Stabilization by means of State Space Depending Switching Rules

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Abstract

In this paper we consider switched systems defined by a pair of linear systems $\dot{x} = A_1 x$, $\dot{x} = A_2 x$, such that there exists a neutrally stable linear combination of the matrices A_1, A_2. Under an additional assumption, we prove that there exists a state space depending switching rule which stabilizes the system in a very general sense. The result is illustrated by some simulations and examples.

Key words. Stabilization, Discontinuous feedback, Switched systems, Bilinear systems, Krasowski solutions.

1 Introduction

In this paper we study stability of bilinear systems of the form

$$\dot{x} = uA_1x + (1-u)A_2x \tag{1}$$

where $x \in \mathbf{R}^n$ represents the state variable, A_1, A_2 are square matrices of order n and the scalar input u is allowed to take only the values 0 and 1. In an open loop philosophy, under the additional requirement that the map $t \mapsto u(t)$ is piecewise constant, systems of this form are usually called (linear) *switched systems*: a great effort has been recently done in order to characterize stability of switched systems under arbitrary switched signals (see the survey papers [13], [8] and the references therein). As a matter of fact, in the spirit of geometric control theory, system (1) is equivalent to the assignment of the family of linear vector fields $\{A_1x, A_2x\}$; the investigation of the asymptotic behavior of families of vector fields under arbitrary switching (polysystems) was actually initiated in the early paper [4]).

Stability under arbitrary switched signals implies that A_1 and A_2 are both Hurwitz matrices. If this is not the case, the problem is to select classes of switched inputs (if any) which give rise to an asymptotically stable flow of trajectories. The selection criterion may be based either on the imposition of constraints about switching times, or on the specification of some switching rule depending on the position of the state variable. In the latter case, one should make a further distinction: the switching rule may or may not depend on the information about the past evolution. In this paper, we are interested in state static (memoryless) feedback laws, which excludes for instance hysteresis phenomena.

More precisely, we give a sufficient condition for the existence of a feedback law $u = k(x)$ which takes values in the set $\{0, 1\}$ and such that the closed loop system (1) (with $u = k(x)$) is asymptotically stable in a very general sense to be specified later. The construction of $k(x)$ is suggested by the natural interpretation of (1) as a bilinear system. Implications of our approach, and relationships with other known results, especially those of [15], will be discussed in the last section.

2 State space depending switching rule

Since the values of the input are restricted to a finite set, any feedback law is expected to be not continuous. Hence, the closed loop system will have a discontinuous right hand side. In general, let $f(x)$ be a locally bounded vector field of $\mathbf{R}^n$. Recall that a map $\varphi(t) : I \to \mathbf{R}^n$ is a *Krasowski solution* of the equation

$$\dot{x} = f(x) \tag{2}$$

if $\varphi(t)$ is absolutely continuous and satisfies a.e.

$$\dot{\varphi}(t) \in \mathbf{K}f(\varphi(t)) \qquad \text{where} \qquad \mathbf{K}f(x) = \bigcap_{\delta>0} \overline{\text{co}}\,\{f(\mathcal{B}(x,\delta))\} \tag{3}$$

(here $\mathcal{B}(x,\delta)$ denotes the ball of radius δ centered at x, and I is an interval of real numbers, possibly unbounded). This is not the unique way to define generalized solutions of systems with a discontinuous right hand side (see [5], [9], [10] for other possible notions). It is clear that using a different notion of solution one can obtain different results. However, the notion of Krasowski includes many other known notions, and it is adequate for the purposes of this paper.

The "operator" $\mathbf{K}$ admits a different definition.

Lemma 1 *Let $f(x) : \mathbf{R}^n \to \mathbf{R}^n$ be locally bounded. Then $\mathbf{K}f(x) = \text{co}\,\{v \in \mathbf{R}^n : \ \exists x_j \to x \text{ s.t. } f(x_j) \to v\}$.*

Proof For each $\delta > 0$, $\overline{\text{co}}\,\{f(\mathcal{B}(x,\delta))\}$ is a compact set, so that

$$\bigcap_{\delta>0} \overline{\text{co}}\,\{f(\mathcal{B}(x,\delta))\} = \text{co} \bigcap_{\delta>0} \overline{\{f(\mathcal{B}(x,\delta))\}}\ .$$

Let $v \in \text{co} \bigcap_{\delta>0} \overline{\{f(\mathcal{B}(x,\delta))\}}$. Then $v = \lambda u + (1-\lambda)w$ with $\lambda \in [0,1]$ and $u, w \in \bigcap_{\delta>0} \overline{\{f(\mathcal{B}(x,\delta))\}}$. Take $\delta = 1/k$, for any positive integer k. Then we can find points z_k and y_k such that

$$|z_k - x| < \frac{1}{k}\ , \quad |y_k - x| < \frac{1}{k}\ , \quad |f(z_k) - u| < \frac{1}{k}\ , \quad |f(y_k) - w| < \frac{1}{k}\ .$$

Of course, $z_k \to x$, $y_k \to x$, $f(z_k) \to u$ and $f(y_k) \to w$. This proves that $v \in \text{co}\,\{v \in \mathbf{R}^n : \ \exists x_j \to x \text{ s.t. } f(x_j) \to v\}$. Vice-versa, let $v \in \text{co}\,\{v \in \mathbf{R}^n : \ \exists x_j \to x \text{ s.t. } f(x_j) \to v\}$. Then for some sequences $z_k \to x$, $y_k \to y$, we have $v = \lambda u + (1-\lambda)w$ with $\lambda \in [0,1]$ and $u = \lim_k f(z_k)$, $w = \lim_k f(y_k)$. Let us choose an arbitrary $\delta > 0$. For any sufficiently large k we have $|x - z_k| < \delta$, $|x - y_k| < \delta$, so that $f(z_k), f(y_k) \in f(\mathcal{B}(x,\delta))$. It follows

$$\lambda f(z_k) + (1-\lambda) f(y_k) \in \text{co}\, f(\mathcal{B}(x,\delta))$$

and hence

$$v = \lim_k [\lambda f(z_k) + (1-\lambda) f(y_k)] \in \overline{\text{co}}\, f(\mathcal{B}(x,\delta))\ .$$

The proof is complete.

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The previous Lemma has an immediate consequence.

Lemma 2 *Let $f_1(x), f_2(x)$ be two continuous vector fields of $\mathbf{R}^n$. Let $f(x)$ be defined in such a way that for each $x \in \mathbf{R}^n$ either $f(x) = f_1(x)$ or $f(x) = f_2(x)$. Then,*

$$\mathbf{K}f(x) = \begin{cases} \{f(x)\} & \textit{if } f \textit{ is continuous at } x \\ \text{co}\{f_1(x), f_2(x)\} & \textit{otherwise.} \end{cases}$$

The set valued map $x \mapsto \mathbf{K}f(x)$ is upper semicontinuous, compact and convex valued. Hence, for each initial state there exists at least one maximal solution. We adopt the following definitions. We say that the discontinuous system (2) is *(strongly) stable* at the origin with respect to Krasowski solutions if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each Krasowski solution $\varphi(t)$ of (2), $\varphi(t)$ is continuable for $t \in [0, +\infty)$ and

$$|\varphi(0)| < \delta \implies |\varphi(t)| < \varepsilon\ , \quad \forall t \geq 0\ .$$

We say that system (2) is *globally (strongly) asymptotically stable* at the origin with respect to Krasovski solutions if it is stable and in addition, the following attractivity property holds:

$$\lim_{t\to+\infty} |\varphi(t)| = 0$$

for all the Krasowski solutions $\varphi(t)$ of (2). We need the following version of the invariance principle, which can be easily deduced from Theorem 3 of [3].

Lemma 3 *Let $V(x)$ be a positive definite, proper function of class C^1, and let*

$$\nabla V(x)v \le 0$$

for each $v \in \mathbf{K}f(x)$ and each $x \in \mathbf{R}^n$. Then, the origin is stable with respect to the Krasowski solutions of (2). Moreover, all the Krasowski solutions of (2) are attracted by the maximal invariant set contained in

$$Z = \{x \in \mathbf{R}^n : \exists v \in \mathbf{K}f(x) \text{ s.t. } \nabla V(x)v = 0\}\ .$$

Next we introduce our basic assumption about (1). We agree to say that a matrix A is *neutrally stable* if all the eigenvalues of A have nonpositive real part, and possible eigenvalues with zero real part are simple. When all the eigenvalues of A have negative real part, then A is said to be *Hurwitz*.

(H1) There exists a real number $\alpha \in (0,1)$ such that the matrix $A = \alpha A_1 + (1-\alpha)A_2$ is neutrally stable.

The following proposition is straightforward.

Proposition 1 *If $A = \alpha A_1 + (1-\alpha)A_2$ is neutrally stable, then there exists a symmetric, positive definite matrix P such that $x^{\mathbf{t}}(PA + A^{\mathbf{t}}P)x = 2x^{\mathbf{t}}PAx \le 0$ for each $x \in \mathbf{R}^n$.*

Let $B = A_1 - A_2$. Our second assumption is as follows.

(H2) Let P be as in Proposition 1. Then,

$$x^{\mathbf{t}}PA_1x = x^{\mathbf{t}}PA_2x = 0\ , \quad x \neq 0 \quad \Longrightarrow \quad \forall \beta \in [0,1]\ , \quad x^{\mathbf{t}}(PB + B^{\mathbf{t}}P)(\beta A_1 + (1-\beta)A_2)x \neq 0\ .$$

Now we can state the main result of this paper.

Theorem 1 *Concerning system (1), assume that* **(H1)** *and* **(H2)** *holds. Then there exists a static state (discontinuous) feedback $u = k(x)$ with values in the set $\{0,1\}$ such that the closed loop system is globally asymptotically stable at the origin with respect to Krasowski solutions.*

Proof Let α be as in Assumption **(H1)** and let P be as in Proposition 1. Recalling the definition of A, it is clear that

$$A_1 = A + (1-\alpha)B\ , \qquad A_2 = A - \alpha B\ .$$

System (1) can be therefore rewritten in the more familiar form

$$\dot{x} = (A + wB)x \tag{4}$$

where the new input w is allowed to take only the values $\{1-\alpha, -\alpha\}$. Systems with such a structure are usually named *bilinear systems*. The standard approach to their asymptotic stabilization exploits the so-called *damping feedback*, defined by the formula $w = -x^{\mathbf{t}}PBx$ ([11]). We just need to adapt this idea in order to met the required restrictions about w. Let

$$w = k(x) = \begin{cases} 1-\alpha & \text{if } x^{\mathbf{t}}PBx \le 0 \\ -\alpha & \text{if } x^{\mathbf{t}}PBx > 0\ . \end{cases} \tag{5}$$

In order to investigate stability with respect to Krasowski solutions of the closed loop system

$$\dot{x} = f(x) = (A + k(x)B)x \tag{6}$$

we make use of the (smooth) Liapunov function $V(x) = x^{\mathbf{t}}Px$. Let $\mathcal{O}^- = \{x \in \mathbf{R}^n : x^{\mathbf{t}}PBx < 0\}$. The set $\mathcal{O}^-$ is open and

$$f(x) = (A + (1-\alpha)B)x = A_1x$$

is obviously continuous on $\mathcal{O}^-$. Hence, by Lemma 2, $\mathbf{K}f(x) = \{f(x)\}$ and

$$\nabla V(x)f(x) = x^{\mathbf{t}}P(A + (1-\alpha)B)x + x^{\mathbf{t}}(A + (1-\alpha)B)^{\mathbf{t}}Px = 2x^{\mathbf{t}}PAx + 2(1-\alpha)x^{\mathbf{t}}PBx < 0$$

by virtue of **(H1)** and Proposition 1 (recall that $1-\alpha > 0$). A similar reasoning yields the same conclusion on $\mathcal{O}^+ = \{x \in \mathbf{R}^n : x^{\mathbf{t}}PBx > 0\}$. It remains to consider the closed set $\mathcal{O}^0 = \{x \in \mathbf{R}^n : x^{\mathbf{t}}PBx = 0\}$. Note that $x \in \mathcal{O}^0$ is equivalent to $x^{\mathbf{t}}PA_1x = x^{\mathbf{t}}PA_2x$. By Lemma 2 we now have $\mathbf{K}f(x) = \text{co}\,\{A_1x, A_2x\}$. Let $v = \gamma A_1x + (1-\gamma)A_2x$ (with $\gamma \in [0,1]$) be any vector in $\mathbf{K}f(x)$. Then,

$$\nabla V(x)v = 2x^{\mathbf{t}}Pvx = 2\gamma x^{\mathbf{t}}PA_1x + 2(1-\gamma)x^{\mathbf{t}}PA_2x = 2x^{\mathbf{t}}PA_2x = 2\alpha x^{\mathbf{t}}PA_1x + 2(1-\alpha)x^{\mathbf{t}}PA_2x = 2x^{\mathbf{t}}PAx \leq 0$$

again by **(H1)**and Proposition 1. Now Lemma 3 implies that the origin is stable, and that all the Krasowski solutions of (6) are attracted by the maximal invariant set $M \subseteq Z$. In order to complete the proof, we still have to verify that $M = \{0\}$. The previous computation clearly shows that

$$Z = \{x \in \mathbf{R}^n : \exists v \in \mathbf{K}f(x) \text{ s.t. } \nabla V(x)v = 0\} \subseteq \{x \in \mathbf{R}^n : x^{\mathbf{t}}PBx = 0\}\ . \tag{7}$$

Moreover, if $x \in Z$ then

$$x^{\mathbf{t}}PA_1x = x^{\mathbf{t}}PA_2x = x^{\mathbf{t}}PAx = 0\ . \tag{8}$$

Assume that there exists a nontrivial Krasowski solution $\varphi(t)$ of (6) lying on M. According to (7), then we have the identity

$$\varphi(t)^{\mathbf{t}}PB\varphi(t) \equiv 0\ . \tag{9}$$

Let t_0 be a point where φ is differentiable and $\dot{\varphi}(t_0) \in \mathbf{K}f(\varphi(t_0))$. Assume that $\varphi(t_0) \neq 0$. From (9) we obtain $\varphi(t_0)^{\mathbf{t}}(PB + B^{\mathbf{t}}P)\dot{\varphi}(t_0) = 0$ and hence, for some $\beta \in [0,1]$,

$$\varphi(t_0)^{\mathbf{t}}(PB + B^{\mathbf{t}}P)(\beta A_1 + (1-\beta)A_2)\varphi(t_0) = 0\ .$$

Taking into account of (8), this is a contradiction to **(H2)**.

∎

The following examples illustrate how the previous theorem can be applied. They are also emblematic in order to show certain features of switched systems.

Example 1 Let $n = 2$, and let us denote the state vector by $x = (x_1, x_2)$. The pair of linear systems defined by the matrices

$$A_1 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}\ , \qquad A_2 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

represent (both counterclockwise) harmonic oscillators. Every convex combination of A_1 and A_2 is neutrally stable. We can choose $\alpha = 1/2$ and $P = Id$. We also have

$$B = A_1 - A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $x^{\mathbf{t}}PA_1x = -x^{\mathbf{t}}PA_2x = x_1x_2$. In order to check condition **(H2)** we have therefore to focus on the set of points where $x_1 = 0, x_2 \neq 0$ or $x_1 \neq 0, x_2 = 0$. It is easy to compute

$$x^{\mathbf{t}}(PB + B^{\mathbf{t}}P)(\beta A_1 + (1-\beta)A_2)x = 2[(1+\beta)x_1^2 + (-2+\beta)x_2^2]\ .$$

Since $\beta \in [0,1]$, condition **(H2)** holds and Theorem 1 is applicable. As shown by Figure 1, for this example the flow of trajectories of the closed loop system can be equivalently described in terms of a genuine switching procedure. The locus of the switching points is given by $x^{\mathbf{t}}PBx = 2x_1x_2 = 0$.

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Example 2 As before, let $n = 2$ and let

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -7 \end{pmatrix}.$$

The convex combination $\alpha A_1 + (1-\alpha)A_2$ is Hurwitz for $\alpha \in (\frac{1}{2}, \frac{7}{8})$ and neutrally stable when α is at the endpoints of the same interval. We can choose again $\alpha = 1/2$ and $P = Id$. An easy computation leads to $x^{\mathbf{t}}PA_1x = -x_1^2 + x_2^2$ and $x^{\mathbf{t}}PA_2x = x_1^2 - 7x_2^2$. Thus, the unique common zero is the origin and assumption **(H2)** is trivially fulfilled. As far as matrix B is concerned, we have

$$B = A_1 - A_2 = \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}.$$

and $\mathcal{O}^0 = \{x^{\mathbf{t}}PBx = 2(2x_2 - x_1)(2x_2 + x_1) = 0\}$. The trajectories of the closed loop system reach the set $\mathcal{O}^0$ in finite time and then go toward the origin sliding on $\mathcal{O}^0$. As shown by the simulation of Figure 2, these solutions can be actually approximated by fast switching between integral curves of the two given vector fields.

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Although the statement of Theorem 1 is valid for arbitrary state dimension, the previous examples are two-dimensional: we point out that a quite deep analysis of two-dimensional switched systems is performed in [16].

3 Discussion and other examples

There are apparent analogies between Theorem 1 of this paper and the main result of [15] (see also Theorem 11 of [13] and Theorem 4.1 of [8]): however, both the assumptions and the conclusions differ. The result of [15] requires the existence of a Hurwitz convex combination of A_1, A_2, while in Theorem 1 a neutrally stable convex combination suffices. In this sense our theorem is more general (the result of [15] cannot applied to Example 1, for instance). Note that if there exists a Hurwitz convex combination of A_1, A_2, then there exists also a matrix P such that $x^{\mathbf{t}}PAx$ is negative definite. Hence, condition **(H2)** is automatically fulfilled. The second difference between Theorem 1 and [15] is the adopted notion of solution (Krasowski versus switched). The use of Krasowski solutions may originate sliding mode behavior (like in Example 2). In [15], occurrence of sliding modes is avoided by introducing hysteresis, which cannot be obtained by a genuine static state (memoryless) feedback. The problem of approximating a sliding mode motion by fast switching trajectories is classical in variable structure control.

The idea exploited in the proof of Theorem 1 is also different from the stabilization method proposed in [1], which is based on the existence of a quadratic control Liapunov function. For instance, it is possible to see that there is no quadratic control Liapunov functions for the system of Example 1. Stabilizability of bilinear systems is also studied in [2] by means of constant feedback, in [12] by means of piecewise constant feedback, and in [6], [14] by means of homogeneous (of zero order) feedback.

It is well known that the conclusion stated in Theorem 1 may be valid even if condition **(H1)** fails. An example can be constructed by taking a pair of unstable foci, as in [13]. We suggest here another interesting example.

Example 3 Let

$$A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

An arbitrary convex combination has the form

$$A = \begin{pmatrix} 1 & \alpha \\ -\alpha & 4\alpha - 1 \end{pmatrix}.$$

It is easy to check that for each $\alpha \in [0, 1]$, there is at least one eigenvalue with positive real part. Hence, condition **(H1)** is not satisfied. Nevertheless, the rule

$$f(x) = \begin{cases} A_1x & \text{if } x(2y - x) < 0 \\ A_2x & \text{if } x(2y - x) \geq 0 \end{cases}$$

generates an asymptotically stable flow of trajectories. This example deserves one more comment. The phase portrait of the first system is an unstable improper node, while the phase portrait of the second one is a saddle. There is only one pair of trajectories converging to the origin: those on the stable manifold of the saddle configuration. Any stabilizing switching rule must ultimately make use of one of these trajectories or of a very close one (see Figure 3). It is so clear that such a stabilizing policy suffers of a serious lack of robustness: a too large delay of the switching time may result in an unstable behavior.

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We now show that even condition **(H2)** is not necessary for the conclusion of Theorem 1.

Example 4 Let

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad A_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} .$$

A convex combination of A_1, A_2 takes the form

$$A = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix} .$$

It is neutrally stable for each $\alpha \in (0, 1)$. The matrix P is necessarily diagonal and uniquely defined, apart from a multiplicative constant:

$$P = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} .$$

Then $x^{\mathbf{t}}PBx = -x_1x_2$. We now look at condition **(H2)**. It amount to say that

$$x_1x_2 = 0 \quad (x_1^2 + x_2^2 \neq 0) \Longrightarrow \beta x_1^2 - x_2^2 \neq 0$$

which is false for $\beta = 0$. On the other hand, for any small $\varepsilon > 0$ the rule

$$f(x) = \begin{cases} A_1x & \text{if } x(y - \varepsilon x) \geq 0 \\ A_2x & \text{if } x(y - \varepsilon x) < 0 \end{cases}$$

defines a discontinuous vector field which is asymptotically stable in the Krasowski sense (the trajectories are plotted in Figure 4, with $\varepsilon = 1/2$). Notice that the switching locus of this rule can be considered a perturbation of the switching locus corresponding to the previous choice of P. In other words, in this case a delay of the switching time helps.

∎

The possibility of using the method suggested by the proof of Theorem 1 may depend on the choice of α and P.

Example 5 Let

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} , \qquad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

The convex combination of these matrices is

$$A = \begin{pmatrix} -\alpha & \alpha - 1 \\ 1 - \alpha & 0 \end{pmatrix}$$

whose eigenvalues have negative real part for each $\alpha \in (0, 1)$. One possible choice is $P = Id$. Then $x^{\mathbf{t}}PBx = -x_1^2$. In this case condition **(H2)** is not fulfilled. Indeed, it is easy to see that $x^{\mathbf{t}}PA_1x$, $x^{\mathbf{t}}PA_2x$, $x^{\mathbf{t}}(PB + B^{\mathbf{t}}P)(\beta A_1 + (1 - \beta)A_2)x$ all vanish for $x_2 \neq 0$, $x_1 = 0$. However, there are other possible choices for P. Letting $\alpha = 1/2$ for simplicity, we can take

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

(note that with this matrix P the inequality $x^{\mathbf{t}}PAx \leq 0$ is not strict; in fact, we do not need to dispose of a strict Liapunov function not even when we know it exists). Now we have

$$x^{\mathbf{t}}PBx = (x_2 - \sqrt{3}x_1)(x_2 + \sqrt{3}x_1) .$$

Condition (**H2**) is fulfilled since

$$x^{\mathbf{t}}PA_1x = -x(y+2x) \quad \text{and} \quad x^{\mathbf{t}}PA_1x = (x - \frac{1+\sqrt{5}}{2}y)(x - \frac{1-\sqrt{5}}{2}y)$$

have a common zero only at the origin. The trajectories of the closed loop system are plotted in Figure 5. ∎

We finally remark that (**H2**) is essentially a first order condition. It can be improved by iterating the reasoning and computing higher order derivatives, as in the classical Jurdjevic-Quinn method (see [11]).

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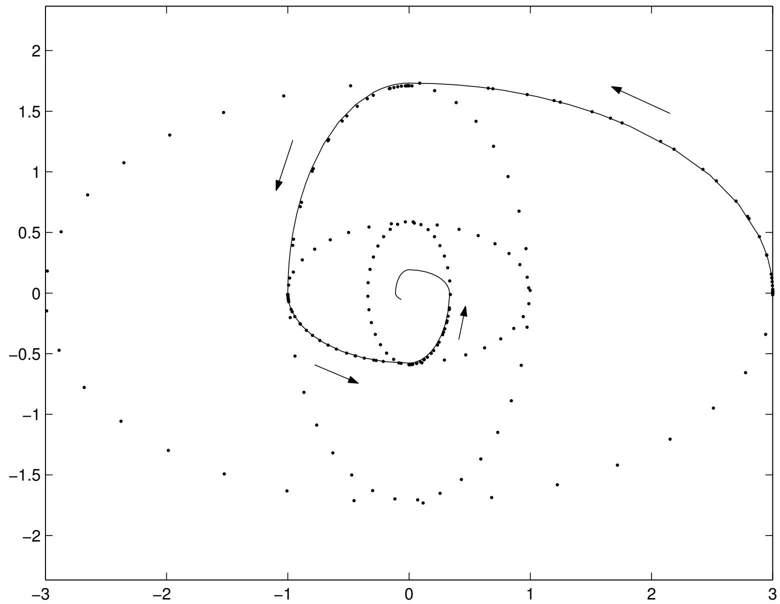

Figure 1: Example 1

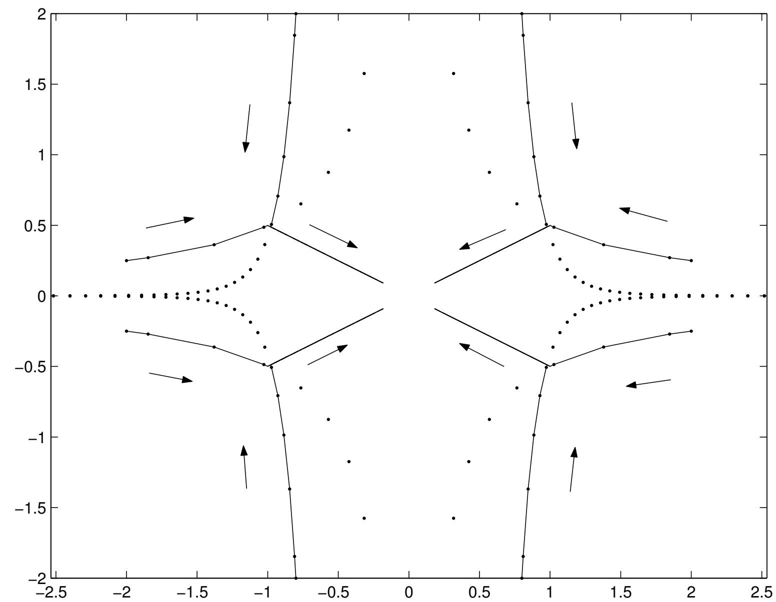

Figure 2: Example 2

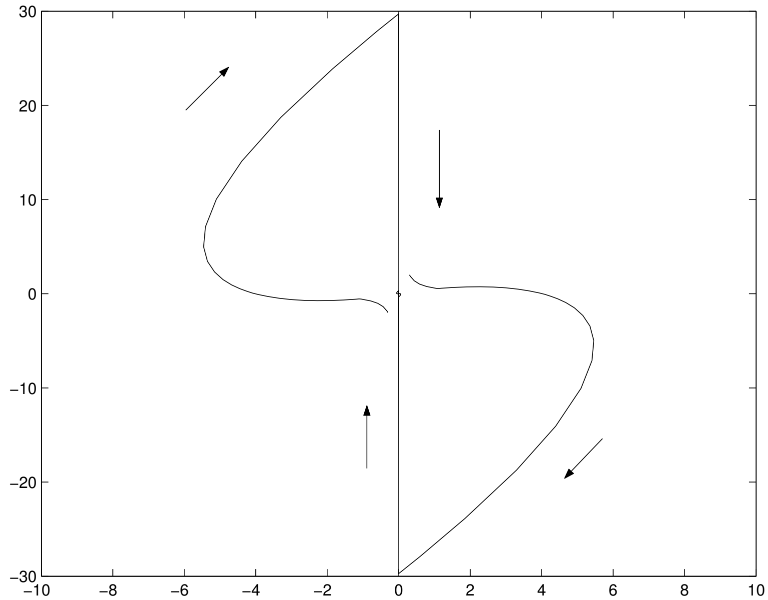

Figure 3: Example 3

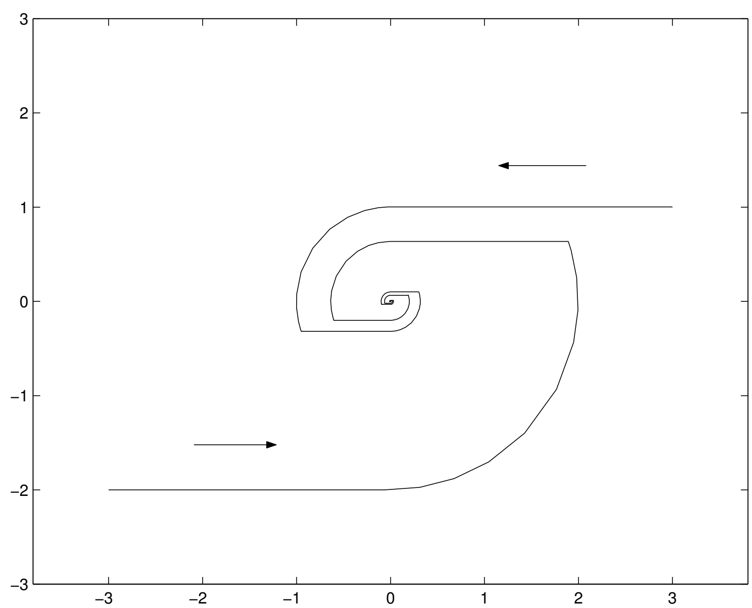

Figure 4: Example 4

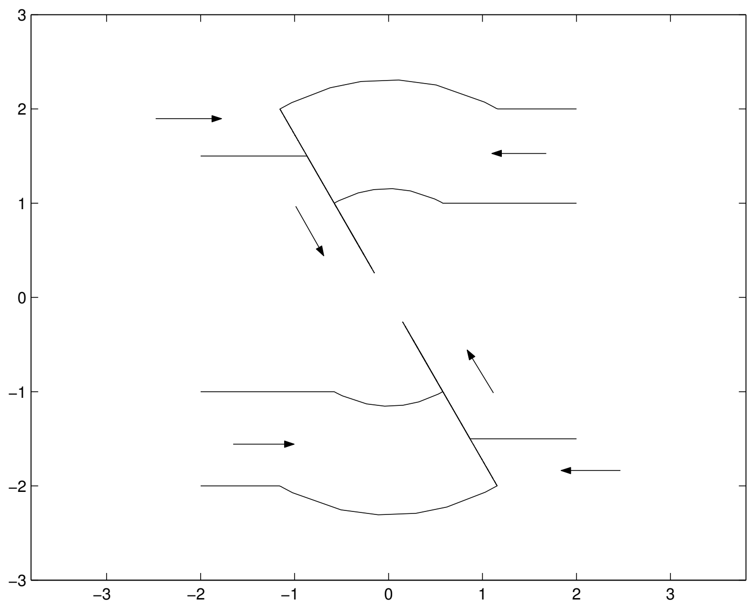

Figure 5: Example 5