvable* if there exists an admissible input, denoted by $u^*_{x_0}(t)$, such that $J(x_0, u^*_{x_0}(\cdot)) < \infty$ and

$$J(x_0, u^*_{x_0}(\cdot)) \leq J(x_0, u(\cdot))$$

for any other admissible input $u(t)$. The *value function* is defined, as usual, by

$$V(x_0) = \inf_u J(x_0, u(\cdot)) \ . \tag{11}$$

$V(x_0)$ is actually a minimum if and only if the minimization problem is solvable for x_0. We are ready to state the main result.

Theorem 3 *The following properties are equivalent.*

(3.i) There exist a continuous, positive definite, radially unbounded function $h(x)$ and a positive number γ such that the minimization problem (10) is solvable for each x_0. Moreover, $V(x_0)$ is radially unbounded and of class C^1.

(3.ii) There exist a radially unbounded, positive definite, C^1 function $W(x)$ and a positive number γ such that (9) is stabilizable by means of the continuous feedback

$$u = k(x) = -\frac{\gamma}{2}(\nabla W(x)G(x))^{\mathbf{t}} \ . \tag{12}$$

Moreover, the closed-loop system admits $W(x)$ as a strict Liapunov function.

(3.iii) There exist a continuous, positive definite, radially unbounded function $h(x)$ and a positive number γ such that the following first order partial differential equation (of the Hamilton-Jacobi type)

$$\nabla U(x)f(x) - \frac{\gamma}{2}\left|\nabla U(x)G(x)\right|^2 = -\frac{h(x)}{2} \tag{13}$$

has a solution $U(x)$ which is radially unbounded, positive definite and of class C^1.

A feedback law of the form (12) is called a *damping control* (or even a feedback of the Jurdjevic-Quinn type). Note that it is the natural generalization of (7) in a nonlinear setting.

Remark 2 A. As a by-product of the proof of Theorem 3, we shall see that if the property (3.i) holds, then for each x_0 and each $t \geq 0$

$$u^*_{x_0}(t) = -\gamma(\nabla V(\varphi(t, x_0, u^*_{x_0}(\cdot)))G(\varphi(t, x_0, u^*_{x_0}(\cdot))))^{\mathbf{t}} \ .$$

In other words, $u^*_{x_0}(t)$ is the composition of the continuous function

$$-\gamma(\nabla V(x)G(x))^{\mathbf{t}}$$

and the solution $\varphi(t, x_0, u^*_{x_0}(\cdot))$. This means that the minimization problem has a solution in feedback form.

B. If the property (3.ii) holds, then every feedback

$$u = -\alpha(\nabla W(x)G(x))^{\mathbf{t}}$$

with $\alpha \geq \gamma/2$ stabilizes system (9).

∎

5 Proof of the main Theorem

In this section we give a proof of Theorem 3.

5.1 Some useful lemmas

The following lemma will be repeatedly used in the sequel.

Lemma 1 *Let x_0 be fixed. Assume that there exists a function $a \in \mathcal{K}_0$ such that $h(x) \geq a(|x|)$ for each $x \in \mathbb{R}^n$. Assume also that $J(x_0, u(\cdot)) < \infty$ for some admissible input $u(t)$. Then,*

$$\lim_{t \to +\infty} \varphi(t; x_0, u(\cdot)) = 0 \ .$$

Proof. Since $u(t)$ and x_0 are fixed, we shall write simply $\varphi(t)$ instead of $\varphi(t; x_0, u(\cdot))$. From the assumption, it follows that both the integrals

$$\int_0^{+\infty} h(\varphi(t))\, dt \quad \text{and} \quad \int_0^{+\infty} |u(t))|^2\, dt \tag{14}$$

converge. It follows in particular that $u(t)$ is square integrable on $[0, +\infty)$ and on every subinterval of $[0, +\infty)$. It also follows that $\liminf_{t \to +\infty} h(\varphi(t)) = 0$, and since $h(x)$ is continuous, positive definite and bounded from below by class $\mathcal{K}_0$ function, this in turn implies

$$\liminf_{t \to +\infty} |\varphi(t)| = 0 \ .$$

Assume, by contradiction, that $\limsup_{t \to +\infty} |\varphi(t)| > 0$, and let

$$l = \min\{1, \limsup_{t \to +\infty} |\varphi(t)|\} \ .$$

Let L be a Lipschitz constant for $f(x)$, valid on the sphere $|x| \leq l$. Moreover, let $b > 0$ be a bound for the norm of the matrix $G(x)$ for $|x| \leq l$. By the definition of l, there exists a strictly increasing, divergent sequence $\{t_j\}$ such that for each j,

$$|\varphi(t_j)| > \frac{3l}{4} \ .$$

Without loss of generality, we can assume that $t_{j+1} - t_j > 1/(4L)$ for each $j \in \mathbb{N}$. The existence of some $\tau > 0$ and $k \in \mathbb{N}$ such that for each $j > k$ and each $t \in [t_j - \tau, t_j]$

$$|\varphi(t)| \geq \frac{l}{4}$$

is excluded, since in this case the first integral in (14) would be divergent (here again, we use the facts that h is bounded from below by $a(\cdot)$). Hence, for $\tau = 1/(4L)$ and each $k \in \mathbb{N}$, we can find an index $j_k > k$ and an instant s_k such that

$$t_{j_k} - \tau \le s_k \le t_{j_k} \quad \text{and} \quad |\varphi(s_k)| < \frac{l}{4} \ .$$

Since the solution $\varphi(t)$ is continuous, for each k there exist two instants σ_k, θ_k such that

$$s_k < \sigma_k < \theta_k < t_{j_k} \ , \qquad |\varphi(\sigma_k)| = \frac{l}{4} \ , \qquad |\varphi(\theta_k)| = \frac{3l}{4} \ ,$$

and

$$\frac{l}{4} < |\varphi(t)| < \frac{3l}{4} \ , \qquad \forall t \in (\sigma_k, \theta_k) \ .$$

We have, for each k,

$$|\varphi(\theta_k) - \varphi(\sigma_k)| \ge |\varphi(\theta_k)| - |\varphi(\sigma_k)| = \frac{l}{2} \ . \tag{15}$$

On the other hand,

$$|\varphi(\theta_k) - \varphi(\sigma_k)| \le \int_{\sigma_k}^{\theta_k} |f(\varphi(t))| \, dt + \int_{\sigma_k}^{\theta_k} |G(\varphi(t))| \cdot |u(t)| \, dt \ .$$

By construction, for $t \in [\sigma_k, \theta_k]$ we have $|\varphi(t)| \le l$. Hence, on the interval $[\sigma_k, \theta_k]$ the following inequalities hold:

$$|f(\varphi(t))| \le L|\varphi(t)| \quad \text{and} \quad |G(x)| \le b \ .$$

This yields

$$|\varphi(\theta_k) - \varphi(\sigma_k)| \le lL\tau + b \int_{t_{j_k} - \tau}^{t_{j_k}} |u(t)| \, dt \ . \tag{16}$$

Now, taking into account (15), (16) and recalling that $\tau = 1/(4L)$, we infer

$$\frac{l}{2} \le \frac{l}{4} + b \int_{t_{j_k} - \tau}^{t_{j_k}} |u(t)| \, dt$$

that is,

$$\int_{t_{j_k} - \tau}^{t_{j_k}} |u(t)| \, dt \ge \frac{l}{4b} \ .$$

Using Hölder inequality and the fact that $u(t)$ is square integrable on $[t_{j_k} - \tau, t_{j_k}]$, we also have

$$C\left(\int_{t_{j_k}-\tau}^{t_{j_k}} |u(t)|^2\, dt\right)^{\frac{1}{2}} \geq \int_{t_{j_k}-\tau}^{t_{j_k}} |u(t)|\, dt$$

where C is a positive constant independent of $u(\cdot)$. This yields

$$\int_{t_{j_k}-\tau}^{t_{j_k}} |u(t)|^2\, dt \geq \frac{l^2}{16b^2C^2} > 0\ .$$

But this is impossible, since $u(t)$ is square integrable on $[0,+\infty)$ by virtue of (14). Thus, we conclude that

$$\liminf_{t\to+\infty} |\varphi(t)| = \limsup_{t\to+\infty} |\varphi(t)| = 0$$

which implies

$$\lim_{t\to+\infty} |\varphi(t)| = 0$$

as required.

∎

Next lemma is actually a remark about Jurdjevic-Quinn method of stabilization.

Lemma 2 *Assume that there exists a positive definite, radially unbounded* C^1 *function* $W(x)$ *such that the affine system (9) can be globally stabilized by the feedback law*

$$u = -\omega(\nabla W(x)G(x))^{\mathbf{t}} \tag{17}$$

where ω *is a positive constant, and assume also that the closed loop system admits* $W(x)$ *as a strict global Liapunov function. Then there exists a function* $\tilde{W}(x)$ *such that (9) can be globally stabilized by the feedback law*

$$u = -\omega(\nabla \tilde{W}(x)G(x))^{\mathbf{t}} \tag{18}$$

and the closed loop system admits $\tilde{W}(x)$ *as a strong global Liapunov function.*

Proof. Let $\alpha(r)$ be any function of class $\mathcal{K}_0^\infty$ such that

$$\alpha(|x|) \leq W(x)\ .$$

Moreover, let $\rho \in \mathcal{K}_0^\infty$ be such that

$$\lim_{|x|\to\infty} \dot{W}(x)\rho(|x|) = -\infty \tag{19}$$

where $\dot{W}(x)$ denotes the derivative of W with respect to the closed loop system (with u given by (17)) i.e.,

$$\dot{W}(x) = \nabla W(x) f(x) - \omega |\nabla W(x) G(x)|^2 \ .$$

The argument which proves the existence of such ρ is easy and can be found in [12], p. 440.

Since $\rho, \alpha^{-1} \in \mathcal{K}_0^\infty$, there exists $R > 0$ such that $\rho(\alpha^{-1}(R)) = 1$. Let

$$\beta(r) = \begin{cases} 1 & \text{for } r \in [0, R] \\ \rho(\alpha^{-1}(r)) & \text{for } r > R \end{cases}$$

We are now ready to introduce $\tilde{W}(x)$, by a suitable modification of an idea borrowed from [12]. Let

$$\tilde{W}(x) = \int_0^{W(x)} \beta(r)\, dr \ .$$

We see immediatly that $\tilde{W}(x)$ is positive definite, radially unbounded and satisfies $\tilde{W}(0) = 0$. Let C be the compact set $\{x : \ W(x) \leq R\}$. We have

$$\nabla \tilde{W}(x) = \beta(W(x)) \nabla W(x) = \begin{cases} \nabla W(x) & \text{if } x \in C \\ \rho(\alpha^{-1}(W(x))) \nabla W(x) & \text{if } x \notin C \end{cases} \ .$$

In fact, we have $\tilde{W}(x) = W(x)$ if $x \in C$. Now consider the derivative of $\tilde{W}$ with respect to the closed loop system, with u given by (18), denoted by $\dot{\tilde{W}}(x)$. We have

$$\begin{aligned} \dot{\tilde{W}}(x) &= \nabla \tilde{W}(x)[f(x) - \omega (\nabla \tilde{W}(x) G(x))^{\mathbf{t}} G(x)] \\ &= \begin{cases} \dot{W}(x) & \text{if } x \in C \\ \rho(\alpha^{-1}(W(x)))\big[\nabla W(x) f(x) & \\ \qquad - \omega \rho(\alpha^{-1}(W(x))) |\nabla W(x) G(x)|^2\big] & \text{if } x \notin C \end{cases} \ . \end{aligned}$$

In order to achieve the proof, we need to prove that $\dot{\tilde{W}}(x)$ is negative definite and radially unbounded. Since $\dot{\tilde{W}} = \dot{W}$ for $x \in C$, from now on we can limit ourselves to the case $x \notin C$. This implies in particular that $\rho(\alpha^{-1}(W(x))) > 1$. Hence,

$$-\omega \rho(\alpha^{-1}(W(x))) |\nabla W(x) G(x)|^2 \leq -\omega |\nabla W(x) G(x)|^2 \ .$$

It follows

$$
\begin{aligned}
& \nabla W(x) f(x) - \omega\rho(\alpha^{-1}(W(x)))|\nabla W(x) G(x)|^2 \\
\leq\ & \nabla W(x) f(x) - \omega|\nabla W(x) G(x)|^2 \\
=\ & \dot{W}(x) \ .
\end{aligned}
$$

From $\alpha(|x|) \leq W(x)$ we infer $\rho(|x|) \leq \rho(\alpha^{-1}(W(x)))$. Finally,

$$
\begin{aligned}
\dot{\tilde{W}}(x) &= \rho(\alpha^{-1}(W(x)))\left[\nabla W(x) f(x) - \omega\rho(\alpha^{-1}(W(x)))|\nabla W(x) G(x)|^2\right] \\
&\leq \rho(\alpha^{-1}(W(x)))\left[\nabla W(x) f(x) - \omega|\nabla W(x) G(x)|^2\right] \\
&\leq \rho(|x|)\left[\nabla W(x) f(x) - \omega|\nabla W(x) G(x)|^2\right] \\
&= \rho(|x|)\dot{W}(x) \ .
\end{aligned}
$$

The proof is achieved by virtue of (19).

∎

5.2 (3.iii) $\Longrightarrow$ (3.ii)

This is the easiest part of the theorem. Indeed, we can identify $W(x)$ with the solution $U(x)$ of the equation (13). Using the feedback law (12) in (9), the derivative of U with respect to the closed-loop systems is easily computed:

$$
\dot{U}(x) = \nabla U(x) \cdot [f(x) - \frac{\gamma}{2} G(x)(\nabla U(x) G(x))^{\mathbf{t}}]
$$

which is negative definite by virtue of (13). The global asymptotic stability of the closed-loop system then follows according to the well known extention of the second Liapunov theorem to systems with continuous right-hand side.

5.3 (3.ii) $\Longrightarrow$ (3.i)

The following proof is inspired by, but at the same time is a generalization of, analogous proofs available in [5], [3], [11].

According to Lemma 2 it is not restrictive to assume that $W(x)$ is actually a strong Liapunov function for the closed loop system. We define an appropriate cost functional of the form (10), by taking the same γ as in (12), and

$$
h(x) = -2\nabla W(x) f(x) + \gamma|\nabla W(x) G(x)|^2 \ .
$$

Clearly, h is continuous, positive definite and radially unbounded. Thus, the functional to be minimized takes the form

$$
\begin{aligned}
J(x_0, u(\cdot)) &= \int_0^{+\infty} \Big[-\nabla W(\varphi(t; x_0, u(\cdot))) f(\varphi(t; x_0, u(\cdot))) \\
&+ \frac{\gamma}{2} \big| \nabla W(\varphi(t; x_0, u(\cdot))) G(\varphi(t; x_0, u(\cdot))) \big|^2 \\
&+ \frac{1}{2\gamma} \big| u(t) \big|^2 \Big] \, dt \ .
\end{aligned} \tag{20}
$$

Let us consider the Cauchy problem

$$
\begin{cases} \dot{x} = f(x) + G(x) k^o(x) \\ x(0) = x_0 \end{cases} \tag{21}
$$

where

$$
k^o(x) = -\gamma (\nabla W(x) G(x))^{\mathbf{t}} \ . \tag{22}
$$

According to Remark 2B above, (22) is a stabilizer for (9). If we select any solution $\varphi^o(t)$ of (21) we have therefore

$$
\varphi^o(t) \to 0 \quad \text{when} \quad t \to +\infty
$$

which implies

$$
\lim_{t \to +\infty} W(\varphi^o(t)) = 0 \ . \tag{23}
$$

Let $u^o(t) = -\gamma (\nabla W(\varphi^o(t)) G(\varphi^o(t)))^{\mathbf{t}}$. Clearly the (unique) solution of (9) with $u = u^o(t)$ is $\varphi^o(t) = \varphi(t; x_0, u^o(\cdot))$. Let us compute

$$
\begin{aligned}
J(x_0, u^o(\cdot)) &= \int_0^{+\infty} \left[-\nabla W(\varphi^o(t)) f(\varphi^o(t)) + \gamma |\nabla W(\varphi^o(t)) G(\varphi^o(t))|^2 \right] dt \\
&= \int_0^{+\infty} -\nabla W(\varphi^o(t)) \left[f(\varphi^o(t)) + G(\varphi^o(t)) u^o(t) \right] dt \\
&= \int_0^{+\infty} -\nabla W(\varphi^o(t)) \dot{\varphi}^o(t) \, dt = W(x_0)
\end{aligned}
$$

by virtue of (23).

Now, let $u(t)$ be any admissible input. For simplicity, we use here the shortned notation $\varphi(t) = \varphi(t; x_0, u(\cdot))$. Let us distinguish two cases.

1) The integral in (20) diverges. In this case, it is obvious that $J(x_0, u^o(\cdot)) < J(x_0, u(\cdot))$.

2) The integral in (20) converges. According to Lemma 1, we conclude that $\lim_{t \to +\infty} \varphi(t) = 0$, and since $W(x)$ is radially unbounded, continuous and

positive definite, this in turn implies $\lim_{t\to+\infty} W(\varphi(t)) = 0$. Finally, from (20) we have

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^{+\infty} -\nabla W(\varphi(t)) \left[f(\varphi(t)) + G(\varphi(t))u(t)\right] dt \\ &+ \int_0^{+\infty} \Big[\frac{|u(t)|^2}{2\gamma} + \frac{\gamma}{2}|\nabla W(\varphi(t))G(\varphi(t))|^2 \\ &\qquad + \nabla W(\varphi(t))G(\varphi(t))u(t)\Big] dt \\ &= \int_0^{+\infty} -\nabla W(\varphi(t))\dot{\varphi}(t)\, dt \\ &+ \frac{1}{2}\int_0^{+\infty} \left|\frac{u(t)}{\sqrt{\gamma}} + \sqrt{\gamma}(\nabla W(\varphi(t))G(\varphi(t)))\right|^2 dt \geq W(x_0) \ . \end{aligned}$$

Summing up, we have $J(x_0, u^o(\cdot)) = W(x_0) \leq J(x_0, u(\cdot))$ for each x_0. This achieves the proof. In particular, we see that $u^o(t) \equiv u^*_{x_0}(t)$ and that $W(x)$ coincides with the value function of the minimization problem (20).

5.4 (3.i) $\Longrightarrow$ (3.iii)

Let us fix an initial state x_0, a positive time T and a constant input u_0. Let

$$\xi = \varphi(T; x_0, u_0) \ .$$

Define a new input $u(t)$ as

$$u(t) = \begin{cases} u_0 & t \in [0, T] \\ u^*_\xi(t-T) & t > T \ . \end{cases}$$

Of course,

$$V(x_0) \leq J(x_0, u(\cdot)) = V(\xi) + \frac{1}{2}\int_0^T (h(\varphi(t; x_0, u_0)) + \frac{|u_0|^2}{\gamma})\, dt \ .$$

Let us rewrite it in the form

$$-\frac{1}{2}\int_0^T \left(h(\varphi(t, x_0, u_0)) + \frac{|u_0|^2}{\gamma}\right) dt \leq V(\xi) - V(x_0) \ .$$

Dividing by T and using the mean value theorem we get

$$-\frac{1}{2}\left(h(\varphi(\theta; x_0, u_0)) + \frac{|u_0|^2}{\gamma}\right) \leq \frac{V(\varphi(T; x_0, u_0)) - V(x_0)}{T}$$

for some $\theta \in [0, T]$. Taking the limit for $T \to 0^+$, we obtain

$$-\frac{h(x_0)}{2} - \frac{|u_0|^2}{2\gamma} \leq \nabla V(x_0)[f(x_0) + G(x_0)u_0] \ . \tag{24}$$

On the other hand, by definition we have

$$V(x_0) = J(x_0, u^*_{x_0}(\cdot)) \ .$$

Let $\eta = \varphi(T; x_0, u^*_{x_0}(\cdot))$. Now we have

$$V(x_0) = V(\eta) + \frac{1}{2}\int_0^T \left(h(\varphi(t, x_0, u^*_{x_0}(\cdot))) + \frac{|u^*_{x_0}(t)|^2}{\gamma} \right) dt \ .$$

By repeating the same computation as before, we get

$$-\frac{h(x_0)}{2} - \frac{|u^*_{x_0}(0)|^2}{2\gamma} = \nabla V(x_0)[f(x_0) + G(x_0)u^*_{x_0}(0)] \tag{25}$$

(here we are using right-continuity of the admissible inputs). Comparing (24) and (25), we find that

$$-\nabla V(x)G(x)u^*_x(0) - \frac{|u^*_x(0)|^2}{2\gamma} = \max_{u \in \mathbb{R}^m} \left[-\nabla V(x)G(x)u - \frac{|u|^2}{2\gamma} \right]$$

for each $x \in \mathbb{R}^n$ (the subscript 0 has been suppressed, since x_0 and u_0 were arbitrary). The expression to be maximized at the right hand side can be rewritten as

$$-\left| \frac{u}{\sqrt{2\gamma}} + \frac{\sqrt{2\gamma}(\nabla V(x)G(x))^{\mathbf{t}}}{2} \right|^2 + \gamma \frac{|\nabla V(x)G(x)|^2}{2} \ .$$

Now it is evident that the maximum is reached for $u = -\gamma(\nabla V(x)G(x))^{\mathbf{t}} = u^*_x(0)$. Substituting in (25) for a generic $x \in \mathbb{R}^n$, after semplification we finally check that V satisfies the desired equation (13).

∎

Remark 3 A. Concerning the map h, in the proof that $(3.i) \Longrightarrow (3.iii)$ only the assumption that it is continuous is really needed. It is interesting to discuss what happens when h is positive semidefinite (i.e., $h(x) \geq 0 \ \forall x \in \mathbb{R}^n$), instead of positive definite and radially unbounded. In this case, we can still apply the feedback law (12) with $W = U$, but we don't have W as a strict Liapunov function for the closed loop system any more. Indeed, the derivative of W with respect to the closed loop system is only negative semidefinite, and the classical Liapunov theory cannot be applied. However, we can obtain some stabilizability results under additional assumptions by invoking LaSalle's invariance principle.

A remarkable example of this approach is the so-called Jurdjevic-Quinn method (see [8]; see also [5] where the connection with optimal regulation was pointed out for the first time, and [2] for a "nonsmooth" extension). Jurdjevic-Quinn condition can be interpreted as an observability condition (see again [5]).

Concerning the relationship between optimal regulation and stabilization when $h(x) \geq 0$ see also [10].

B. If we assume that statement $(3.i)$ holds, then the value function $V(x)$ is a *control Liapunov function* for system (9). Hence, by means of this V one can construct other types of stabilizing feedback. For instance, one can use the so-called *universal formula* (see [13]).

∎

6 A remark about radial unboundedness

From the point of view of stability theory, the implications $(3.i) \Longrightarrow (3.iii) \Longrightarrow (3.ii)$ of Theorem 3 are the most interesting. In this section, we prove that under reasonable assumptions about the vector fields $f, g_1, \ldots, g_m$, the radial unboundedness of the value function $V(x)$ imposed in $(3.i)$ is in fact automatically satisfied. First of all, we prove the following lemma.

Lemma 3 *Assume that there exist a map* $a(\cdot) \in \mathcal{K}_0^\infty$ *and positive numbers* A_0, A_1, b, R *such that*

(i) $h(x) \geq a(|x|)$ *for* $|x| > R$,

(ii) $|f(x)| \leq A_0 a(|x|) + A_1$ *for* $|x| > R$,

(iii) $||G(x)|| \leq b$.

Assume further that for each $x_0 \in \mathbb{R}^n$, *it has been specified an admissible input* $u_{x_0}(t)$ *such that*

$$J(x_0, u_{x_0}(\cdot)) < \infty \ .$$

Then, the function

$$v(x) = J(x_0, u_{x_0}(\cdot))$$

is radially unbounded.

Proof.

Let us assume that the conclusion is false. Then, there exists $L > 0$ such that for each $K > 0$ we can find a point x_K with $|x_K| > K$ for which $v(x_K) \leq L$. Without loss of generality, we can assume that $L > a(R)$. Let us take $K > 2A_1 + a^{-1}(L) + 2A_0 L + 2b\sqrt{\gamma L}$. We remark that

$$K > a^{-1}(L) \tag{26}$$

and that

$$a(|x_K|) > a(K) > L \ . \tag{27}$$

Let us set for simplicity $\varphi_K(t) = \varphi(t; x_K, u_{x_K}(\cdot))$. Using Lemma 1, we have that $\varphi_K(t) \to 0$ for $t \to +\infty$ and since $a \in \mathcal{K}_0^\infty$, we also have $a(\varphi_K(t)) \to 0$ for $t \to +\infty$. Recall now (27). Since a is continuous, there exists an instant $T_K > 0$ such that $a(|\varphi_K(T_K)|) = L$ while

$$a(|\varphi_K(t)|) \geq L \quad \forall t \in [0, T_K] \ . \tag{28}$$

This is the same thing as

$$|\varphi_K(t)| \geq a^{-1}(L) > R \quad \forall t \in [0, T_K] \ . \tag{29}$$

By assumption (i), we have

$$h(\varphi_K(t)) \geq a(|\varphi_K(t)|) \geq L \quad \forall t \in [0, T_K] \ . \tag{30}$$

On the other hand we have

$$L \geq v(x_K) = \frac{1}{2}\int_0^\infty \left(h(\varphi_K(t)) + \frac{|u_{x_K}(t)|^2}{\gamma} \right) dt \geq \frac{1}{2}\int_0^\infty h(\varphi_K(t))\, dt \geq \frac{1}{2} L T_K$$

from which we deduce that $T_K \leq 2$. By virtue of (ii) and (iii), we also have

$$\begin{aligned} K - a^{-1}(L) &\leq |x_K| - |\varphi_K(T_K)| \\ &\leq |\varphi_K(T_K) - x_K| \\ &\leq \int_0^{T_K} \left(|f(\varphi_K(t))| + \|G(\varphi_K(t))\| \cdot |u_{x_K}(t)| \right) \, dt \\ &\leq \int_0^{T_K} \left(A_0 a(|\varphi_K(t)|) + A_1 \right) \, dt + b \int_0^{T_K} |u_{x_K}(t)|\, dt \ . \end{aligned}$$

Using Hölder inequality and (i), we obtain therefore

$$\begin{aligned} K - a^{-1}(L) &\leq A_0 \int_0^{T_K} h(|\varphi_K(t)|)\, dt + A_1 T_K + b\sqrt{T_K} \left(\gamma \int_0^{T_K} \frac{|u_{x_K}(t)|^2}{\gamma}\, dt \right)^{\frac{1}{2}} \\ &\leq 2A_0 L + T_K A_1 + 2b\sqrt{\gamma L} \ . \end{aligned}$$

Finally, we get

$$T_K \geq \frac{K - a^{-1}(L) - 2A_0 L - 2b\sqrt{\gamma L}}{A_1} > 2 \ ,$$

a contradiction.

∎

Remark 4 Note that since h is radially unbounded and positive definite, then h must satisfy an equality like that in (i) for some $a(\cdot) \in \mathcal{K}_0^\infty$. The requirement in Lemma 3 is actually that, apart from a multiplicative constant, the same $a(\cdot) \in \mathcal{K}_0^\infty$ can be used to estimate h from below and the norm of f from above.

∎

Now we can state the announced result.

Proposition 1 *Assume that f, G and h satisfy the same conditions (i), (ii) and (iii) as in Lemma 3. Assume further that the optimization problem (10) is solvable. Then, the value function $V(x)$ defined by (11) is radially unbounded.*

Proof. If for each $x_0 \in \mathbb{R}^n$ we specify $u_{x_0}(t) = u^*_{x_0}(t)$, then the assumptions of Lemma 3 are fulfilled and, in addition, we have $v(x_0) = V(x_0)$. The conclusion is immediate.

∎

7 The generality of damping control

As noticed in Section 3, if a linear system is stabilized by a continuous feedback, it is also stabilized by means of a linear feedback and, more precisely, by a feedback of the form (7). This fact has an analogue for the nonlinear case. We actually prove that if an affine system admits a C^1 stabilizer satisfying certain additional conditions, then it also admits a stabilizing feedback in damping form.

The material of this section is basically borrowed from [9], where the main result is obtained by exploiting in a clever way the connections among stabilization, optimal regulation and the so-called Zubov's equation. However, in authors' opinion the argument requires assumptions which are not explicitly stated in [9]. Thus, we find convenient to expose here a more precise and slightly improved version.

Proposition 2 *Consider the affine system (9) and assume that*

$$|f(x)| \leq A_0|x|^2 + A_1 \quad \text{and} \quad ||G(x)|| \leq b \tag{31}$$

for some positive constants A_0, A_1, b. Assume further that (9) admits a stabilizer $u = k(x)$ such that:

(i) $k(x)$ is of class C^1 and $k(0) = 0$,

(ii) for each compact set $C \subset \mathbb{R}^n$, there exists a map $\lambda_C(t) : [0, +\infty) \to [0, +\infty)$ such that

$$\int_0^{+\infty} \lambda_C(t)\, dt < +\infty \tag{32}$$

and

$$\|\frac{\partial}{\partial x}\varphi_{k(\cdot)}(t;x)\| \leq \lambda_C(t) \tag{33}$$

for each $t \geq 0$ and each $x \in C$.

Then, there exists a map $W(x)$ with the property stated in Theorem 3, (3.ii). In other words, the system can be also stabilized by a damping control.

Proof. Let us set for simplicity $\varphi(t;x) = \varphi_{k(\cdot)}(t;x)$. Using the mean value theorem and the fact that $\varphi(t;0) = 0$, it is not difficult to infer from Assumption (ii) that

$$\int_0^{+\infty} |\varphi(t;x)|\, dt < +\infty \tag{34}$$

for each $x \in \mathbb{R}^n$. According to global asymptotic stability, we have $|\varphi(t;x)| \to 0$ as $t \to +\infty$. Hence, (34) implies that also the integral

$$\int_0^{+\infty} |\varphi(t;x)|^2\, dt \tag{35}$$

converges. Assumption (i) implies in particular that $k(x)$ is locally Lipschitz continuous. Taking again into account asymptotic stability, from (35) we therefore obtain that the integral

$$\int_0^{+\infty} |k(\varphi(t;x_0))|^2\, dt \tag{36}$$

converges, as well. Now, let us introduce the function

$$L(x) = \frac{|x|^2 + |k(x)|^2}{2}\ .$$

Notice that $L(x)$ is C^1, positive definite and radially unbounded. Moreover, by virtue of (35), (36), the function

$$\Psi(x) = \int_0^{\infty} L(\varphi(t;x))\, dt \tag{37}$$

is well defined for each $x \in \mathbb{R}^n$. Global asymptotic stability of the closed loop system also implies that for each compact set C there exists $R > 0$ such that $|\varphi(t;x)| \leq R$ for each $t \geq 0$ and each $x \in C$ ([1]). Using this fact, it is not difficult to see that

$$\|\frac{\partial}{\partial x}L(\varphi(t;x))\| \leq M\lambda_C(t) \tag{38}$$

for some $M > 0$, each $t \geq 0$ and each $x \in C$. It follows by standard arguments that Ψ is of class C^1 on $\mathbb{R}^n$. Note that $\Psi(0) = 0$. We also note that by virtue of

(31), Lemma 3 is applicable, if we specify for each x the input $u_x(t) = k(\varphi(t;x))$. Hence, $\Psi(x)$ is radially unbounded.

Now, let

$$\Phi(x) = 1 - e^{-\Psi(x)} \ .$$

It is immediate to check that Φ is of class C^1, $\Phi(0) = 0$ and $0 < \Phi(x) < 1$ for $x \neq 0$, and that $\lim_{|x|\to\infty} \Phi(x) = 1$. Moreover, for each $s \in \mathbb{R}$ we have

$$\begin{aligned} \Phi(\varphi(s;x)) &= 1 - e^{-\int_0^\infty L(\varphi(t;\varphi(s;x)))\,dt} \\ &= 1 - e^{-\int_s^\infty L(\varphi(\tau;x))\,d\tau} \ . \end{aligned}$$

This yields

$$\begin{aligned} \frac{d}{ds}\Phi(\varphi(s;x)) &= \nabla\Phi(\varphi(s;x))\dot{\varphi}(s;x) \\ &= -L(\varphi(s;x))e^{-\int_s^\infty L(\varphi(\tau;x))\,d\tau} \ . \end{aligned}$$

Finally, taking $s = 0$, we obtain the equation

$$\nabla\Phi(x)[f(x) + G(x)k(x)] = -L(x)(1 - \Phi(x)) \tag{39}$$

for each $x \in \mathbb{R}^n$. Next, we set

$$V(x) = -\log(1 - \Phi(x)) \ .$$

We have that V is of class C^1 and $V(0) = 0$. Moreover, V is positive definite and radially unbounded. From (39) we also get

$$\nabla V(x)[f(x) + G(x)k(x)] = -L(x) \tag{40}$$

for each $x \in \mathbb{R}^n$. Let

$$h(x) = |x|^2 + (k(x) + (\nabla V(x)G(x))^{\mathbf{t}})^2 \ .$$

From (40) we see that

$$\nabla V(x)f(x) - \frac{1}{2}|\nabla V(x)G(x)|^2 = -\frac{h(x)}{2} \tag{41}$$

for each $x \in \mathbb{R}^n$.

We are now ready to get the conclusion. Consider the affine system (9) with the feedback law $u = -\frac{1}{2}(\nabla V(x)G(x))^{\mathbf{t}}$. The left hand side of (41) coincides with the derivative of V with respect to the closed loop system. Hence, (9) with the feedback law $u = -\frac{1}{2}(\nabla V(x)G(x))^{\mathbf{t}}$ admits V as a global, strict Liapunov function. This implies that the origin is globally asymptotically stable for the closed loop system and the statement is proved.

∎

Remark 5 If the system is linear and $k(x)$ is any linear stabilizer, then the closed loop system exhibits exponential decay so that Assumption (ii) is automatically fulfilled.

According to (34), we see that Assumption (ii) guarantees sufficiently fast decay of solutions of the closed loop system (with $u = k(x)$).

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