On Several Notions of Generalized Solutions for Discontinuous Differential Equations and their Relationships

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Abstract

In this report we collect some results and counterexamples, with the aim of giving a complete representation of relationships among various possible definitions of solution for ordinary differential equations with a discontinuous right hand side.

1 Introduction

In this report we review and compare several approaches to the problem of defining solutions of the ordinary differential equation

$$\dot{x} = f(x) \ , \qquad x \in \mathbf{R}^n \tag{1}$$

under the standing assumption that the vector field $f : \mathbf{R}^n \to \mathbf{R}^n$ is locally bounded and (Lebesgue) measurable. We emphasize that f is not required to be continuous. In order to motivate our study, we limit ouselves to recall that discontinuous differential equations often arise in applied mathematics: one of the traditional examples is the motion of a body in presence of dry friction ([16]); more recent sources of interest come from control theory and games theory. Ideally, we continue the program started many years ago by the important paper [18]. Other remarkable work was done in [6], [11] and [8] (see also the reference therein).

The organization of the report is as follows. Sections 2, 3, 4, 5 and 6 contain the definitions of the basic notions. Our main contribution consists of a number of counterexamples. Some of them are trivial and well known, and are collected here for the sake of completeness and tutorial reasons. On the contrary, the examples described in Sections 7 and 8 are new. They show in particular that there may exist Sentis and Carathéodory solutions which are not Forward Euler. In Section 9 we give a rather complete map of the relationships among the various concepts. Recent developments of the engineering literature enlighten the interest in solutions which have a relay or switching nature. In Section 10 we propose and comment a rigorous definition of switched solution for systems of discontinuous differential equations. Finally, in Section 11 we recall the notion of patchy vector field.

The definition of Sentis solution involves in particular the construction of a set valued map $F_S(x)$. An additional contribution of this report is a representation formula for $F_S(x)$ in terms of limit points of the map $f(x)$ (see the Appendix).

2 Classical notions

In this section we shortly recall the more classical notions. Let $I = [a, b] \subseteq \mathbf{R}$ be any closed interval, with $a < b$. According to the terminology adopted in [18], a function $\varphi(t) : I \to \mathbf{R}^n$ is a *Newton solution* of (1) if it admits an extension on some open interval (a', b') (with $[a, b] \subset (a', b')$) which is everywhere differentiable and satisfies $\dot{\varphi}(t) = f(\varphi(t))$ for each $t \in I$. The set of all Newton solutions of (1) is denoted by $\mathcal{N}$. Peano's Theorem states that if f is continuous, then for each $c \in \mathbf{R}$ and for each $\bar{x} \in \mathbf{R}^n$ there exist $a, b \in \mathbf{R}$ with $a < c < b$, and there exists at least one Newton solution of the initial value problem

$$\begin{cases} \dot{x} = f(x) \\ x(c) = \bar{x} \end{cases} \tag{2}$$

defined on the interval $[a, b]$. A function $\varphi(t) : I \to \mathbf{R}^n$ is a *Carathéodory solution* of (1) if it is differentiable a.e. on the interval I, and it satisfies $\dot{\varphi}(t) = f(\varphi(t))$ a.e. on I. The set of all Carathéodory solutions is denoted by $\mathcal{C}$.

It is obvious that any Newton solution is a Carathéodory solution. The converse is true if f is continuous but not in general.

Example 1 Consider the one dimensional differential equation (1) with

$$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 2 & \text{if } x > 0 \ . \end{cases}$$

The function

$$\varphi(t) = \begin{cases} t & \text{if } t \leq 0 \\ 2t & \text{if } t > 0 \end{cases}$$

is the unique Carathéodory solution such that $\varphi(0) = 0$. Clearly, there is no Newton solution defined on a complete neighborhood of $t = 0$, and satisfying the same condition $\varphi(0) = 0$.

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When $f(x)$ is not continuous, there may be some initial state $\bar{x}$ for which existence of Carathéodory solutions is not guaranteed. Sufficient conditions for forward existence of Carathéodory solutions can be found in [23] and [7] (see also [8]).

3 Differential inclusions

In what follows, by *ordinary solution* of a differential inclusion

$$\dot{x} \in F(x) \ . \tag{3}$$

we mean any function $\varphi(t) : [a, b] \to \mathbf{R}^n$ which is differentiable a.e., and satisfies $\dot{\varphi}(t) \in F(\varphi(t))$ a.e. on the interval I. The most popular approach to ordinary differential equations with a discontinuous right hand side consists of replacing (1) by a suitable differential inclusion (3). Namely, the generalized solutions of (1) are defined as the ordinary solutions of (3). Of course, the notions of solution obtained in this way depend on the construction of the set valued map F. For instance, Krasowski solutions of (1) are the ordinary solutions of (3) where

$$F(x) = F_K(x) = \bigcap_{\delta > 0} \overline{\text{co}} \{ f(\mathcal{B}(x, \delta)) \} \tag{4}$$

while Filippov solutions of (1) are the ordinary solutions of (3) where

$$F(x) = F_F(x) = \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} \overline{\text{co}} \{ f(\mathcal{B}(x, \delta) \backslash N) \} \tag{5}$$

(here μ is the Lebesgue measure of $\mathbf{R}^n$, $\overline{\text{co}}$ denotes the closure of the convex hull, and $\mathcal{B}(x, r)$ is the ball of radius r centered at x). Of course, every Filippov solution is a Krasowski solution, as well. Examples of Krasowski solutions which are not Filippov solutions are trivial (see for instance the following Example 7). The set of Filippov (respectively, Krasowski) solutions of (1) will be denoted by $\mathcal{F}$ (respectively, $\mathcal{K}$).

Since f is measurable and locally bounded, then the set valued map $F_F(x)$ is upper semicontinuous, locally bounded, compact and convex valued. The same is true for $F_K(x)$. In follows that for each $c \in \mathbf{R}$ and each initial state $\bar{x}$, the initial value problem (2) has a Filippov solution (and hence, also a Krasowski solution) on some interval $[a, b]$, with $a < c < b$ ([17]).

4 g-solutions

It is commonly recognized that Filippov solutions are "too many" for some applications (this is true in particular in the stabilization problem for nonlinear systems by means of discontinuous feedback, see [5], [9], [10], [25]). To overcome this drawback, in [26] the author replaces (1) by a differential inclusion (3) where

$$F(x) = F_S(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\{f(\mathcal{B}(x,\delta)\backslash N)\}} \ . \tag{6}$$

The set valued map $F_S(x)$ turns out to be upper semicontinuous, locally bounded and compact (but in general not convex) valued. Of course, $F_S(x) \subseteq F_F(x)$; in fact, we prove in the Appendix that $F_F(x) = \text{co}F_S(x)$.

Unfortunately, an ordinary solution with a given initial condition may not exist in general, for a differential inclusion (3) with nonconvex right hand side. Thus, in [26] a new class of solutions, called g-solutions, is introduced: for reader's convenience, we report the definition.

Let $\bar{x}$ be a fixed point of $\mathbf{R}^n$, and $I = [a,b]$ be a closed interval. Given $m \in \mathbf{N}$, let us take a partition of I

$$a = t_{m,0} < t_{m,1} < \ldots < t_{m,k_m-1} < t_{m,k_m} = b$$

where k_m is some positive integer, and let $l_m = \max\{t_{m,i+1} - t_{m,i}\ ,\ i = 0,\ldots,k_m-1\}$. Then, for each $i = 0,\ldots,k_m-1$ choose $\varepsilon_{m,i} \in \mathbf{R}^n$ and construct a right-continuous, piecewise affine function $\psi_m(t)$ on the interval $[a,b]$ in such a way that $\psi_m(t_{m,0}) = \bar{x}$, and

$$\psi_m(t_{m,i+1}) = \psi_m(t_{m,i}) + v_{m,i}(t_{m,i+1} - t_{m,i}) + \varepsilon_{m,i}$$

where $v_{m,i}$ is any element in $F(\psi_m(t_{m,i}))$. Any such function $\psi_m(t)$ will be called a *discontinuous polygonal approximation.*

Definition 1 *According to [26], we say that a function $\varphi(t) :\ [a,b] \to \mathbf{R}^n$ is a* g-solution *of (3) if for each $\sigma > 0$ there exists an integer m and a discontinuous polygonal approximation such that*

$$|\varphi(t) - \psi_m(t)| < \sigma \quad \forall t \in [a,b]$$

$0 < l_m < \sigma$ *and* $0 \leq \sum_{i=0}^{k_m-1} |\varepsilon_m^i| < \sigma$.

Definition 1 can be rephrased by saying that φ is the uniform limit of a sequence of discontinuous polygonal approximations $\{\psi_m\}$, such that l_m is decreasing, $\lim_m l_m = 0$ and

$$\lim_m \sum_{i=0}^{k_m-1} |\varepsilon_m^i| = 0 \ .$$

Note that by construction, $\varphi(a) = \bar{x}$. In [26], it is proven that if $F(x)$ is upper semicontinuous, compact valued and locally bounded, then for each $\bar{x}$ there exist $b > 0$ and a g-solution $\varphi(t) : [a,b] \to \mathbf{R}^n$ with $\varphi(a) = \bar{x}$. In [26] it is also proven that ordinary solutions of (3), when they exists, are g-solutions. If $\varphi(t)$ is a g-solution of (3) on the interval $[a,b]$, then $\varphi(t)$ is locally Lipschitz continuous and

$$\dot{\varphi}(t) \in \text{co}F(\varphi(t)) \ , \qquad \text{a.e. } t \in [0,T] \ .$$

However, in general, we do not have $\dot{\varphi}(t) \in F(\varphi(t))$ for a.e. $t \in [a,b]$: this is the main difference between g-solutions and ordinary solutions of differential inclusions (for other properties about g-solutions the reader is referred to [26] and [19]).

We propose to say that a function $\varphi(t)$ is a Sentis solution of (1) if it is a g-solution of (3) with $F(x) = F_S(x)$. The set of all Sentis solutions of (1) will be denoted by $\mathcal{S}$.

Proposition 1 *([26]) Let f be locally bounded and measurable. Then every Sentis solution of (1) is a Filippov solution.*

We know from [26] that the converse of the previous proposition is false in general (see for instance the following Example 5).

5 Forward-Euler solutions

Recently, in order to introduce notions of solution for discontinuous differential equations more suitable for applications to control theory, new approaches have been tried ([1], [6], [10], [11]). These approaches do not involve differential inclusions: rather, they make use of certain polygonal approximations defined by means of the mere values of the vector field $f(x)$, and possibly allowing inner and/or outer perturbations. The main difference with respect to the previous section, is that now the polygonal approximations will be continuous. We start by explaining how to construct this new type of approximations.

Let $m \in \mathbf{N}$, and let $I = [a, b]$ be a given interval. Consider, as in the previous section, a partition of I

$$a = t_{m,0} < t_{m,1} < \ldots < t_{m,k_m-1} < t_{m,k_m} = b$$

and let again $l_m = \max\{t_{m,i+1} - t_{m,i}\ ,\ i = 0, \ldots, k_m - 1\}$. Let us choose a point $x_{m,0} \in \mathbf{R}^n$, and certain vectors $p_{m,i}$, $q_{m,i} \in \mathbf{R}^n$ $(i = 0, \ldots, k_m - 1)$. Then construct a continuous, piecewise affine function $\psi_m(t)$ on the interval $[a, b]$ in such a way that $\psi_m(t_{m,0}) = x_{m,0}$, and

$$\psi_m(t_{m,i+1}) = \psi_m(t_{m,i}) + [f(\psi_m(t_{m,i}) + p_{m,i}) + q_{m,i}]\,(t_{m,i+1} - t_{m,i})$$

$(i = 0, \ldots, k_m - 1)$. Any such function $\psi_m(t)$ will be called a *continuous polygonal approximation.* The vectors $p_{m,i}$'s are called *inner perturbations*, while the vectors $q_{m,i}$'s are called *outer perturbations.*

Definition 2 *A function $\varphi(t) :\ [a, b] \to \mathbf{R}^n$ is said to be a* perturbed Euler solution *of (1) if for each $\sigma > 0$ there exists an integer m and a continuous polygonal approximation such that*

$$|\varphi(t) - \psi_m(t)| < \sigma \qquad \forall t \in [a, b]$$

$0 < l_m < \sigma$, $0 \le |p_{m,i}| < \sigma$, $0 \le |q_{m,i}| < \sigma$, for each $i = 0, \ldots, k_m - 1$.

Depending on the choice of the $p_{m,i}$'s and the $q_{m,i}$'s, we can distinguish interesting subsets of solutions.

- An *Euler* solution is obtained by limiting ourselves to continuous polygonal approximations with $p_{m,i} = q_{m,i} = 0$ for all m and i, and imposing $x_{m,0} = \varphi(t_0)$ for all m (see [11]).
- An *externally perturbed Euler* solution is obtained by limiting ourselves to continuous polygonal approximations with $p_{m,i} = 0$ for all m and i.

Externally perturbed solutions are also called *Forward Euler* solutions (see [6]). In fact, the term Forward Euler solution (in short, FE-solution) is preferable for notational reasons.

The symbols adopted in this report for perturbed Euler, Euler and FE-solutions will be, respectively, $\mathcal{PE}$, $\mathcal{E}$ and $\mathcal{FE}$ [1].

Several examples and properties concerning Euler solutions can be found in [11]. Continuous polygonal approximations without inner and outer perturbations seem to be a very poor way to produce solutions. For instance, in the classical (scalar) example with $f(x) = \sqrt{|x|}$, $\bar{x} = 0$, $t_0 = 0$, the unique Euler solution is $\varphi(t) \equiv 0$. Note that if we allow $x_{m,0} \ne \varphi(0)$, then we can approach also the solution $\varphi(t) = (\operatorname{sgn} t)t^2$. However, it is by no means possible to approach solutions of the type

$$\varphi(t) = \begin{cases} 0 & \text{for } t \in [0, \tau] \\ (t - \tau)^2 & \text{for } t > \tau\ . \end{cases}$$

Euler solutions are not discussed in this paper; we prefer to focus on some significant relationships among FE-solutions and the other notions.

Proposition 2 *Let f be locally bounded and measurable. Then every Newton solution of (1) is a FE-solution.*

Proof Let $\varphi(t) :\ [a, b] \to \mathbf{R}^n$ be a Newton solution of equation (1), with $f(x)$ locally bounded and measurable. Let us fix an integer $N \ge 2$. For each $\bar{t} \in [a, b]$ we can write

$$\varphi(t) = \varphi(\bar{t}) + f(\varphi(\bar{t}))(t - \bar{t}) + o(t - \bar{t}) \qquad t \to \bar{t}\ . \tag{7}$$

[1] In principle, we could introduce another set of solutions by allowing only inner perturbations, but this issue will not be pursued in this report (note that if f is continuous, any inner perturbation can be easily reduced to an outer one).

Hence, for each $\bar{t}$ we can select a $\delta_{\bar{t}} > 0$ such that

$$t \in (\bar{t} - \delta_{\bar{t}}, \bar{t} + \delta_{\bar{t}}) \quad \Longrightarrow \quad \left| \frac{\varphi(t) - \varphi(\bar{t})}{t - \bar{t}} - f(\varphi(\bar{t})) \right| < \frac{1}{N} \ . \tag{8}$$

The open sets $(\bar{t} - \delta_{\bar{t}}, \bar{t} + \delta_{\bar{t}})$ for $\bar{t} \in [a,b]$ constitute a cover of the interval $[a,b]$. By the compactness argument, we can extract a finite cover. The centers of the intervals of this finite cover, together with the endpoints a, b, forms a partition of the interval $[a,b]$:

$$a = \tau_0 < \tau_1 < \ldots < \tau_m = b \ .$$

For each $i = 1, \ldots, m$ let us fix a point θ_i such that $\tau_{i-1} < \theta_i < \tau_i$ and $\tau_i - \delta_{\tau_i} < \theta_i < \tau_{i-1} + \delta_{\tau_{i-1}}$.

Now we are able to construct a continuous polygonal approximation on the base of the partition

$$a = \tau_0 < \theta_1 < \tau_1 < \theta_2 < \tau_2 \ldots < \theta_m < \tau_m = b \ .$$

We define $\psi(t)$ as the piecewise affine function joining the points

$$\varphi(\tau_0), \varphi(\theta_1), \varphi(\tau_1), \varphi(\theta_2), \varphi(\tau_2), \ldots, \varphi(\theta_m), \varphi(\tau_m)$$

in this order. For $t \in [\tau_{i-1}, \theta_i]$ $(i = 1, \ldots, m)$ we have

$$\psi(t) = \varphi(\tau_{i-1}) + \frac{\varphi(\theta_i) - \varphi(\tau_{i-1})}{\theta_i - \tau_{i-1}}(t - \tau_{i-1}) \ .$$

Let $p_i = \frac{\varphi(\theta_i) - \varphi(\tau_{i-1})}{\theta_i - \tau_{i-1}} - f(\varphi(\tau_{i-1}))$. Since $\theta_i - \tau_{i-1} < \delta_{\tau_{i-1}}$ we can apply (8) and we obtain

$$|p_i| < \frac{1}{N} \ . \tag{9}$$

Moreover, for $t \in [\tau_{i-1}, \theta_i]$, we also have

$$\begin{aligned} |\psi(t) - \varphi(t)| &= |\varphi(\tau_{i-1}) + [f(\varphi(\tau_{i-1})) + p_i](t - \tau_{i-1}) - \varphi(\tau_{i-1}) - f(\varphi(\tau_{i-1}))(t - \tau_{i-1}) + o(t - \tau_{i-1})| \\ &\le |p_i(t - \tau_{i-1})| + o(t - \tau_{i-1}) \\ &\le \frac{2}{N^2} \ . \end{aligned}$$

Since $N \ge 2$, $\frac{2}{N^2} \le \frac{1}{N}$. Similar arguments can be repeated for the intervals of the type $[\theta_{i-1}, \tau_{i-1}]$ and for each $i = 1, \ldots, m$. In conclusion, we constructed a continuous polygonal approximation $\psi(t)$ such that

$$|\psi(t) - \varphi(t)| < \frac{1}{N} \qquad \forall t \in [a,b] \ .$$

Every subinterval of the partition has a length less than $\frac{1}{N}$ and the absolute value of all the outer perturbations is less than $\frac{1}{N}$, as well. The statement is so proven.

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6 Hermes solutions

One can consider more general types of inner or outer perturbations. In [18], a function $\varphi(t) : [a,b] \to \mathbf{R}^n$ is said to be a *Hermes solution* of (1) if for each $\sigma > 0$ there exist a measurable function $p(t) : [a,b] \to \mathbf{R}^n$ and a Carathéodory solution $\chi(t) : [a,b] \to \mathbf{R}^n$ of the perturbed equation

$$\dot{x} = f(x + p(t)) \tag{10}$$

such that $|p(t)| < \sigma$ and $|\varphi(t) - \chi(t)| < \sigma$ for each $t \in [a,b]$.

Proposition 3 *Every perturbed Euler solution is a Hermes solution.*

Proof The proof is built up over some ideas already present in [18]. Let $\varphi : [a,b] \to \mathbf{R}^n$ be a perturbed Euler solution, and let $\sigma > 0$ be fixed. Let K be a compact subset of $\mathbf{R}^n$, such that

$$\cup_{t\in[a,b]}\mathcal{B}(\varphi(t),2) \subset K \ .$$

Let $|f(x)| \le M$ for $x \in K$. Let finally

$$\eta = \min\left\{1, \frac{\sigma}{3(M+1)}, \frac{\sigma}{3(b-a)}\right\} \ .$$

As in Definition 2, we can find a continuous polygonal approximation $\psi(t)$ such that $|\varphi(t) - \psi(t)| < \eta$ for all $t \in [a,b]$, $t_{i+1} - t_i < \eta$, $|p_i| < \eta$ and $|q_i| < \eta$, for each $i = 0, \ldots, k-1$. Note that

$$\dot\psi(t) = f(\psi(t_i) + p_i) + q_i \qquad t \in (t_i, t_{i+1}) \ .$$

Define the piecewise constant function $Q(t) = q_i$ for $t \in (t_i, t_{i+1})$. Then define $\chi(t) = \psi(t) - \int_a^t Q(s)\,ds$, for $t \in [a,b]$. It is clear that $\chi(t)$ is Lipschitz continuous and

$$\dot\chi(t) = \dot\psi(t) - q_i = f(\psi(t_i) + p_i) = f(\chi(t) + P(t)) \qquad t \in (t_i, t_{i+1})$$

where $P(t) = p_i + \psi(t_i) - \psi(t) + \int_a^t Q(s)\,ds$. This means that $\chi(t)$ is a Carathéodory solution of a suitably perturbed equation of the form (10). We now give an estimation of $P(t)$. For $t \in [t_i, t_{i+1}]$ we have

$$|P(t)| \le |p_i| + |f(\psi(t_i) + p_i) + q_i| \cdot |t - t_i| + \int_a^t |Q(s)|\,ds \ .$$

Now, by construction $|p_i| < \eta < \sigma/3$. Moreover, it is clear that $|\psi(t_i)+p_i-\varphi(t_i)| < 2\eta \le 2$, so that $\psi(t_i)+p_i \in K$. Recall that f is bounded by M aon K; since $|q_i| < \eta \le 1$ we have

$$|f(\psi(t_i) + p_i) + q_i| \cdot |t - t_i| < (M+1)\eta \le \sigma/3 \ .$$

Finally, $|Q(s)| \le \max_i |q_i| < \eta$, and hence

$$\int_a^t |Q(s)|\,ds \le \eta(b-a) \le \sigma/3 \ .$$

In conclusion, we see that $|P(t)| < \sigma$. In order to conclude the proof, it remains to give an estimation of $|\varphi(t) - \chi(t)|$. We have

$$|\varphi(t) - \chi(t)| \le |\varphi(t) - \psi(t)| + |\psi(t) - \chi(t)| \le \eta + \int_a^t |Q(s)|\,ds \le \frac{2\sigma}{3} < \sigma$$

as required.

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On the other hand, the set of Hermes solutions of (1) coincides with the set of its Krasowski solutions (see again [18]). Hence, it follows that every perturbed Euler solution is a Krasowski solution: in [6] it is claimed that also the converse is true.

7 Comparison between S-solutions and FE-solutions

It is not difficult to give examples of FE-solutions which are not Sentis solutions (see again, for instance, Example 5 below). The main contribution of this section is the following example. It shows that there may exist Sentis solutions which are not FE-solutions, as well.

Example 2 We construct a two-dimensional vector field $f(x,y)$, as illustrated by Figure 3. The construction starts by taking on the positive x-axis the points $(2^{-n}, 0)$, with $n \in \mathbf{Z}$ (but only the case $n > 0$ is really of interest). Each interval $[2^{-n}, 2^{-n+1}]$ is thought of as divided in three parts by the points $2^{-n}(4/3)$, $2^{-n}(5/3)$. Let us draw the lines joining the points

$$\begin{cases} (2^{-n}(5/3), 2^{-n}(5/3)) \text{ and } (2^{-n+1}(4/3), -2^{-n+1}(4/3)) & \text{if } n \text{ is even} \\ (2^{-n}(5/3), -2^{-n}(5/3)) \text{ and } (2^{-n+1}(4/3), 2^{-n+1}(4/3)) & \text{if } n \text{ is odd.} \end{cases}$$

Note that the absolute value of the slopes of these lines is constant and, to be precise, equal to 13/3. Finally consider for $x \geq 0$ the lines $y = \pm x$. The intersections of all these lines define the vertices of two sequences of triangles (one sequence above the line $y = x$, the other below the line $y = -x$) both "converging" toward the origin. Let us define

$$f(x, y) = \begin{cases} \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{inside the triangles above the line } y = x \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{inside the triangles below the line } y = -x \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{elsewhere.} \end{cases}$$

We claim that the origin is an equilibrium point in the sense of Sentis. To prove it, let us fix $\sigma > 0$, and let $n_0 \in \mathbf{N}$ be such that $n_0 > -\log_2(3\sigma/8)$. We construct a discontinuous piecewice affine approximation in the following way. Starting from the origin, we first move to right, until the point $(2^{-n_0}(4/3), 0)$ is reached. Without loss of generality, we may assume that n_0 is even. Then we jump to the point $(2^{-n_0}(4/3), 2^{-n_0}(4/3))$. Next we move downward, and stop at the point $(2^{-n_0}(4/3), -2^{-n_0-1}(4/3))$. Then we make another jump to the point $(2^{-n_0-1}(4/3), -2^{-n_0-1}(4/3))$. From this point, let us move upward and stop at $(2^{-n_0-1}(4/3), 2^{-n_0-2}(4/3))$. We continue in this way. We obtain a piecewice affine curve on every interval $[0, T]$ whose image is contained in the square $[-\sigma, \sigma] \times [-\sigma, \sigma]$. It is not difficult to estimate that the maximal length of time subintervals is less than $2^{-n_0}(8/3)$. In a similar way, we see that the sum of the distances covered by the jumps is less than the same quantity $2^{-n_0}(8/3)$. The conclusion is an easy consequence of these remarks.

If we would try to prove that the origin is an FE-solution, we should repeat the previous construction, by substituting any discontinuous piecewise affine approximation by a continuous one. But this is impossible. Indeed, in order to reach a triangle of the upper half plane from a triangle of the lower half plane (and vice versa) we should apply an outer perturbation q with $|q| = \cos(\operatorname{arctg}(13/3))$.

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8 Comparison between Carathéodory and FE-solutions

In the next section we will see a simple example of an FE-solution which is not a Carathéodory one (Example 4). Here, we deal with the opposite case.

Example 3 We define a planar vector field in the following manner (see Figure 4).

- for $y < 0$ we set $f(x, y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$;
- for $y = 0$ and $x \leq 0$, we set $f(x, 0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$;
- for $y > 0$, $x \leq 0$, we set $f(x, y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;
- for $y \geq 0$, $n \in \mathbf{N}$, we set $f(\frac{1}{2^n}, y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$;
- for $y \geq 0$, $x > 0$, $x \neq \frac{1}{2^n}$ $(n \in \mathbf{N})$, we set $f(x, y) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Next we define a curve $\varphi(t)$ for $t \in [0, 3]$. consider the two sequences of points $s_0 = 0, s_1 = 1, \ldots, s_{n+1} = 2 - \frac{1}{2^n}, \ldots$ and $\sigma_n = \frac{s_{n+1} + s_n}{2}$. Note that $\lim s_n = \lim \sigma_n = 2$. Then we set

$$\varphi(t) = \begin{cases} \begin{pmatrix} -t+2-\dfrac{1}{2^n} \\ t-2+\dfrac{1}{2^{n-1}} \end{pmatrix} & t \in [s_n, \sigma_n], \quad n = 0, 1, \dots \\ \begin{pmatrix} \dfrac{1}{2^{n+1}} \\ -t+2-\dfrac{1}{2^n} \end{pmatrix} & t \in [\sigma_n, s_{n+1}], \quad n = 0, 1, \dots \\ \begin{pmatrix} -t+2 \\ 0 \end{pmatrix} & t \in [2, 3] \ . \end{cases} \tag{11}$$

The vector field $f(x, y)$ is bounded and measurable. The function $\varphi(t)$ is Lipschitz continuous and such that $\dot{\varphi}(t) = f(\varphi(t))$ a.e. $t \in [0, 3]$, so that it is a Carathéodory solution. The restriction of $\varphi(t)$ on the interval $[0, 2]$ is clearly an FE-solution, as well as the restriction of $\varphi(t)$ on $[2, 3]$, but the entire trajectory $\varphi(t)$ on $[0, 3]$ is not. Indeed, assume by contradiction that there is a continuous polygonal approximation $\psi(t)$ such that

$$|\varphi(t) - \psi(t)| < \sigma \ , \quad t \in [0, 3]$$

for sufficiently small $\sigma > 0$. Then, there must exist a subinterval $[\tau', \tau''] \subset [0, 3]$ such that $\psi(t)$ is affine on $[\tau', \tau'']$ and, in addition,

$$\psi(\tau') \in \{(x, y) : x > 0, y > -\sigma\} \ , \quad \psi(\tau'') \in \{(x, y) : x \le 0, \ y = 0\} \ .$$

Taking into account the values of the vector field available for $x > 0$, we see that this is impossible, if only small outer perturbations are allowed.

By the way, this example shows that in general, FE-solutions cannot be "cut and pasted" together.

∎

9 Other relationships and examples

So far, we have recalled several notions of solution for differential equations with a discontinuous right hand side and we have compared some of them. In this section we go on and provide new examples and results.

Example 4 Consider the one dimensional differential equation with

$$f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ -1 & \text{if } x > 0 \ . \end{cases}$$

Clearly, there are no Carathéodory solutions such that $\varphi(0) = 0$. However, the function $\varphi(t) \equiv 0$ is a Sentis-solution and also an FE-solution.

∎

Example 5 Consider the one dimensional differential equation with

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \ . \end{cases}$$

Now, the function $\varphi(t) \equiv 0$ is a Newton solution and also a Filippov solution, but it is not a solution in the sense of Sentis.

∎

Example 6 Consider the following modification of Example 5

$$f(x) = \begin{cases} -1 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \ . \end{cases}$$

The function $\varphi(t) \equiv 0$ remains a Filippov solution, but it is no more a Carathéodory solution. Moreover, it is neither a solution in the sense of Sentis nor an FE-solution

∎

Example 7 Consider the one-dimensional differential equation with

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} .$$

In this case, the function $\varphi(t) \equiv 0$ is a Newton solution but now, it is not a Filippov solution.

∎

Example 8 Consider the one-dimensional differential equation with

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ -1 & \text{otherwise} \end{cases} .$$

The function $\varphi(t) \equiv t$ is an FE-solution, but it is neither a Carathéodory solution nor a Filippov one.

∎

Example 9 Starting from Example 1 we now construct a a two-dimensional differential system. We define a vector field $f(x, y)$ in such a way that $f(x, y)$ vanishes if $y \neq 0$, while $f(x, 0)$ coincides with the function $f(x)$ of Example 1. We see that the function

$$\varphi(t) = \begin{cases} \begin{pmatrix} t \\ 0 \end{pmatrix} & \text{if } t \leq 0 \\ \begin{pmatrix} 2t \\ 0 \end{pmatrix} & \text{if } t > 0 \end{cases} \tag{12}$$

is a Carathéodory solution and also an FE-solution. It is neither a Newton solution nor a Filippov one.

∎

Example 10 We can further modify Example 9, in order to render the function (12) a Filippov solution, but not a solution in the sense of Sentis. Take

$$f(x, y) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } y = 0,\ x \leq 0 \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \text{if } y = 0,\ x > 0 \\ \begin{pmatrix} \sqrt{2} \\ (\operatorname{sgn} y)\sqrt{2} \end{pmatrix} & \text{if } y \neq 0,\ x \leq 0 \\ \begin{pmatrix} 2\sqrt{2} \\ (\operatorname{sgn} y)2\sqrt{2} \end{pmatrix} & \text{if } y \neq 0,\ x > 0 \end{cases} .$$

∎

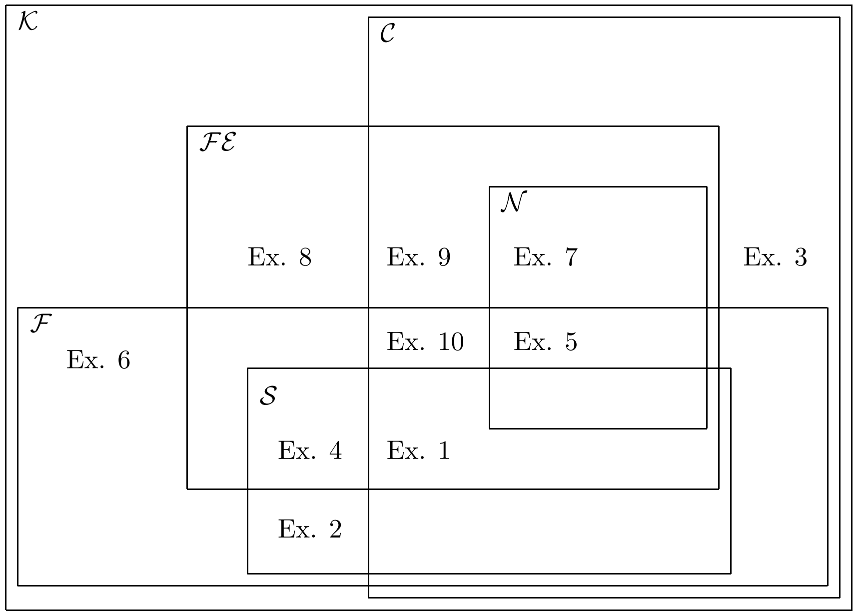
Example 11 We can finally construct a two-dimensional example starting from Example 8. Take

$$f(x, y) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } y = 0,\ x \in \mathbf{Q} \\ \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \text{if } y = 0,\ x \notin \mathbf{Q} \\ \begin{pmatrix} \sqrt{2} \\ (\operatorname{sgn} y)\sqrt{2} \end{pmatrix} & \text{if } y \neq 0 \end{cases} .$$

The function $\varphi(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$ is now a Filippov solution and an FE-solution. It is neither a Carathéodory, nor a Sentis solution.

∎

The information provided so far can be visualized in the following map.



We do not insist further on this boring list of counterexamples. We limit ourselves to point out that by an easy, technical modification of Example 3, we can render the function (11) a Filippov (not Sentis, not FE) solution or also, by a further modification, a Sentis (not FE) one. Inspired by Example 2, we can finally construct an example with a Krasowski (not Filippov, not Carathéodory, not FE) solution. Examples in $\mathcal{S} \cap \mathcal{N}$ are obvious.

10 Switched vector fields and switched solutions

Sometimes, dynamics of nonlinear systems are conveniently described by means of trajectories which are continuous and exhibit a piecewise structure. For instance, trajectories obtained by piecing together solutions of smooth vector fields of a given family were the basic tool in Geometric Control Theory, one of the major developments in nonlinear systems theory of the last two decades of the past century. A second example is given by the so called switched systems theory, which is gathering great attention in the recent engineering literature. This type of trajectories are easy to identify in an open loop approach: they correspond to the assumption that only piecewise constant inputs are admissible. The task seems more involved when the switching policy is determined by the position in the state space. For instance, if the system acts under a closed loop connection and the feedback is discontinuous, it is not clear how to select solutions which correspond to the intuitive idea of "switching" and how to prove that they exist.

In this section we propose some definitions and examples. However, we point out that switching policies exploit in general hybrid techniques ([3], [13], [21]) which may be hard to reproduce as a pure state space feedback.

We consider again differential equations of the form (1), but we assume now a particular structure for the right hand side.

Definition 3 *Let us be given a finite*[2] *family of* C^1 *vector fields* $\{f_i(x)\}_{i=1,\dots,N}$ *of* $\mathbf{R}^n$*. We say that a function* $f : \mathbf{R}^n \to \mathbf{R}^n$ *is a switched vector field (in short, SVF) if it is measurable, locally bounded, and there exists a function* $j(x) : \mathbf{R}^n \to \{1, \dots, N\}$ *such that*

$$\forall x \in \mathbf{R}^n, \quad f(x) = f_{j(x)}(x) \ .$$

To any SVF we can associate a partition of $\mathbf{R}^n$ in the following way. For each $i = 1, \dots, N$, let

[2]For the sake of simplicity, we prefer to work with a finite set of indices but this restriction can be easily dropped out.

$$\Omega_i = \{x \in \mathbf{R}^n : \ j(x) = i\} \ .$$

Clearly, the sets Ω_i are pairwise disjoint and $\cup_{i=1,\ldots,N}\Omega_i = \mathbf{R}^n$. Our purpose now is to identify a notion of solution in such a way to capture the intuitive idea of switching. Recall that a subset $E \subset \mathbf{R}$ is *locally finite* if for each closed interval $[a,b]$, the set $[a,b] \cap E$ is finite.

Definition 4 *Let f be a SVF, let I be an open interval and let $\varphi(t) : I \to \mathbf{R}^n$ be a continuous curve. We say that $T \in I$ is a* regular point *of φ with respect to f if $\exists i \in \{1,\ldots,N\}$, $\exists \varepsilon > 0$ such that for each $t \in (T-\varepsilon, T+\varepsilon)$*

(i) $f(\varphi(t)) = f_i(\varphi(t))$

(ii) $\dot\varphi(t) = f(\varphi(t))$.

Then, we say that φ is a switched solution *of f if the subset $E \subset I$ of points which are not regular is locally finite. Points $T \in E$ will be called* switching points.

Definition 4 is clearly equivalent to the following one.

Definition 5 *Let f be a SVF. The map $\varphi(t) : [a,b] \to \mathbf{R}^n$ is a switched solution of f if it is continuous and, in addition, there exist a partition $a = t_0 < t_1 < \ldots < t_K = b$ and a finite sequence of (possibly not distinct) indices $i_1,\ldots,i_K \in \{1,\ldots,N\}$ such that for each $k = 1,\ldots,K$:*

(i) $\varphi(t) \in \Omega_{i_k}$, for all $t \in (t_{k-1}, t_k)$

(ii) $\varphi(t)$ is of class C^1 on (t_{k-1}, t_k) and $\dot\varphi(t) = f_{i_k}(\varphi(t))$.

According with this definition, it is clear that each switched solution is a Carathéodory solution. It is also clear that there may be switched solutions which are not Filippov solutions.

Proposition 4 *Every switched solution on a bounded interval $[a,b]$ is a FE-solution.*

Proof Let $\varphi(t)$ be a switched solution on $[a,b]$, and let $a = t_0 < t_1 < \ldots < t_K = b$, $i_1,\ldots,i_K$ as required by Definition 5. Then, $\varphi(t)$ is a Newton solution of the C^1 vector field f_{i_k} on the interval $[t_{k-1}, t_k]$. The argument of Proposition 2 can be repeated. The partial polygonal approximations constructed on the intervals $[t_{k-1}, t_k]$ can be globally reviewed as a polygonal approximation on $[t_0, t_K]$. No problem arises at the points $t_1,\ldots,t_{k-1}$ since, according to the costruction of Proposition 2, the values of the polygonal approximations at these points are exactly $\varphi(t_1),\ldots,\varphi(t_{k-1})$.

∎

Example 12 Consider the family formed by the two planar vector fields

$$f_1(x,y) = \begin{pmatrix} -y \\ x+1 \end{pmatrix} , \qquad f_2(x,y) = \begin{pmatrix} -y \\ x-1 \end{pmatrix}$$

and let

$$f(x,y) = \begin{cases} f_1(x,y) & \text{if } x \le 0 \\ f_2(x,y) & \text{if } x > 0 \end{cases}$$

(see Figure 1). The curve

$$\begin{cases} x = 1 - \sin t \\ y = \cos t \end{cases}$$

with $t \in (0,\pi)$ is a switched solution, and $T = \pi/2$ is a switching (non regular) point. Note that the curve

$$\begin{cases} \begin{cases} x = 1 - \sin t \\ y = \cos t \end{cases} & \text{for } t \in (0, \pi/2) \\ \\ \begin{cases} x = -1 + \sin t \\ y = -\cos t \end{cases} & \text{for } t \in (\pi/2, \pi) \end{cases}$$

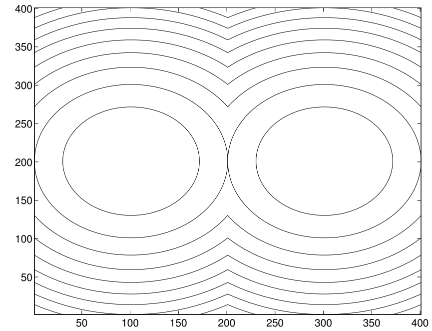


Figure 1: Example 12

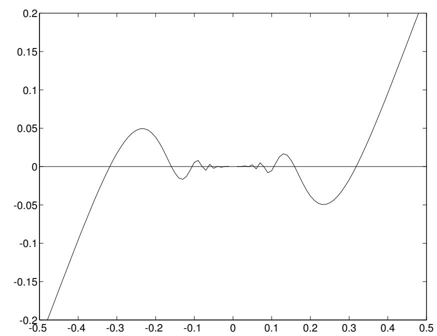


Figure 2: Example 14

is another switched solution which coincides with the previous one for $t \in (0, \pi/2)$. This confirm the obvious intuition that a SVF does not have the uniqueness of solution property.

Reviewed as a discontinuous vector field, this system has an equilibrium solution in the sense of Filippov (whose image is the origin) which is not a switched solution.

Example 13 The previous example can be modified by setting

$$f(x,y) = \begin{cases} f_1(x,y) & \text{if } x < 0 \\ f_0(x,y) & \text{if } x = 0 \\ f_2(x,y) & \text{if } x > 0 \end{cases}$$

where $f_0(x,y)$ is the constant vector field $(0,1)$. Then, the curve $x = 0$, $y = t$ is a switched solution which is not a Filippov solution.

Example 14 Let S_+ and S_- be respectively the upper and lower part of the plane separated by the graph of the function $y = g(x) = x^r \sin \frac{1}{x}$ $(g(0) = 0)$, see Figure 2. Note that this graph is a manifold which can be taken as regular as desired, by increasing r. Let

$$f(x,y) = \begin{cases} (1,0) & \text{if } (x,y) \in \overline{S_-} \\ (2,0) & \text{if } (x,y) \in S_+ \end{cases} .$$

There is a Carathéodory solution which starts from $(-1,0)$ and ends in $(1,0)$ whose image is contained in the x_1 axis (and whose velocity is greater than 1 a.e.). It is not a switched solution, since the switching points accumulate at the origin.

11 Patchy vector fields

The definition of SVF can be strengthened. According to [1], [6], we say that a pair (Ω, g) is a *patch* if:

(1) Ω is an open, connected subset of $\mathbf{R}^n$ with a smooth boundary

(2) g is a smooth vector field defined on an open neighborhood of $\overline{\Omega}$ such that for all $x \in \partial\Omega$

$$g(x) \cdot \mathbf{n}(x) < 0$$

where $\mathbf{n}(x)$ is the outer normal of Ω at x.

Definition 6 *We say that $f : \mathbf{R}^n \to \mathbf{R}^n$ is a* patchy vector field *if there exists a finite*[3] *number of patches (Ω_i, f_i) $(i = 1, \ldots, N)$ such that $\cup\Omega_i = \mathbf{R}^n$ and*

$$f(x) = f_i(x)$$

for each $x \in \Omega_i \setminus (\Omega_{i+1} \cup \ldots \cup \Omega_N)$.

In [6], A. Bressan collects a number of interesting properties of patchy vector fields. In particular, it turns out that for each initial state there exists a switched solution. Moreover, the set of Carathèodory solutions, the set of switched solutions and the set of the FE-solutions, all coincide.

12 Appendix: a limiting representation formula

Throughout this section, N denotes any subset of $\mathbf{R}^n$ with zero measure and N_0 denotes the set of points where f is not approximately continuous (see [14], for the definition of approximate continuity). Moreover, let

$$L_N(x) = \{v : \ \exists\{x_i\} \text{ with } x_i \to x \text{ such that } x_i \notin N \text{ and } v = \lim_i f(x_i)\}$$

and

$$L(x) = \{v : \ \forall N \text{ with } \mu(N) = 0 \ \ \exists\{x_i\} \text{ with } x_i \to x \text{ such that } x_i \notin N \text{ and } v = \lim_i f(x_i)\} \ .$$

It is not difficult to check that $L(x) = \bigcap_{\mu(N)=0} L_N(x)$. In fact, from next Proposition 5 it will be clear that $L(x) = L_{N_0}(x)$.

The construction of the set valued map $F_F(x)$ can be better understood if one observes that

$$F_F(x) = \bigcap_{\delta>0} \overline{\text{co}}\, \{f(\mathcal{B}(x, \delta) \backslash N_0)\} \tag{13}$$

(see [17]). Based on this remark, the following limiting representation formula is given in [22]:

$$F_F(x) = \text{co}\, L_{N_0}(x) \ . \tag{14}$$

Our aim is to determine similar formulas for $F_S(x)$.

Proposition 5 *For each $x \in \mathbf{R}^n$, $F_S(x) = \bigcap_{\delta>0} \overline{f(\mathcal{B}(x, \delta) \backslash N_0)} = L_{N_0} = L(x)$.*

Proof We proceed according to the following scheme:

$$F_S(x) \subseteq \bigcap_{\delta>0} \overline{f(\mathcal{B}(x, \delta) \backslash N_0)} \subseteq L_{N_0} \subseteq L(x) \subseteq F_S(x) \ .$$

Let $v \in F_S(x)$. Then $v \in \overline{f(\mathcal{B}(x, \delta) \backslash N)}$ for each $\delta > 0$ and each N. In particular we have $v \in \overline{f(\mathcal{B}(x, \delta) \backslash N_0)}$ for each $\delta > 0$. This immediately implies the first inclusion.

[3] Even in this case the finiteness assumption is not essential.

Now let $v \in \bigcap_{\delta>0} \overline{f(\mathcal{B}(x,\delta)\backslash N_0)}$. Then $v \in \overline{f(\mathcal{B}(x,\frac{1}{i})\backslash N_0)}$ for each positive integer i. This means that we can find y_i such that $|v - y_i| < \frac{1}{i}$ and

$$y_i \in f\left(\mathcal{B}\left(x, \frac{1}{i}\right)\backslash N_0\right) \ . \tag{15}$$

From (15) it follows the existence of some $x_i \in \mathcal{B}(x,\frac{1}{i})\backslash N_0$ such that $y_i = f(x_i)$. Clearly, $\lim_i x_i = x$ (with $x_i \notin N_0$) and $\lim_i f(x_i) = v$. By definition, $v \in L_{N_0}(x)$.

Next, we prove that $L_{N_0}(x) \subseteq L(x)$. Let $v \in L_{N_0}(x)$, and let $\{x_i\}$ be a sequence such that $x_i \to x$, $x_i \notin N_0$, $v = \lim_i f(x_i)$. Such a sequence exists by the definition of $L_{N_0}(x)$. Take any set $N \subset \mathbf{R}^n$ with zero measure. For each positive integer k, there exists an index $i(k)$ such that

$$|x_{i(k)} - x| < \frac{1}{k} \quad \text{and} \quad |f(x_{i(k)}) - v| < \frac{1}{k} \ .$$

Since f is approximately continuous at any $x_{i(k)}$, we can find points $\tilde{x}_k \notin N$ in such a way that

$$|x_{i(k)} - \tilde{x}_k| < \frac{1}{k} \quad \text{and} \quad |f(x_{i(k)}) - f(\tilde{x}_k)| < \frac{1}{k} \ .$$

We have

$$|\tilde{x}_k - x| \le |x_{i(k)} - \tilde{x}_k| + |x_{i(k)} - x| \le \frac{2}{k}$$

and

$$|f(\tilde{x}_k) - v| \le |f(x_{i(k)}) - f(\tilde{x}_k)| + |f(x_{i(k)}) - v| < \frac{2}{k} \ .$$

Hence, $\tilde{x}_k \to x$ and $f(\tilde{x}_k) \to v$. Since $\tilde{x}_k \notin N$, we conclude that $v \in L(x)$.

Finally, let $v \in L(x)$ and fix $\delta > 0$. By definition, for each N with $\mu(N) = 0$, there exists a sequence $\{x_i\}$ with $x_i \notin N$ such that $x_i \to x$ and $f(x_i) \to v$. For a sufficiently large index i, we must have $x_i \in \mathcal{B}(x,\delta)$. This implies $v \in \overline{f(\mathcal{B}(x,\delta)\backslash N)}$. The choice of δ and N being arbitrary, we obtain that $v \in F_S(x)$, as required.

∎

Proposition 5 and formula (14) enable us to point out the relationship between F_F and F_S.

Proposition 6 *For each $x \in \mathbf{R}^n$, $F_F(x) = \mathrm{co}\, F_S(x)$.*

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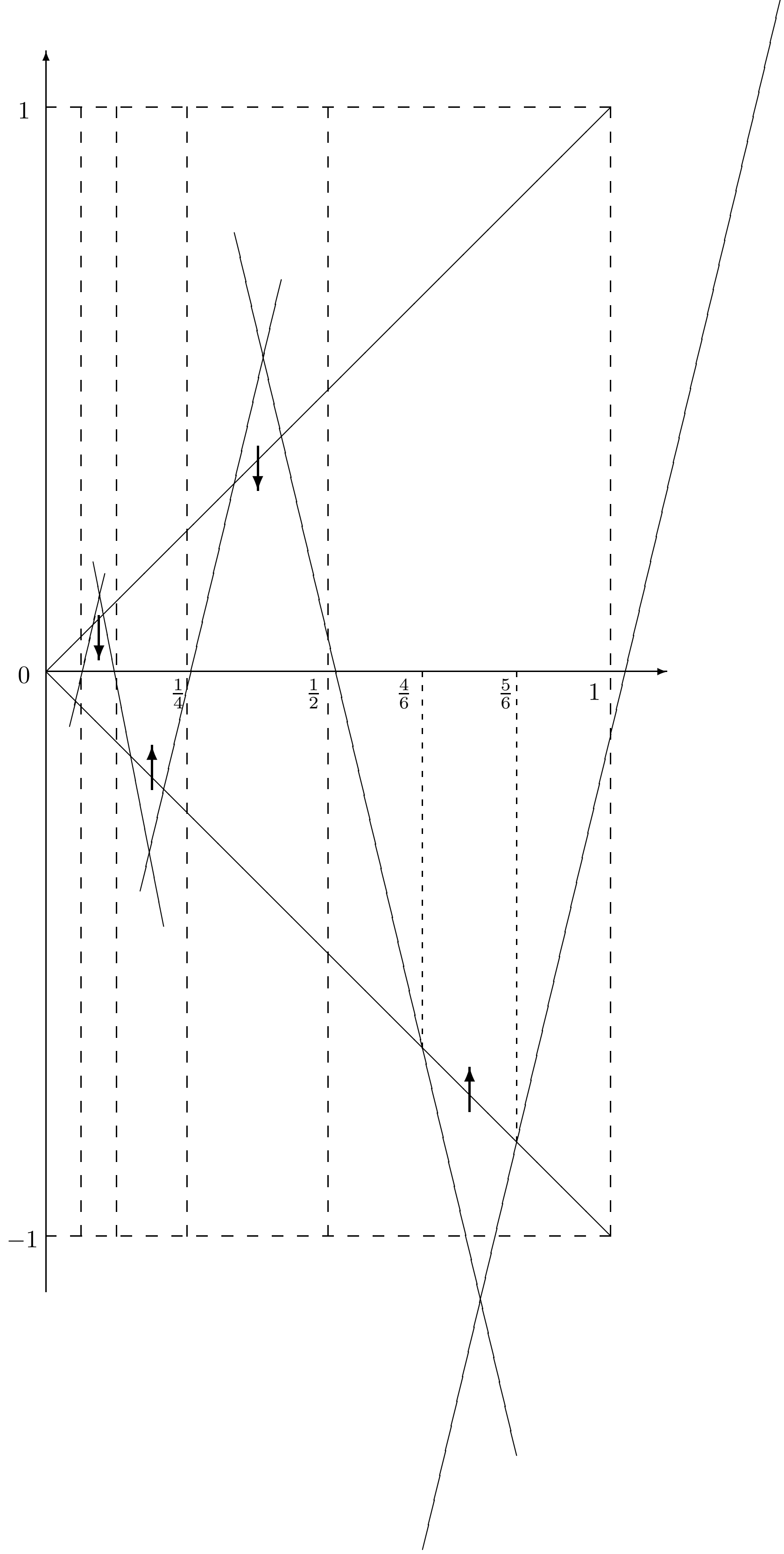

Figure 3: Example 2

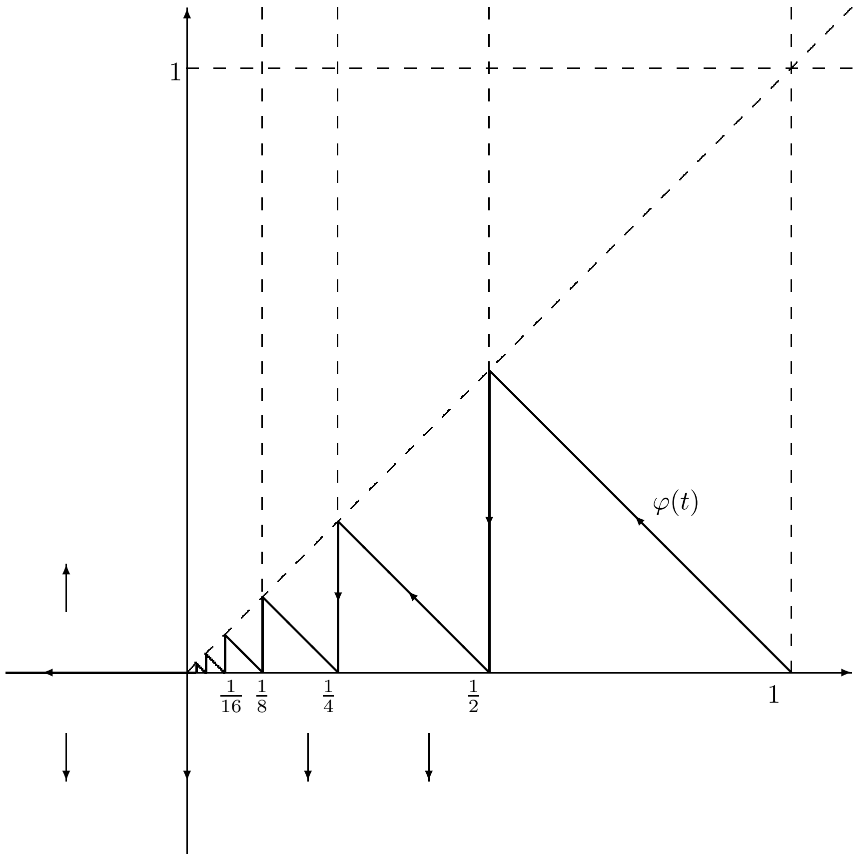

Figure 4: Example 3