External Stability and Continuous Liapunov Functions

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Abstract. It is well known that external stability of nonlinear input systems can be investigated by means of a suitable extension of the Liapunov functions method. We prove that a complete characterization by means of continuous Liapunov functions is actually possible, provided that the definition of external stability is appropriately strengthened.

1 Introduction

A finite dimensional autonomous nonlinear system

$$\dot{x} = f(x, u) \ , \qquad x \in \mathbf{R}^n, \ u \in \mathbf{R}^m \tag{1}$$

is said to be *bounded input bounded state stable* (in short, BIBS stable) if for each initial state and each bounded input $u(t) : [0, +\infty) \to \mathbf{R}^m$ the corresponding solution is bounded for $t \geq 0$ (see [1] for a formal definition and comments). In the recent paper [3], uniform BIBS stability has been characterized by means of certain upper semi-continuous Liapunov functions. In fact, it is known that continuous Liapunov functions may not to exist for BIBS stable systems of the form (1).

The situation is exactly the same as in the theory of stability for equilibrium positions of systems without inputs (see [2], [4]). In this note we prove that the analogy can be further pursued. We extend to systems with inputs the theory developed in [2]. We show in particular that the existence of continuous Liapunov functions with suitable properties is equivalent to a type of external stability which is more restrictive than uniform BIBS stability.

In the next section we recall the basic notions (prolongations and prolongational sets associated to a dynamical system). Then we show how they generalize to systems with inputs. In Section 3 we introduce the definition of absolute bounded input bounded state stability (our strengthened form of external stability) and state the main result. The last section contains the proof.

2 Prerequisites

As already mentioned, for a locally stable equilibrium of a system without inputs

$$\dot{x} = f(x) \ , \qquad f \in C^1 \tag{2}$$

not even the existence of a continuous Liapunov function can be given for sure. In 1964, Auslander and Seibert ([2]) discovered that the existence of a continuous generalized Liapunov function is actually equivalent to a stronger form of stability. In order to illustrate the idea, it is convenient to begin with some intuitive considerations. Roughly speaking, stability is a way to describe the behavior of the system in presence of small perturbations of the initial state. More generally, let us assume that perturbations are allowed also at arbitrary positive times: under the effect of such perturbations, the system may jump from the present trajectory to a nearby one. Now, it may happens that an unfortunate superposition of these jumps results in an unstable behavior even if the system is stable and the amplitude of the perturbations tends to zero.

This phenomenon is technically described by the notion of *prolongation*, due to T. Ura and deeply studied in [2]. The existence of a continuous Liapunov function actually prevents the unstable behavior of the prolongational sets. On the other hand, the possibility of taking under control the growth of the prolongational sets leads to the desired strengthened notion of stability.

We proceed now formally to precise what we means for prolongation. First of all, we recall that from a topological point of view, very useful tools for stability analysis are provided by certain sets associated to the given system. These sets depend in general on the initial state. Thus, they can be reviewed as set valued maps. The simplest examples are the *positive trajectory* issuing from a point x_0

$$\Gamma^+(x_0) = \{y \in \mathbf{R}^n : \ y = x(t; x_0) \text{ for some } t \geq 0\} \tag{3}$$

where $x(\cdot; x_0)$ represents the solution of (2) such that $x(0; x_0) = x_0$, and the *positive limit set*.

We adopt the following agreements about notation. The open ball of center x_0 and radius $r > 0$ is denoted by $B(x_0, r)$. If $x_0 = 0$, we simply write B_r instead of $B(0, r)$. For $M \subset \mathbf{R}^n$, we denote $|M| = \sup_{x \in M} |x|$. Let $Q(x)$ be a set valued map from $\mathbf{R}^n$ to $\mathbf{R}^n$. For $M \subseteq \mathbf{R}^n$, we denote $Q(M) = \cup_{x \in M} Q(x)$. Powers of Q will be defined iteratively:

$$Q^0(x) = Q(x) \quad \text{and} \quad Q^k(x) = Q(Q^{k-1}(x))$$

for $k = 1, 2, \ldots$. Next, we introduce two operators, denoted by $\mathcal{D}$ and $\mathcal{I}$, acting on set valued maps. They are defined according to

$$(\mathcal{D}Q)(x) = \cap_{\delta > 0} \overline{Q(B(x, \delta))}$$

$$(\mathcal{I}Q)(x) = \cup_{k=0,1,2,\ldots} Q^k(x) \ .$$

The following characterizations are straightforward.

Proposition 1 *a) $y \in (\mathcal{D}Q)(x)$ if and only if there exist sequences $x_k \to x$ and $y_k \to y$ such that $y_k \in Q(x_k)$ for each $k = 1, 2, \ldots$.*

b) $y \in (\mathcal{I}Q)(x)$ if and only if there exist a finite sequence of points $x_0, \ldots, x_K$ such that $x_0 = x$, $y = x_K$ and $x_k \in Q(x_{k-1})$ for $k = 1, 2, \ldots, K$.

The operators $\mathcal{D}$ and $\mathcal{I}$ are idempotent. Moreover, for every set valued map Q, $\mathcal{D}Q$ has a closed graph, so that for every x the set $(\mathcal{D}Q)(x)$ is closed. However, $(\mathcal{I}Q)(x)$ is not closed in general, not even if $Q(x)$ is closed for each x.

When $\mathcal{I}Q = Q$ we say that Q is *transitive*. The positive trajectory is an example of a transitive map. In general, $\mathcal{D}Q$ is not transitive, not even if Q is transitive. In conclusion, we see that the construction

$$(\mathcal{D}(\ldots(\mathcal{I}(\mathcal{D}(\mathcal{I}(\mathcal{D}Q))))\ldots))(x) \tag{4}$$

gives rise in general to larger and larger sets.

Definition 1 *A* prolongation *associated to system (2) is a set valued map $Q(x)$ which fulfils the following properties:*

(i) for each $x \in \mathbf{R}^n$, $\Gamma^+(x) \subseteq Q(x)$

(ii) $(\mathcal{D}Q)(x) = Q(x)$

(iii) If K is a compact subset of $\mathbf{R}^n$ and $x \in K$, then either $Q(x) \subset K$, or $Q(x) \cap \partial K \neq \emptyset$.

If Q is a prolongation and it is transitive, it is called a *transitive prolongation*. The following proposition will be used later (see [2]).

Proposition 2 *Let K be a compact subset of $\mathbf{R}^n$ and let Q be a transitive prolongation. Then $Q(K) = K$ if and only if K possesses a fundamental system of compact neighborhoods $\{K_i\}$ such that $Q(K_i) = K_i$.*

Starting from the map Γ^+ and using repeatedly the operators $\mathcal{D}$ and $\mathcal{I}$, we can construct several prolongational sets associated to (2). For instance, it is not difficult to see that

$$D_1(x) := (\mathcal{D}\Gamma^+)(x)$$

is a prolongation, the so called *first prolongation* of (2). The first prolongation characterizes stability. Indeed, it is possible to prove that an equilibrium x_0 of (1) is stable if and only if $D_1(x_0) = \{x_0\}$. The first prolongation in general is not transitive.

The intuitive construction (4) can be formalized by means of transfinite induction. This allows us to speak about higher order prolongations. More precisely, let α be an ordinal number and assume that the prolongation $D_\beta(x)$ of order β has been defined for each ordinal number $\beta < \alpha$. Then, we set

$$D_\alpha(x) = (\mathcal{D}(\cup_{\beta<\alpha}(\mathcal{I}D_\beta)))(x) \ .$$

The procedure saturates when $\alpha = \gamma$, the first uncountable ordinal number. Indeed, it is possible to prove that $\mathcal{I}D_\gamma = D_\gamma$, which obviously implies $D_\alpha(x) = D_\gamma(x)$ for each $\alpha \geq \gamma$.

Since, as already mentioned, (2) is stable at an equilibrium x_0 if and only if $D_1(x_0) = \{x_0\}$, it is natural to give the following definition.

Definition 2 *Let α be an ordinal number. The equilibrium x_0 is* stable of order α *(or α-stable) if $D_\alpha(x_0) = \{x_0\}$. The equilibrium x_0 is said to be* absolutely stable *when it is γ-stable.*

The main result in the Auslander and Seibert paper [2] is as follows.

Theorem 1 *The equilibrium x_0 is absolutely stable for system (2) if and only if there exists a generalized Liapunov function which is continuous in a whole neighborhood of the origin.*

3 Systems with input

The notion of prolongation applies also to systems with inputs ([5]). Let us adopt the following agreement: throughout this note an *admissible input* is any piecewise constant function $u(\cdot) : [0, +\infty) \to U$, where U is a preassigned constraint set of $\mathbf{R}^m$. In other words, for each admissible input there are sequences $\{t_k\}$ and $\{u_k\}$ such that

$$0 = t_0 < t_1 < t_2 < \ldots < t_k \ldots$$

and $u(t) \equiv u_k \in U$ for $t \in [t_{k-1}, t_k)$. Assume that for each $u \in U$, the vector field $f(\cdot, u)$ is of class C^1. A *solution* of (1) corresponding to an admissible input $u(\cdot)$ and an initial state x_0 is a continuous curve $x(\cdot; x_0, u(\cdot))$ such that $x(0; x_0, u(\cdot)) = x_0$ and coinciding with an integral curve of the vector field $f(\cdot, u_k)$ on the interval (t_{k-1}, t_k). The *reachable set* $A(x_0, U)$ relative to the system (1) and the constraint set U, is the set of all points lying on solutions corresponding to the initial state x_0 and any admissible input.

Reachable sets are the most natural candidate to play the role of the positive trajectories (3) in the case of systems with inputs. More precisely, let R be a positive real number, and let $U = \overline{B_R}$. We adopt the simplified notation $A^R(x_0) = A(x_0, \overline{B_R})$, and introduce the prolongations

$$D_1^R(x_0) = (\mathcal{D}A^R)(x_0) \ , \quad D_2^R(x_0) = (\mathcal{D}(\mathcal{I}(\mathcal{D}A^R)))(x_0) \ \text{ and so on.}$$

Definition 3 *We say that the system (1) is* absolutely bounded input bounded state *stable (in short ABIBS stable) if for each* $R > 0$*, there exists* $S > 0$ *such that*

$$|x_0| \le R \ , \implies |D_\gamma^R(x_0)| \le S \ ,$$

$\forall t \ge 0$.

The following characterization is easy. The proof is omitted.

Proposition 3 *System (1) is ABIBS-stable if and only if for each* $R > 0$ *there exists a compact set* $K \subset \mathbf{R}^n$ *such that* $B_R \subset K$ *and* $D_\gamma^R(K) = K$.

Definition 4 *A (generalized)* ABIBS-Liapunov function *for (1) is an everywhere continuous, radially unbounded function* $V : \mathbf{R}^n \to \mathbf{R}$ *which enjoys the following monotonicity property:*

(**MP**) *for all* $R > 0$*, there exists* $\rho > 0$ *such that for each admissible input* $u(\cdot) : [0, +\infty) \to \overline{B_R}$ *and each solution* $x(\cdot)$ *of (1) defined on an interval* I *and corresponding to* $u(\cdot)$*, one has that the composite map* $t \mapsto V(x(t))$ *is non-increasing on* I*, provided that* $|x(t)| \ge \rho$ *for each* $t \in I$.

We are now ready to state our main result.

Theorem 2 *System (1) is ABIBS-stable if and only if there exists an ABIBS-Liapunov function.*

The proof of Theorem 2 is given in the following section. We conclude by the remark that in general an ABIBS stable system does not admit ABIBS-Liapunov functions of class C^1. As an example, consider a system of the form (2) for which there exists a continuous function $V(x)$ which is radially unbounded and non-increasing along solutions, but not a C^1 function with the same properties. It is proved in [4] that such systems exist, even with $f \in C^\infty$. Of course, $f(x)$ can be thought of as a function of x and u, constant with respect to u. Any $V(x)$ which is radially unbounded and non-increasing along solutions, can be reinterpreted as an ABIBS Liapunov function.

4 The proof

Sufficient part

Assume that there exists a function $V(x)$ with the required properties. In what follows, we adopt the notation

$$W_\lambda = \{x \in \mathbf{R}^n : V(x) \le \lambda\} \ .$$

Fix $R_0 = 1$. According to (**MP**) we can associate to R_0 a number ρ_0. In fact, without loss of generality we can take $\rho_0 > R_0$. Let $m_0 = \max_{|y| \le \rho_0} V(y)$, and pick any $\lambda > m_0$. We note that

$$|x| \leq \rho_0 \Longrightarrow V(x) \leq m_0 \Longrightarrow x \in W_\lambda$$

that is, $B_{\rho_0} \subset W_\lambda$. In fact, there exist some $\eta > 0$ such that $B_{\rho_0+\eta} \subset W_\lambda$.

Lemma 1 *For each $\lambda > m_0$, we have $D_\gamma^{R_0}(W_\lambda) = W_\lambda$.*

Proof Of course, it is sufficient to prove that $D_\gamma^{R_0}(W_\lambda) \subseteq W_\lambda$.

Step 1. *For each $\lambda > m_0$ we have $A^{R_0}(W_\lambda) \subseteq W_\lambda$.*

Indeed, in the opposite case we could find $\bar\lambda > m_0$, $\bar x \in W_{\bar\lambda}$, $\bar y \notin W_{\bar\lambda}$, an admissible input $u(\cdot)$ with values in B_{R_0}, and a positive time T such $x(T; x_0, u(\cdot)) = \bar y$. Set for simplicity $x(t) = x(t; x_0, u(\cdot))$. Let $\tau \in (0,T)$ such that $x(\tau) \in W_{\bar\lambda}$, while $x(t) \notin W_{\bar\lambda}$ for $t \in (\tau, T]$. Such a τ exists since the solutions are continuous. By construction, $V(x(\tau)) = \bar\lambda < V(\bar y)$. On the other hand, $|x(t)| \geq \rho_0$ on the interval $[\tau, T]$, so that $V(x(t))$ is non-increasing on this interval. A contradiction.

Step 2. *For each $\lambda > m_0$ we have $(\mathcal{D}A^{R_0})(W_\lambda) \subseteq W_\lambda$.*

Even in this case we proceed by contradiction. Assume that it is possible to find $\bar\lambda > m_0$, $\bar x \in W_{\bar\lambda}$, and $\bar y \in (\mathcal{D}A^{R_0})(W_{\bar\lambda})$ but $\bar y \notin W_{\bar\lambda}$. This means $V(\bar x) \leq \bar\lambda < V(\bar y)$. Let $\varepsilon > 0$ be such that $\bar\lambda + 3\varepsilon \leq V(\bar y)$. Since V is continuous, there exists $\delta > 0$ such that

$$V(x) \leq \bar\lambda + \varepsilon < \bar\lambda + 2\varepsilon < V(y)$$

for all $x \in B(\bar x, \delta)$ and $y \in B(\bar y, \delta)$. By the definition of the operator $\mathcal{D}$, we can now take $\tilde x \in B(\bar x, \delta)$ and $\tilde y \in B(\bar y, \delta)$ in such a way that $\tilde y \in A^{R_0}(\tilde x)$. This is a contradiction to Step 1: indeed, since $\tilde x \in W_{\bar\lambda+\varepsilon}$, we should have $\tilde y \in W_{\bar\lambda+\varepsilon}$, as well. On the contrary, the fact that $\bar\lambda + 2\varepsilon < V(\tilde y)$ implies $\tilde y \notin W_{\bar\lambda+\varepsilon}$.

Thus, we have shown that $D_1^{R_0}(W_\lambda) = W_\lambda$ for each $\lambda > m_0$. To end the proof, we need to make use of transfinite induction. Let α be an ordinal number, and assume that the statement

$$D_\beta^{R_0}(W_\lambda) = W_\lambda \quad \textit{for each} \quad \lambda > m_0$$

holds for every ordinal number $\beta < \alpha$. It is not difficult to infer that also

$$(\mathcal{I}(D_\beta^{R_0}))(W_\lambda) = W_\lambda \quad \textit{for each} \quad \lambda > m_0$$

and, hence,

$$\cup_{\beta<\alpha}(\mathcal{I}(D_\beta^{R_0}))(W_\lambda) = W_\lambda \quad \textit{for each} \quad \lambda > m_0 \ . \tag{5}$$

For sake of convenience, let us set $E_\alpha^{R_0} = \cup_{\beta<\alpha}(\mathcal{I}(D_\beta^{R_0}))$. The final step is to prove that

$$(\mathcal{D}E_\alpha^{R_0})(W_\lambda) \subseteq W_\lambda \quad \textit{for each} \quad \lambda > m_0 \ .$$

Assume that there are $\bar\lambda > m_0$, $\bar x \in W_{\bar\lambda}$, and $\bar y \in (\mathcal{D}E_\alpha^{R_0})(W_{\bar\lambda})$ but $\bar y \notin W_{\bar\lambda}$. As before, we have $V(\bar x) \le \bar\lambda < V(\bar y)$ and, by continuity, for sufficiently small ε we can find δ such that

$$V(x) \le \bar\lambda + \varepsilon < \bar\lambda + 2\varepsilon < V(y)$$

for all $x \in B(\bar x, \delta)$ and $y \in B(\bar y, \delta)$. Let us choose $\tilde x$ and $\tilde y$ satisfying this last conditions, and such that $\tilde y \in E_\alpha^{R_0}(\tilde x)$. This is possible because of the definition of $\mathcal{D}$. In conclusion, we have $\tilde x \in W_{\bar\lambda+\varepsilon}$, $\tilde y \notin W_{\bar\lambda+\varepsilon}$, and $\tilde y \in E_\alpha^{R_0}(\tilde x)$. A contradiction to (5). The proof of the lemma is complete.

We are finally able to prove the sufficient part of Theorem 2. Fix $\lambda_0 > m_0$. Note that W_{λ_0} is closed (since V is continuous) and bounded (since V is radially unbounded). Hence, $W_{\lambda_0} \subset B_{R_1}$ for some $R_1 > \rho_0 > R_0 = 1$. In addition, it is not restrictive to take $R_1 \ge 2$. Using the properties of V, we find $\rho_1 > R_1$ and define $m_1 = \max_{|x|\ge\rho_1} V(x) \ge m_0$.

By repeating the previous arguments, we conclude that $D_\gamma^{R_1}(W_\lambda) = W_\lambda$ for each $\lambda > m_1$.

Fix $\lambda_1 > m_1$, and iterate again the procedure. We arrive to define a sequence of compact sets $\{W_{\lambda_i}\}$ such that $B_{R_i} \subset W_{\lambda_i}$, with $R_i \to +\infty$, and $D_\gamma^{R_i}(W_{\lambda_i}) = W_{\lambda_i}$.

Let finally R be an arbitrary positive number, and let R_i be the smallest number of the sequence $\{R_i\}$ such that $R \le R_i$. Set $K = W_{\lambda_i}$. We clearly have

$$B_R \subset B_{R_i} \subseteq K \qquad \text{and} \qquad D_\gamma^R(K) \subseteq D_\gamma^{R_i}(K) = K \ .$$

The proof of the sufficient part is complete, by virtue of Proposition 3.

Necessary part

The idea is to construct a Liapunov function V by assigning its level sets for all numbers of the form

$$\frac{2^k}{j} \qquad j = 1, \ldots, 2^k \ , \quad k = 0, 1, 2, \ldots \tag{6}$$

namely, the reciprocals of the so called dyadic rationals. Note that they are dense in $[0, +\infty)$.

Let us start by setting $R_0 = 1$. According to Proposition 3, we can find a compact set denoted by W_{2^0} such that $B_{R_0} \subset W_{2^0}$ and $D_\gamma^{R_0}(W_{2^0}) = W_{2^0}$. Let $R_1 \ge \max\{2, |W_{2^0}|\}$. Using again Proposition 3, we find a compact set W_{2^1} such that $B_{R_1} \subset W_{2^1}$ and $D_\gamma^{R_1}(W_{2^1}) = W_{2^1}$. This procedure can be iterated. Assuming that W_{2^k} has been defined, we take $R_{k+1} \ge \max\{k +$

$2, |W_{2^k}|\}$ and the compact set $W_{2^{k+1}}$ in such a way that $B_{R_{k+1}} \subset W_{2^{k+1}}$ and $D_\gamma^{R_{k+1}}(W_{2^{k+1}}) = W_{2^{k+1}}$. The sequence $\{W_{2^k}\}$ satisfies the conditions

$$W_{2^k} \subset B_{R_{k+1}} \subset W_{2^{k+1}} \quad \text{and} \quad \cup_k W_{2^k} = \mathbf{R}^n \ .$$

We have so assigned a set to any dyadic reciprocals $\frac{2^k}{j}$ with $j = 1$, $k = 0, 1, 2, \ldots$. Next, consider pairs k, j such that $k \geq 1$ and $2^{k-1} \leq j \leq 2^k$, that is all dyadic reciprocals such that $2^0 \leq \frac{2^k}{j} \leq 2^1$.

By virtue of Proposition 2, there exists a compact neighborhood K of W_{2^0} such that K is properly contained in W_{2^1} and $D_\gamma^{R_0}(K) = K$. Call it $W_{4/3}$. Note that (beside the endpoints 1 and 2) 4/3 is the unique dyadic reciprocal with $k = 2$ included in the interval $[1, 2]$. Using again Proposition 2 applied to W_{2^0} and $W_{4/3}$ we define two new sets

$$W_{2^0} \subset W_{8/7} \subset W_{4/3} \subset W_{8/5} \subset W_{2^1}$$

such that $D_\gamma^{R_0}(W_{8/7}) = W_{8/7}$ and $D_\gamma^{R_0}(W_{8/5}) = W_{8/5}$. By repeating the procedure, we arrive to assign a compact set W_λ to any dyadic reciprocal $\lambda = \frac{2^k}{j}$ with $k \geq 1$ and $2^{k-1} \leq j \leq 2^k$, in such a way that $D_\gamma^{R_0}(W_\lambda) = W_\lambda$ and

$$W_{2^0} \subset W_\lambda \subset W_\mu \subset W_{2^1}$$

if $\lambda < \mu$. Then we turn our attention to dyadic reciprocals $\frac{2^k}{j}$ with $k \geq 2$ and $2^{k-2} \leq j \leq 2^{k-1}$, that is $2 \leq \frac{2^k}{j} \leq 4$. We proceed as above. This time, we obtain sets W_λ such that $D_\gamma^{R_1}(W_\lambda) = W_\lambda$ and

$$W_{2^1} \subset W_\lambda \subset W_\mu \subset W_{2^4}$$

if $\lambda < \mu$. This construction can be repeated for all k and j. We finally obtain an increasing family of compact sets $\{W_\lambda\}$ with the property that if $2^k \leq \lambda < 2^{k+1}$ then $D_\gamma^{R_k}(W_\lambda) = W_\lambda$.

We are now ready to define the Liapunov function $V(x)$ for all $x \in \mathbf{R}^n$ as

$$V(x) = \inf\{\lambda : x \in W_\lambda\} \ .$$

Claim A. *For each R there exists ρ such that if $|x| \geq \rho$ and $y \in A^R(x)$ then $V(y) \leq V(x)$.*

Let R be given and pick the integer k in such a way that $R_k < R \leq R_{k+1}$. We prove that the choice $\rho = R_{k+2}$ works.

First of all, we remark that if $|x| \geq \rho$ then $x \notin W_{2^{k+1}}$, so that $V(x) > 2^{k+1}$. Let the integer p be such that $2^{k+1} \leq 2^p \leq V(x) < 2^{p+1}$, and let λ be a dyadic reciprocal such that $V(x) < \lambda < 2^{p+1}$. Of course $x \in W_\lambda$, and hence

$$A^R(x) \subseteq D_\gamma^R(x) \subseteq D_\gamma^{R_{k+1}}(x) \subseteq D_\gamma^{R_p}(x) \subseteq W_\lambda \ .$$

It follows that if $y \in A^R(x)$, then $V(y) \leq \lambda$. Since λ can be taken arbitrarily close to $V(x)$, Claim A is proved.

Note that Claim A implies property (**MP**).

Claim B. *$V(x)$ is radially unbounded.*

Let $N > 0$, and let k be an integer such that $N < 2^k$. For $|x| > R_{2^{k+1}}$, we have $x \notin W_{2^k}$, that is $V(x) \geq 2^k$.

Claim C. *$V(x)$ is continuous.*

First, we remark that by construction, V is locally bounded. Assume that we can find a point $\bar{x} \in \mathbf{R}^n$ and a sequence $x_\nu \to \bar{x}$ such that $V(x_\nu)$ does not converge to $V(\bar{x})$. By possibly taking a subsequence, we have

$$\lim_\nu V(x_\nu) = l \neq V(\bar{x}) \ . \tag{7}$$

Assume first that $V(x) < l$ and pick a dyadic reciprocal λ in such a way that $V(x) < \lambda < l$ and $x \in \operatorname{Int} W_\lambda$. For all sufficient large ν, we should have $x_\nu \in W_\lambda$ as well. But then, $V(x_\nu) < \lambda$, and this is a contradiction to (7).

The case $V(x) > l$ is treated in a similar way.

Also the proof of the necessary part is now complete.

References

1. Andriano V., Bacciotti A. and Beccari G. (1997) *Global Stability and External Stability of Dynamical Systems*, Journal of Nonlinear Analysis, Theory, Methods and Applications **28**, 1167-1185
2. Auslander J. and Seibert P. (1964) *Prolongations and Stability in Dynamical Systems*, Annales Institut Fourier, Grenoble **14**, 237-268
3. Bacciotti A. and Mazzi L., *A Necessary and Sufficient Condition for Bounded Input Bounded State Stability of Nonlinear Systems*, SIAM Journal Control and Optimization, to appear
4. Bacciotti A. and Rosier L., *On the Converse of First Liapunov Theorem: the Regularity Issue*, Systems and Control Letters, to appear
5. Tsinias J., Kalouptsidis N. and Bacciotti A. (1987) *Lyapunov Functions and Stability of Dynamical Polysystems*, Mathematical Systems Theory **19**, 333-354