Remarks on dwell time solutions and stability of families of nonlinear vector fields

Andrea Bacciotti and Luisa Mazzi

***Abstract*—In this note we discuss the problem of stability of (finite or infinite) families of continuous vector fields, all of them asymptotically stable but, in general, not exponentially stable. Under a multiple Liapunov function condition and an average dwell time constraint, we prove that the system possesses a form of stability, weaker than the standard one. The advantage of our results is that the conditions imposed on the Liapunov functions can be verified a priori, with no previous knowledge of the integral curves of the family.**

***Index terms*—Dwell time, families of vector fields, Liapunov functions, stability.**

I. INTRODUCTION

Families of vector fields have been fruitfully employed as a model for representing input systems both in geometric control theory and in the recent theory of switched systems. Here we consider a family of finite dimensional, continuous vector fields $\mathcal{F} = \{f_p(x)\}_{p\in P}$, which possess a common equilibrium position at the origin. In general terms, by a solution of $\mathcal{F}$ we mean any continuous, piecewise differentiable curve which is obtained by gluing together integral curves of the vector fields $f_p(x)$; however, the solutions of $\mathcal{F}$ which are interesting from the point of view of applications, are often required to satisfy additional constraints, as it will be explained later.

The problem of finding conditions under which the origin is stable or asymptotically stable with respect to certain solutions of $\mathcal{F}$ has been addressed in many papers (see the monographs [9], [16] for a general introduction, basic results and lists of references). We emphasize that such a stability problem cannot be solved in general by individually looking at the vector fields $f_p(x)$; indeed, it is well known that the origin may be unstable for $\mathcal{F}$ even if every vector field $f_p(x)$ is asymptotically stable [4], [9]. In order to establish whether a family of vector fields is stable at the origin, it is natural to seek some suitable extension of Liapunov direct method.

A rather naive approach relies on the condition that the vector fields $f_p(x)$ share a common smooth Liapunov function; under this assumption, it is easy to prove that the origin is asymptotically stable with respect to arbitrary solutions of $\mathcal{F}$. From a theoretical point of view, such a result is not too restrictive: indeed, under some reasonable uniformity assumptions, it has been proved that the converse is true, as well [5], [12]. However, in general it is hard to find an explicit common Liapunov function; obviously, the difficulties grow with the number of vector fields of $\mathcal{F}$. Moreover, converse theorems do not hold in general for simple (non asymptotic) stability. Thus, at least from a practical point of view, it is advisable to explore other ways.

To reduce the difficulty of finding a common Liapunov function, in [15] the authors suggested the idea that stability of families of vector fields can be studied by using multiple Liapunov functions. This means that each vector field $f_p(x)$ has its own Liapunov function $V_p(x)$. Of course, this assumption is too weak to imply stability of $\mathcal{F}$ (see again [4], [9] for counterexamples); it is actually necessary to add a condition ensuring compatibility among the Liapunov functions $V_p(x)$. A variety of compatibility conditions has been proposed in the literature.

The simplest idea is to ask that for each solution $\varphi(t)$ and each switching instant t_k one has

$$V_{p_{k-1}}(\varphi(t_k)) \geq V_{p_k}(\varphi(t_k)) \tag{1}$$

where p_k is the index of the vector field in charge after t_k, and p_{k-1} is the index of the vector field in charge before t_k. Under this condition, in [6] the author obtains a generalized versions of LaSalle's invariance principle (see also [1], [13] for related results).

A more sophisticated compatibility condition was introduced in [4] (see also [2]) in order to generalize Liapunov first and second theorem to switched systems; it states that

$$V_p(\varphi(t')) \geq V_p(\varphi(t'')) \tag{2}$$

if t' and t'' are left endpoints of the intervals where the same vector field $f_p(x)$ is in charge ($t' < t''$).

Conditions (1) and (2) can be practically applied only if the switching times are determined by a positional rule, or if the solutions of $\mathcal{F}$ are explicitly known. Thus, we are led to look for alternative compatibility conditions, perhaps more conservative, but verifiable without solving differential equations.

Such a condition has been proposed in [7], [8] (Theorem 4). It states that for some $\mu > 1$

$$\mu V_q(x) \geq V_p(x) \tag{3}$$

for each pair of indices $q, p \in P$ and each point x in the state space. In [8] the authors prove that if (3) is satisfied, then the origin is asymptotically stable with respect to all the solutions which met an average dwell time constraint, meaning that for each interval I there is a bound (depending on the length of I) on the number of switches occurring in I.

We point out that the proof of Theorem 4 in [8] is essentially based on linear techniques: in fact, if all the Liapunov functions $V_p(x)$ satisfy a Łojasewicz inequality in a neighborhood of the origin [11], then the assumptions of Theorem 4 of [8] actually imply that the origin is exponentially stable for each vector field $f_p(x)$ (see for instance Theorem 5.2 of [3]).

Thus, Theorem 4 of [8] seems to leave open the problem of investigating the dynamical behavior of the solutions of $\mathcal{F}$ near the origin, when some of the vector fields $f_p(x)$ are asymptotically stable but not exponentially stable at the origin[1]. We point out that in this case the rate of convergence becomes slower and slower while the solutions of the vector fields approach the origin, so that we should not expect that, in general, stability is preserved under a constant dwell time constraint (such an intuition seems to be confirmed by simulations: see Section V of this paper).

We are finally able to illustrate the contribution of the present paper. We consider a family of continuous vector fields $\mathcal{F}$, under assumptions which are slightly more general than those of Theorem 4 of [8]: in particular, our assumptions do not imply that all the vector fields $f_p(x)$ are exponentially stable at the origin. We prove that $\mathcal{F}$ possesses a different form of stability, we name it here pseudo-stability. The precise definition of pseudo-stability will be given in Section III; intuitively, it means that the size of excursions of solutions satisfying an average dwell time constraint can be

---

[0] A. Bacciotti and L. Mazzi are with the Department of Mathematics, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy (e-mail: andrea.bacciotti@polito.it, luisa.mazzi@polito.it

[1] Of course, the case of finite-time convergence is covered by Theorem 4 of [8].

controlled, provided that the estimation margin is not too small. The proof relies on the method of multiple Liapunov functions and basically exploits classical arguments.

The outline of the exposition is as follows. In Section II we introduce the notion of dwell time solutions, and we discuss some of their properties. The definition of pseudo-stability is given in Section III, where we state and prove the main result. The case where assumptions hold globally is shortly considered in Section IV; we shall see that average dwell time solutions of $\mathcal{F}$ possess a boundedness property which can be seen as a dual property of pseudo-stability. In Section V we discuss two illustrative examples. We present further comments about the notion of pseudo-stability in the Conclusion.

II. DWELL-TIME SOLUTIONS

In this section we discuss some properties of dwell-time solutions. As far as we know, the first result about stability preservation under dwell time constraints was obtained by Morse [14] in a linear context. Let $\mathcal{F} = \{f_p(x)\}_{p\in P}$ be a family of continuous and complete vector fields of $\mathbf{R}^n$, where P is any set (finite or infinite) of indices.

Definition 1: A continuous function $\varphi : [0, +\infty) \to \mathbf{R}^n$ is said to be a *switched solution* of $\mathcal{F}$ if there exists a divergent sequence of times $0 = t_0 < t_1 < \ldots < t_i < \ldots$ and a sequence of indices $p_0, p_1, \ldots, p_i, \ldots$ such that $\varphi(t)$ is a solution of $\dot{x} = f_{p_i}(x)$ for $t \in [t_i, t_{i+1})$. The set of all the switched solutions of $\mathcal{F}$ is denoted by $\mathcal{S}$.

In the literature, the sequence $0 = t_0 < t_1 < \ldots < t_i < \ldots$ is often called the sequence of *switching times* of the switched solution φ. The piecewise constant, right-continuous function $\sigma : [0, +\infty) \to P$ such that $\sigma(t) = p_i$ for all $t \in [t_i, t_{i+1})$ is also called a *switching law* associated to φ. In general, the switching law associated to a switching solution is not unique.

On the other hand, given any switching law (i.e., a piecewise constant, right-continuous function $\sigma : [0, +\infty) \to P$), a switched solution associated to σ is a continuous function φ_σ such that $\dot{\varphi}_\sigma(t) = f_{\sigma(t)}(\varphi_\sigma(t))$, for all $t \neq t_i$.

In Definition 1 we do not require $p_i \neq p_{i+1}$, because we want to include switching laws which are constant or have finitely many discontinuities. On the other hand, later on we need to consider with special attention the actual discontinuities of a switching law σ. Thus, we set

$$I_\sigma = \left\{ t_i > 0, \sigma(t_i) = \lim_{t\to t_i^+} \sigma(t) \neq \lim_{t\to t_i^-} \sigma(t) \right\} \cup \{t_0 = 0\} . \quad (4)$$

When I_σ contains finitely many points, say $0 = t_0 < t_1 < \ldots < t_K$, we agree to write $t_{K+1} = +\infty$.

The distribution of the switches may be very complicated: we limit ourselves to some cases, by considering some of the definitions introduced by [6] et al.

Definition 2: Given a switching law σ, if there exists $\tau_D > 0$ such that

$$\inf \{t_{k+1} - t_k \ , \ t_k, t_{k+1} \in I_\sigma\} = \tau_D$$

we say that σ is a *dwell-time switching law*. A solution φ is said to be a *dwell-time solution* if there is a dwell-time switching law σ associated to φ. τ_D is called the *dwell-time* of the dwell-time switching law σ.

Following [6], we shall denote by $\mathcal{S}_{dw}[\tau_D]$ the set of switching laws of dwell-time $\tau \geq \tau_D$.

When $\sigma \in \mathcal{S}_{dw}[\tau_D]$, let us observe that, for all $t \geq 0$, for all $l > 0$,

$$N_\sigma(t, t+l) \overset{\text{def}}{=} \# \{I_\sigma \cap (t, t+l)\} \leq 1 + \frac{l}{\tau_D},$$

where $\#$ denotes the cardinality of the considered set. If $l < \tau_D$, $N_\sigma(t, t+l) \leq 1$.

When $\sigma \notin \mathcal{S}_{dw}[\tau_D]$, it may happen that for arbitrarily small $l > 0$ there exists $t \geq 0$ such that $N_\sigma(t, t+l) > 1$. This is the case of the following example:

Example 1: Let $I_\sigma = \left\{0, 1, 2, 2+\frac{1}{2}, 3, 3+\frac{1}{3}, \ldots, n, n+\frac{1}{n}, \ldots\right\}$. It is easy to check that, for all $l > 0$, for all $t \geq 0$, $N_\sigma(t, t+l) \leq 2 + 2l$. On the other hand, for all $l \in (0, 1)$, there exists $t > 0$ such that $N_\sigma(t, t+l) = 2$.

This remark leads to the following definition [6]:

Definition 3: We say that a switching law σ has an *average dwell-time* $\tau_a > 0$ if there exists $N_0 > 0$ such that, for all $t \geq 0$, for all $l > 0$,

$$N_\sigma(t, t+l) \leq N_0 + \frac{l}{\tau_a}.$$

In [6], [7], N_0 is often referred to as a *chatter bound*, since on intervals of length $l < \tau_a$ there are at most N_0 discontinuities of σ. The set of switching laws satisfying Definition 3 is denoted by $\mathcal{S}_a[N_0, \tau_a]$. It is clear that $\mathcal{S}_a[1, \tau_a] = \mathcal{S}_{dw}[\tau_a]$.

In the following proposition we list some properties of these sets we are interested in.

Proposition 1: (1) $\mathcal{S}_a[N_0, \tau_a] \subseteq \mathcal{S}_a[N_0, \tau_a']$, for all $\tau_a' \in (0, \tau_a]$.

(2) $\mathcal{S}_a[N_0, \tau_a] \subseteq \mathcal{S}_a[N_0', \tau_a]$, for all $N_0' \geq N_0$.

(3) If $\sigma \in \mathcal{S}_a[N_0, \tau_a]$, $t_k \in I_\sigma$, $\implies$ $N_\sigma(t_k, t_k+l) \leq N_0 - 1 + \frac{l}{\tau_a}$.

(4) If $\sigma \in \mathcal{S}_a[N_0, \tau_a]$, $t_k \in I_\sigma, l < \tau_a$ $\implies$ $N_\sigma(t_k, t_k+l) \leq N_0 - 1$.

(5) Let $\sigma \in \mathcal{S}_a[N_0, \tau_a]$, $N_0 \geq 1$. If $t_k \in I_\sigma, l \geq \tau_a$ $\implies$ $N_\sigma(t_k, t_k+l) \leq \frac{N_0 l}{\tau_a}$.

Proof: (1), (2) are trivial.

(3) Let $t_k \in I_\sigma$ and $l > 0$. We choose $\varepsilon > 0$ so that $\varepsilon < \frac{1}{2}(t_k - t_{k-1})$ and $\left[\frac{l}{\tau_a}\right] = \left[\frac{l+\varepsilon}{\tau_a}\right]$, where $[\cdot]$ denotes the integer part of a real number. Then,

$$N_\sigma(t_k, t_k+l) = N_\sigma(t_k-\varepsilon, (t_k-\varepsilon)+l+\varepsilon) - 1 \leq N_0 - 1 + \frac{l+\varepsilon}{\tau_a}.$$

Since $\left[\frac{l}{\tau_a}\right] = \left[\frac{l+\varepsilon}{\tau_a}\right]$, we have that $N_\sigma(t_k, t_k+l) \leq N_0 - 1 + \frac{l}{\tau_a}$.

(4) It follows from (3), when $l < \tau_a$.

(5) Let $t_k \in I_\sigma$ and $l \geq \tau_a$. We set $l = \tau_a + h$, with $h \geq 0$. Then

$$\begin{aligned} N_\sigma(t_k, t_k+l) &\leq N_0 - 1 + \frac{l}{\tau_a} = N_0 - 1 + \frac{\tau_a + h}{\tau_a} \\ &= N_0 + \frac{h}{\tau_a} \leq N_0 + \frac{N_0 h}{\tau_a} = \frac{N_0 l}{\tau_a} . \end{aligned}$$

∎

III. STABILITY

In this section we consider a family $\mathcal{F} = \{f_p(x)\}_{p\in P}$, where the vector fields f_p have a common equilibrium position at the origin. We introduce an appropriate notion of stability, and we give a sufficient condition, in terms of an associated family of Liapunov functions, to check whether $\mathcal{F}$ is stable in this sense.

The Euclidean norm of $x \in \mathbf{R}^n$ is denoted by $|x|$, and the ball with radius $r > 0$ and center at the origin is denoted by $B(r) = \{x : |x| < r\}$.

Definition 4: We say that the origin is *pseudo-stable* for $\mathcal{F}$ if for each $N_0 \geq 1$ and each $\varepsilon > 0$ there exists $\tau_a > 0$ such that:

$$\begin{aligned} &\exists \delta > 0 \ : \ \sigma \in \mathcal{S}_a[N_0, \tau_a], \ |\varphi_\sigma(0)| < \delta \\ &\implies |\varphi_\sigma(t)| < \varepsilon, \ \forall t \geq 0. \end{aligned} \quad (5)$$

Remark 1: In the definitions of stability for families of vector fields adopted so far in the literature, possible constraints on the switching laws are imposed *a priori*. In other words, implication (5) must be verified only for trajectories in a preassigned set. On the contrary, Definition 4 introduces a dependence of τ_a (and hence, of the set of admissible switching signals), on the chatter bound N_0 and on the estimation bound ε. In this sense, Definition 4 is weaker than the usual notion of stability.

Recall that a function $\alpha : [0, r_0) \to [0, +\infty)$ (where r_0 is some positive real number possibly dependent on α) is of class $\mathcal{K}$ if it is continuous, strictly increasing and such that $\alpha(0) = 0$.

Lemma 1: Let $\alpha, \beta : [0, r_0) \to \mathbf{R}$ be functions of class $\mathcal{K}$, such that $\alpha(r) < \beta(r)$, for all $r \in (0, r_0)$.

Then $\gamma(r) = \beta^{-1} \circ \alpha(r)$ is well defined on $[0, r_0)$, of class $\mathcal{K}$ and $\gamma(r) < r$, for all $r \in (0, r_0)$.

For sake of simplicity, we shall use the notation $\gamma^N(r)$ to denote the composition of γ with itself N times.

Theorem 1: Let $\mathcal{F} = \{f_p(x)\}_{p \in P}$ be a family of continuous and complete vector fields of $\mathbf{R}^n$, where P is any set (finite or infinite) of indices, such that $f_p(0) = 0$ for each $p \in P$. Let $r_0 > 0$, and let $\mathcal{V} = \{V_p\}_{p \in P}$ be a family of functions from $B(r_0)$ to $\mathbf{R}$, all of class C^1, satisfying the following properties.

(i) There exist maps $\alpha, \beta \in \mathcal{K}$, with $\alpha(r) < \beta(r)$ for $r \in (0, r_0)$, such that $\alpha(|x|) \le V_p(x) \le \beta(|x|)$ for each $p \in P$ and each $x \in B(r_0)$.
(ii) There exists a map $K \in \mathcal{K}$ such that, for each $p \in P$ and each $x \in B(r_0)$

$$\nabla V_p(x) f_p(x) \le -K(|x|).$$

(iii) There exists a map $H \in \mathcal{K}$ such that, for each pair $p, q \in P$, and each $x \in B(r_0)$

$$V_p(x) \le V_q(x) + H(|x|).$$

Then, the origin is pseudo-stable for $\mathcal{F}$.

Proof: For reader's convenience, we first prove the theorem in the particular case $N_0 = 2$. Then we indicate how to adapt the proof to the general case.

So let $N_0 = 2$, and let us fix, without loss of generality, $\varepsilon \in (0, r_0)$. We want to show that

$$\tau_a = \tau_a(\varepsilon) = \frac{2H(\varepsilon)}{K(\gamma^3(\varepsilon))} > 0 \tag{6}$$

is the requested average dwell time. We set $\eta_3 = \varepsilon$, $\eta_2 = \gamma(\eta_3)$, $\eta_1 = \gamma(\eta_2) = \gamma^2(\eta_3)$, $\eta_0 = \delta = \gamma(\eta_1) = \gamma^3(\eta_3)$. By definition of γ, $\eta_3 > \eta_2 > \eta_1 > \eta_0 = \delta$, and

$$\beta(\eta_1) = \beta(\gamma(\eta_2)) = \alpha(\eta_2) < \alpha(\varepsilon) \ . \tag{7}$$

Now, assume by contradiction that there exist $\bar{x} \in B(\delta)$, $\sigma \in \mathcal{S}_a[2, \tau_a]$, a trajectory $\varphi(t)$ associated to σ issuing from $\bar{x}$ and a time $T > 0$ such that $|\varphi(T)| \ge \varepsilon$. In the interval $[0, T]$ we find:

$$\begin{array}{llll}
T_0' & \text{such that} & |\varphi(T_0')| = \delta = \eta_0 & \\
 & \text{and} & |\varphi(t)| > \delta & \forall t > T_0' \\
T_1 & \text{such that} & |\varphi(T_1)| = \eta_1 & \\
 & \text{and} & \delta = \eta_0 < |\varphi(t)| < \eta_1 & \forall t \in (T_0', T_1) \\
T_1' & \text{such that} & |\varphi(T_1')| = \eta_1 & \\
 & \text{and} & |\varphi(t)| > \eta_1 & \forall t > T_1' \\
T_2 & \text{such that} & |\varphi(T_2)| = \eta_2 & \\
 & \text{and} & \eta_1 < |\varphi(t)| < \eta_2 & \forall t \in (T_1', T_2) \\
T_2' & \text{such that} & |\varphi(T_2')| = \eta_2 & \\
 & \text{and} & |\varphi(t)| > \eta_2 & \forall t > T_2' \\
T_3 & \text{such that} & |\varphi(T_3)| = \eta_3 = \varepsilon & \\
 & \text{and} & \eta_2 < |\varphi(t)| < \eta_3 = \varepsilon & \forall t \in (T_2', T_3)
\end{array}$$

Clearly, these points exist and are uniquely defined. It may happen that $T_i' = T_i$ for $i = 1, 2$, but $T_0' < T_1 \le T_1' < T_2 \le T_2' < T_3$. Moreover, we claim that in the intervals $[T_i', T_{i+1})$, $i = 0, 1, 2$ there must be at least one switching time τ_i; otherwise, for some index p we should have

$$\begin{array}{rcl}
V_p(\varphi(T_i')) & \le & \beta(|\varphi(T_i')|) = \beta(\eta_i) = \alpha(\eta_{i+1}) \\
 & = & \alpha(|\varphi(T_{i+1})|) \le V_p(\varphi(T_{i+1})) \ .
\end{array}$$

This is impossible by condition (ii). Indeed,

$$\begin{array}{rcl}
V_p(\varphi(T_{i+1})) & = & V_p(\varphi(T_i')) + \displaystyle\int_{T_i'}^{T_{i+1}} \dot{V}_p(\varphi(s))\, ds \\
 & \le & V_p(\varphi(T_i')) - \displaystyle\int_{T_i'}^{T_{i+1}} K(|\varphi(s)|)\, ds \\
 & \le & V_p(\varphi(T_i')) - K(\eta_i)(T_{i+1} - T_i')
\end{array}$$

which yields $V_p(\varphi(T_i')) > V_p(\varphi(T_{i+1}))$.

Summing up, we have found a triple of switching times $\tau_0, \tau_1, \tau_2 \in I_\sigma \cap [T_0', T_3)$. Hence $N_\sigma(\tau_0, T_3) \ge 2$; by Proposition 1(4), $T_3 - \tau_0 \ge \tau_a$. Now we set: $I_\sigma \cap (\tau_0, T_3) = \{t_1, t_2, \ldots, t_N\}$ where $t_1 < t_2 < \ldots < t_N$. We complete this sequence by adding $t_0 = \tau_0 < t_j$ and $t_{N+1} = T_3 > t_j$, for all $j = 1, \ldots, N$.

By construction, $N \ge 2$ and $t_{N+1} - t_0 \ge \tau_a$. Then, by Proposition 1 (5), $N(t_0, t_{N+1}) = N \le \frac{t_{N+1} - t_0}{\frac{1}{2}\tau_a}$, hence $t_{N+1} - t_0 \ge \frac{1}{2} N \tau_a$.

Let $p_0, \ldots, p_{N+1} \in P$ such that $\sigma(t) = p_j$ if $t \in [t_j, t_{j+1})$, $j = 1, \ldots, N$. We now compute:

$$\begin{array}{rl}
 & V_{p_N}(\varphi(t_{N+1})) - V_{p_0}(\varphi(t_0)) \\
= & V_{p_N}(\varphi(t_{N+1})) - V_{p_N}(\varphi(t_N)) + V_{p_N}(\varphi(t_N)) \\
- & V_{p_{N-1}}(\varphi(t_N)) + V_{p_{N-1}}(\varphi(t_N)) \\
 & \quad \ldots \\
- & V_{p_1}(\varphi(t_1)) + V_{p_1}(\varphi(t_1)) \\
- & V_{p_0}(\varphi(t_1)) + V_{p_0}(\varphi(t_1)) - V_{p_0}(\varphi(t_0)) \ .
\end{array}$$

Using repeatedly (ii), (iii), we have

$$\begin{array}{rl}
 & V_{p_N}(\varphi(t_{N+1})) - V_{p_0}(\varphi(t_0)) \\
\le & -\displaystyle\int_{t_N}^{t_{N+1}} K(|\varphi(s)|)\, ds + H(|\varphi(t_N)|) \\
 & \ldots \\
 & -\displaystyle\int_{t_0}^{t_1} K(|\varphi(s)|)\, ds + H(|\varphi(t_1)|) \\
= & -\displaystyle\int_{t_0}^{t_{N+1}} K(|\varphi(s)|)\, ds + \sum_{j=1}^{N} H(|\varphi(t_j)|) \\
\le & -(t_{N+1} - t_0) K(\delta) + N H(\varepsilon) \le -\dfrac{1}{2} N \tau_a K(\delta) + N H(\varepsilon) \ .
\end{array}$$

The last expression is nonpositive, by (6). This is a contradiction, because from (i) and (7) we have:

$$V_{p_N}(\varphi(t_{N+1})) - V_{p_0}(\varphi(t_0)) \ge \alpha(\varepsilon) - \beta(\eta_1) > 0.$$

Therefore, any solution $\phi_\sigma \in \mathcal{S}_a[2, \tau_a]$, with $|\phi_\sigma(0)| < \delta$ is such that $|\phi_\sigma(t)| < \varepsilon$, for all $t > 0$.

When $N_0 > 2$, the proof follows the same pattern; we sketch here the main modifications. In this case, fixed $\varepsilon \in (0, r_0)$, we define

$$\tau_a = \tau_a(\varepsilon, N_0) = \frac{N_0 H(\varepsilon)}{K(\gamma^{N_0+1}(\varepsilon))} > 0 \ .$$

Then we define $\eta_{N_0+1} = \varepsilon$, $\eta_{N_0} = \gamma(\varepsilon)$, $\ldots$, $\eta_0 = \delta = \gamma^{N_0+1}(\varepsilon)$.

By contradiction, we assume the existence of $\bar{x} \in B(\delta)$, $\sigma \in \mathcal{S}_a[N_0, \tau_a]$, of a trajectory $\varphi(t)$ associated to σ issuing from $\bar{x}$ and a time $T > 0$ such that $|\varphi(T)| \geq \varepsilon$. In the interval $[0, T]$ we find:

$$\begin{array}{lll}
T'_0 & \text{such that} & |\varphi(T'_0)| = \delta = \eta_0 \\
\text{and} & |\varphi(t)| > \delta & \forall t > T'_0 \\
 & \cdots\cdots & \\
T_i & \text{such that} & |\varphi(T_i)| = \eta_i \\
\text{and} & \eta_{i-1} < |\varphi(t)| < \eta_i & \forall t \in (T'_i, T_{i+1}), \\
T'_i & \text{such that} & |\varphi(T'_i)| = \eta_i \\
\text{and} & |\varphi(t)| > \eta_i & \forall t > T'_i \\
 & \cdots\cdots & \\
T_{N_0+1} & \text{such that} & |\varphi(T_{N_0+1})| = \eta_{N_0+1} = \varepsilon \\
\text{and} & \eta_{N_0} < |\varphi(t)| < \eta_{N_0+1} = \varepsilon & \forall t \in (T'_{N_0}, T_{N_0+1})
\end{array}$$

As above, we may prove that, for all $i = 0, \ldots, N_0$, there exists $\tau_i \in I_\sigma \cap [T'_i, T_{i+1})$. Thus, we have $N_\sigma(\tau_0, T_{N_0+1}) \geq N_0$ and $T_{N_0+1} - \tau_0 \geq \tau_a$, by Proposition 3(4).

We complete $I_\sigma \cap (\tau_0, T_{N_0+1}) = \{t_1, t_2, \ldots, t_N\}$ by adding $t_0 = \tau_0$ and $t_{N+1} = T_{N_0+1}$. Since $t_{N+1} - t_0 \geq \tau_a$, we may finally proceed as in the case $N_0 = 2$ to get a contradiction.

Since for any $N_0 > 1$, for any $\varepsilon_0 > 0$ we have found τ_a satisfying the conditions of Def. 4, we have proven that the origin is pseudo-stable for $\mathcal{F}$.

∎

Remark 2: Since γ is a contraction and K is increasing, the average dwell time $\tau_a = \tau_a(\varepsilon, N_0)$ we get in our proof increases with N_0.

Remark 3: In [8] the compatibility condition takes the form (3). This implies assumption (ii) of Theorem 1. Indeed, we may write $V_p(x) \leq V_q(x) + (\mu - 1)V_q(x)$, and take $H(r) = (\mu - 1)\beta(r)$. Moreover, in [8] the authors require that

$$\nabla V_p(x) f_p(x) \leq -2\lambda_0 V_p(x) \tag{8}$$

for some $\lambda_0 > 0$ and for each $p \in P$. Taking into account condition (i), (8) implies that $\nabla V_p(x) f_p(x) \leq -2\lambda_0 \alpha(|x|)$. In other words, if (8) holds, then (ii) holds as well, with $K(r) = 2\lambda_0 \alpha(r)$.

Recall that a positive function $G(x) : \mathbf{R}^n \to \mathbf{R}$ satisfies a Łojasewicz inequality if there exist positive numbers R, c, ν such that $G(x) \geq c|x|^\nu$, $\forall x \in B(R)$. It is well known that if G is analytic then it satisfies a Łojasewicz inequality (see [11]).

If all the Liapunov functions $V_p(x)$ satisfy a Łojasewicz inequality with the same R, c, ν, then we can choose

$$\alpha(r) = \bar{\alpha} r^\nu \ , \tag{9}$$

for some $\bar{\alpha}$ and ν. Assume that we can choose, in addition,

$$\beta(r) = \bar{\beta} r^\nu \tag{10}$$

with the same exponent $\nu > 0$. Then it turns out that $\tau_a(\varepsilon)$ is constant over $[0, r_0]$. Indeed, with $\Gamma = (\bar{\alpha}/4\bar{\beta})^{\frac{N_0+1}{2}}$ and $C = ((\mu - 1)\bar{\beta})/(2\lambda_0 \bar{\alpha} \Gamma^2)$, we have

$$H(r) = (\mu - 1)\bar{\beta} r^2 = CK(\Gamma r) = CK(\gamma^{N_0+1}(r)) \ .$$

As a consequence, in the case where $\alpha(r)$ and $\beta(r)$ can be taken of the form (9) (10), the proof of Theorem 1 shows that the origin is actually stable for $\mathcal{F}$, and the conclusion can be strengthened. In particular, we have another proof of the stability part of Theorem 4 of [8] (the asymptotic stability part could be obtained along similar lines).

Remark 4: Compared with analogous compatibility conditions used in [4], [1] in order to prove stability by means of multiple Liapunov functions, condition (iii) of Theorem 1 has an advantage: it can be checked without computing explicitly the integral curves of the vector fields of $\mathcal{F}$.

Other compatibility conditions not requiring the integration of the vector fields are given in [10] for special classes of switched systems.

If each vector field f_p is asymptotically stable at the origin, Kurzweil converse theorem guarantees the existence of Liapunov functions V_p satisfying (i) and (ii) (see for instance [3]). In general, each V_p has its own α, β and K, but if $\mathcal{F}$ is finite, α, β and K can be made independent of p. If in addition we set

$$h(r) = \max\{|V_p(x) - V_q(x)|, p, q \in P, x \in \overline{\mathcal{B}}_r\}$$

and we choose a function $H(r)$ of class $\mathcal{K}$ such that $h(r) \leq H(r)$, for all $r \in [0, r_0)$, then also (iii) is fulfilled. Thus, we can state the following Corollary.

Corollary 1: If $\mathcal{F}$ is a finite family and the origin is asymptotically stable for all $f_p \in \mathcal{F}$, then the origin is pseudo-stable for $\mathcal{F}$.

IV. BOUNDEDNESS OF SOLUTIONS

Global boundedness of certain switched solutions can be characterized by means of the following notion, which can be interpreted as a dual property of pseudo-stability.

Definition 5: We say that the family $\mathcal{F}$ is *pseudo-Lagrange stable* if for each $N_0 \geq 1$ and each $R > 0$ there exist $\tau_a > 0$ such that

$$\exists S > 0 \ : \ \sigma \in \mathcal{S}_a[N_0, \tau_a], \ |\varphi_\sigma(0)| < R \Longrightarrow |\varphi_\sigma(t)| < S, \ \forall t \geq 0.$$

We agree that all the functions of class $\mathcal{K}$ used in this Section are defined on $[0, +\infty)$ and unbounded. Let $C(r_0) = \{x : |x| > r_0\}$.

Theorem 2: Let $\mathcal{F} = \{f_p(x)\}_{p \in P}$ be a family of continuous and complete vector fields of $\mathbf{R}^n$, where P is any set (finite or infinite) of indices. Let $r_0 > 0$, and let $\mathcal{V} = \{V_p\}_{p \in P}$ be a family of functions from $C(r_0)$ to $\mathbf{R}$, all of class C^1, satisfying the following properties.

(i) There exist maps $\alpha, \beta \in \mathcal{K}$, with $\alpha(r) < \beta(r)$ for $r > r_0$, such that $\alpha(|x|) \leq V_p(x) \leq \beta(|x|)$ for each $p \in P$ and each $x \in C(r_0)$.

(ii) There exists a map $K \in \mathcal{K}$ such that, for each $p \in P$ and each $x \in C(r_0)$,

$$\nabla V_p(x) f_p(x) \leq -K(|x|).$$

(iii) There exists a map $H \in \mathcal{K}$ such that, for each pair $p, q \in P$, and each $x \in C(r_0)$,

$$V_p(x) \leq V_q(x) + H(|x|).$$

Then, $\mathcal{F}$ is pseudo-Lagrange stable.

Sketch of the proof: Fix $R > r_0$ and define

$$\tau_a = \tau_a(R) = \frac{N_0 H((\gamma^{-1})^{N_0+1}(R))}{K(R)}.$$

The remaining part of the proof is similar to that of Theorem 1, apart from some obvious modifications.

∎

V. ILLUSTRATIVE EXAMPLES

The aim of this section is to enlighten by a pair of simple examples the results of Theorem 1 and 2 and, more generally, the complexity of the dynamical behavior of a switched system formed by a pair of planar vector fields $\{f_1, f_2\}$. The first example concerns stability. Let

$$f_1(x,y) = \begin{pmatrix} -2y - 64x^3 \\ x - 32y^3 \end{pmatrix}, \quad f_2(x,y) = \begin{pmatrix} -y - 32x^3 \\ 2x - 64y^3 \end{pmatrix}. \quad (11)$$

Liapunov functions for f_1 and f_2 are, respectively,

$$V_1(x,y) = x^2 + 2y^2 \ , \qquad V_2(x,y) = 2x^2 + y^2 \ .$$

It is immediate to compute that

$$\dot{V}_1(x,y) = \dot{V}_2(x,y) = -128(x^4 + y^4) \ .$$

Thus, it is clear that the origin is asymptotically stable, but not exponentially stable for both f_1 and f_2. With the notation of the previous sections, we can take $\alpha(r) = r^2$, $\beta(r) = 2r^2$, $H(r) = r^2$ and $K(r) = 64r^4$. For sake of simplicity, we take $N_0 = 1$, which means that we limit to non-average dwell time. Then $L(r) = 1/(2r^2)$.

The behavior of the solutions of the switched system has been investigated by computer simulations. We limit ourselves to solutions with switching times $t_k = k\tau$, where $\tau > 0$ is a fixed duration. This is not a severe restriction: simulations with different durations for each vector field can be repeated by simply using different time scale.

Simulations show that starting from a large initial state, the solution is ultimately attracted within a bounded region surrounding the origin: this seems to be independent of the value of τ. However, the size of the attracting region becomes smaller and smaller while τ becomes larger and larger.

The behavior of the switched solutions corresponding to small initial states seems to be more involved, and more sensitive to variations of τ. Roughly speaking, we can distinguish two different situations.

(1) For certain values of τ, solutions starting from a small initial state spiral out of the origin, until they achieve a steady state behavior, similar to a limit cycle. This happens for instance when $\tau = 1$ (Figure 1) and also when $\tau = 10.2$ (Figure 2). But in the second case the attractor has a more complicated geometry and a smaller size. Note that for these values of τ, the origin is not stable (in the classical Liapunov sense).

(2) For other values of τ, for instance when $\tau = 2$, the solutions corresponding to small initial states exhibit a more stable behavior (Figure 3). However, all these solutions present a dead-zone around the origin which cannot be reached.

Situations (1) and (2) arise alternatively, while τ varies. It is not clear from simulations whether there is a regularity in such a repeated behavior.

In any case, this example confirm the intuition that a switched system formed by purely nonlinear asymptotically stable vector fields is not asymptotically stable, in general. However, this switched system satisfies the notion of pseudo-stability introduced by Definition 4, although the estimate for τ_a provided by (6) is far from being sharp.

The second example concerns boundedness of solutions. Let

$$f_1(x,y) = \begin{pmatrix} -\frac{y}{4} \\ 4x - \sqrt[3]{y} \end{pmatrix} \ , \qquad f_2(x,y) = \begin{pmatrix} -4y \\ \frac{x}{4} - \sqrt[3]{y} \end{pmatrix} \ . \quad (12)$$

Figure 4 shows a (reasonably) unbounded trajectory, obtained with initial state $(0,5)$ and switching times $2, 4, 6, \ldots$. With the same initial state and switching times $10, 20, 30, \ldots$ we get the bounded trajectory of Figure 5.

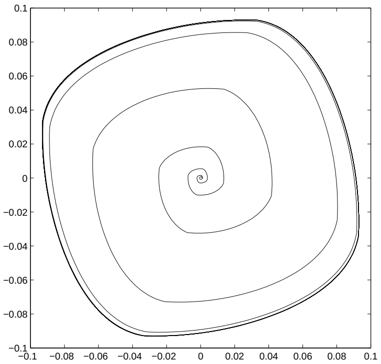

Fig. 1. Trajectory of the switched system (11), with $\tau = 1$ and initial state $(0.001, 0)$. The trajectory spirals out and ultimately it approaches a limit set. The simulation indicates that the origin is not stable.

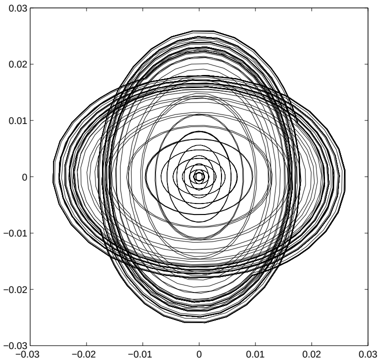

Fig. 2. Trajectory of the switched system (11), with $\tau = 10.2$ and initial state $(0.001, 0)$. As the one of Figure 1, this trajectory spirals out and it approaches a limit set. Again, the origin is not stable but now the attractor is much smaller. This is compatible with the statement of Theorem 1.

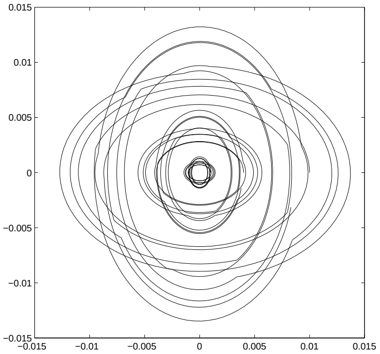

Fig. 3. Some trajectories of the switched system (11) with $\tau = 2$, corresponding to the initial states $(0.001, 0)$, $(0.004, 0)$, $(0.01, 0)$. The picture suggests that the origin is stable, but not asymptotically stable.

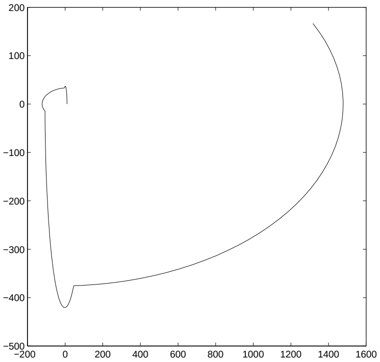

Fig. 4. Trajectory of the switched system (12), with $\tau = 2$ and initial state $(10, 0)$. The norm of the solution increases very rapidly.

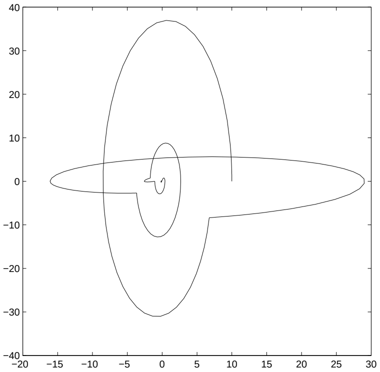

Fig. 5. Trajectory of the switched system (12), with $\tau = 6$ and initial state $(10, 0)$. The solution appears to be bounded.

VI. CONCLUSION

It is well known that unstable trajectories can be generated by suitable switching among the vector fields of a family $\mathcal{F}$, even if all of them are exponentially stable. In fact, looking at very simple examples, this loss of stability seems to be far from being exceptional, especially when the single vector fields are asymptotically stable but not exponentially stable.

On the other hand, to preserve stability is important for control purposes. To face the problem, we can proceed in one of the following ways.

(a) To impose (restrictive) conditions on the vector fields of $\mathcal{F}$.

(b) To impose constraints on the set of admissible switching laws.

(c) To weaken the notion of stability.

In this paper we adopted a mix of these possible approaches. We introduce a notion of stability (pseudo-stability) which differs from the usual one in the sense that the dwell time constraint depend on the desired stability margin. We prove that under reasonable assumptions (multiple Liapunov functions with an easy-to-check compatibility condition) a family of asymptotically stable vector fields is pseudo-stable.

The completeness assumption made in Theorem 1 about the vector fields simplifies the statement, but can be actually removed, since the result has a local nature. Similarly, the completeness assumption in Theorem 2 cam be avoided, since it follows from the other assumptions.

Although pseudo-stability is weaker than the usual notions, we believe that it is strictly related to the nature of switched systems and probably the best one can hope to obtain in general. The illustrative examples of Section V seem to suggest an analogy between pseudo-stability and the notion of practical stability, whose interest in application is relevant.

The examples of Section V suggest also that the bounds for the dwell time provided in the proofs of Theorems 1 and 2 are conservative. This drawback indicates the need for future work.

The authors wish to thank the anonymous referees for many helpful suggestions.

REFERENCES

[1] A. Bacciotti and L. Mazzi, "An invariance principle for nonlinear switched systems", *Syst. Control Lett.*, vol. 54, pp. 1109-1119, 2005.

[2] A. Bacciotti and L. Mazzi, "Stability of dynamical polysystems via families of Liapunov functions", *Jour. Nonlin. Analysis, T.M.A.*, vol. 67, pp. 2167-2179, 2007.

[3] A. Bacciotti and L. Rosier, *Liapunov functions and Stability in Control Theory*, Lecture Notes in Control and Information Sciences 267, London: Springer-Verlag, 2001.

[4] M. S. Branicky, "Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems", *IEEE Trans. Autom. Control*, vol. 43, pp. 475-482, 1998.

[5] W. P. Dayawansa and C. F. Martin, "A Converse Lyapunov Theorem for a Class of Dynamical Systems which Undergo Switching", *IEEE Trans. Autom. Control*, vol. 44, pp. 751-760, 1999.

[6] J. P. Hespanha, "Uniform Stability of Switched Linear Systems: Extensions of LaSalle's Invariance Principle", *IEEE Trans. Automat. Control*, vol. 49, pp. 470-482, 2004.

[7] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time", *Proc.* 38^{th} *CDC* Phoenix, Arizona,1999, pp. 2655-2660.

[8] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell time", *EE-Systems, Univ. South. Cal. / Lab. for Control Sciences and Engineering, Yale Univ.*, Tech. Rep., 1999.

[9] D. Liberzon, *Switching in Systems and Control*, Boston: Birkhäuser, 2003.

[10] J. Lygeros, S. N. Simic, K. H. Johansson, J. Zhang and S. S. Sastry, "Dynamical Properties of Hybrid Automata", *IEEE Trans. Autom. Control*, vol. 48, pp. 2-17, 2003.

[11] B. Malgrange, *Ideals of Differentiable Functions*, Oxford: Oxford University Press, 1966.

[12] J. L. Mancilla-Aguilar and R. A. Garcia, "A converse Lyapunov Theorem for nonlinear switched systems", *Syst. Control Lett.*, vol. 41, pp. 67-71, 2000.

[13] J. L. Mancilla-Aguilar and R. A. Garcia, "An extention of LaSalle's invariance principle for switched systems", *Syst. Control Lett.*, vol. 55, pp. 376-384, 2006.

[14] A. S. Morse, "Supervisory Control of Families of Linear Set-Point Controllers — Part 1: Exact Matching", *IEEE Trans. Autom. Control*, vol. 41, pp. 1413-1431, 1996.

[15] P. Peleties and R. A. deCarlo, "Asymptotic Stability of m-Switched Systems Using Lyapunov-like Functions", *Proc. 1991 Amer. Control Conf.*, pp. 1679-1684.

[16] Z. Sun and S. S. Ge, *Switched Linear Systems*, London: Springer-Verlag, 2005.