An Introduction to Stability and Stabilization

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Abstract

In this lecture we discuss several definitions of stability for systems with inputs and for their unforced associated systems. We consider also the stabilization problem, both from the internal and the external point of view. All these properties are characterized in the framework of second Liapunov's method.

Although nonlinear systems are the main concern of this lecture, we start by a description of what happens in the linear case. By our exposition, we aim also to demonstrate the opportunity of introducing in the treatment differential inclusions and methods of nonsmooth analysis.

1 General framework

We are interested in physical input systems modeled by continuous-time, time-invariant, finite dimensional ordinary differential equations

$$\dot{x} = f(x, u) \tag{1}$$

where $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ represents the state variables, $u = (u_1, \ldots, u_m) \in \mathbf{R}^m$ represents the input variables and $f = (f_1, \ldots, f_n) : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$. Together with (1), we will often consider the *unforced associated system*

$$\dot{x} = f(x, 0)\ . \tag{2}$$

Basically, (2) accounts for the "internal" behavior of the system. More precisely, (2) describes the natural dynamics of (1) when no energy is supplied through the input channels. The analysis of the "external" behavior is rather concerned with the effect of the inputs (disturbances or exogenous signals) on the evolution of the state response of (1).

Physical systems are usually expected to exhibit a "stable" behavior. A first aim of this lecture is to survey some possible mathematical definitions of internal and external stability in a nonlinear setting and to discuss their relationships by the aid of the Liapunov functions method.

We will also consider the problem of achieving a more desirable stability behavior (both from the internal and the external point of view) by means of properly designed feedback laws. To this end, it is convenient to think of the input as a sum $u = u_{\rm e} + u_{\rm c}$. The term $u_{\rm e}$ represents external forces, while $u_{\rm c}$ is actually available for control action. Roughly speaking, (1) is said to be "stabilizable" if there exists a map $u_{\rm c} = k(x)$ such that the closed loop system

$$\dot{x} = f(x, k(x) + u_{\rm e}) \tag{3}$$

exhibits improved (internal and/or external) stability performances.

Summing up, investigation of the internal structure, description of the external behavior, stability analysis and feedback synthesis are therefore the main concern of this lecture. There is some evidence in classical linear systems theory that all these aspects are intimately related. Actually, dealing with nonlinear systems, the connections become weaker and need a more delicate treatment.

We shall see in particular that the approach to stability and stabilizability in the nonlinear case rests much more heavily on the method of Liapunov functions. Thus, we are led to enlighten the importance of having at our disposal a variety of theorems which state, under minimal assumptions, the existence of Liapunov functions with appropriate properties. These theorems are usually called "converse Liapunov theorems". A secondary aim of this lecture is to illustrate the state of the art on this subject, and to present some recent developments.

We have not yet specified what kind of assumptions should be made about the map f which appears at the right hand side of (1) and about the admissible inputs.

Throughout this notes, the class of admissible inputs is constituted by all measurable, essentially bounded functions $u : \ [0, +\infty) \to \mathbf{R}^m$.

To establish the assumptions about f is a more delicate task. In a classical "smooth" setting, it seems natural to ask that f is at least continuous with respect to both variables, though more regularity could be required for certain purposes. On the other hand, further recent developments point out that sometimes the smooth setting is restrictive.

For pedagogical reasons, we prefer not to star directly by the most general approach, which could be felt at this point unmotivated and too abstract. The need of abandoning the classical smooth setting and placing ourselves in a more general "nonsmooth" one, will be left to emerge as long as we proceed in our exposition. This choice will imply a few complications (for instance, the need of a progressive updating of definitions and results) but gives a clearer perspective of problems and theoretical difficulties.

Let us assume therefore, at least for the beginning, that f is continuous on $\mathbf{R}^n \times \mathbf{R}^m$. Then, for each admissible input and each initial state, well known theorems from ordinary differential equations theory guarantee the existence of at least one local solution in Carathéodory sense, that is an absolutely continuous curve which satisfies (1) almost everywhere.

We conclude this introductory section by some notation.

The set of all local solutions corresponding to a given admissible input $u(t)$ and to a given initial state x_0 is denoted by $\mathcal{S}_{x_0,u(\cdot)}$ (when $u \equiv 0$, as in (2), we simply write $\mathcal{S}_{x_0}$). When we need to emphasize the dependence of a solution $\varphi(t) \in \mathcal{S}_{x_0,u(\cdot)}$ on the initial state and the input $u(t)$, we shall use the notation $\varphi(t) = \varphi(t; x_0, u(\cdot))$. We emphasize that since f is assumed to be only continuous, nothing can be said in general about uniqueness of solutions.

The euclidean norm of a finite dimensional vector v is denoted by $|v|$. For $h > 0$, we shall write $\mathcal{B}_h = \mathcal{B}(0,h) = \{x \in \mathbf{R}^n : \ |x| < h\}$ and $\mathcal{B}^h = \{x \in \mathbf{R}^n : |x| > h\} = \mathbf{R}^n \backslash \overline{\mathcal{B}(0,h)}$. Moreover, for each measurable, essentially bounded function $u : [0, +\infty) \to \mathbf{R}^m$, we denote the L_∞ norm of u by

$$||u||_\infty = \operatorname{ess\,sup}_{t \geq 0} |u(t)| < +\infty \ .$$

2 The linear case

Linear systems have been widely studied for a long time and a rather complete picture of the theory is today available. It is natural to take it as a reference model for possible nonlinear developments.

2.1 Stability

In particular, the relationship between internal and external stability is well understood in the case of the finite-dimensional, time-invariant linear system

$$\dot{x} = Ax + Bu \tag{4}$$

(here, A and B are real matrices of appropriate dimensions). The unforced associated system has the form

$$\dot{x} = Ax. \tag{5}$$

It is well known that the asymptotic behavior of its solutions depends on the eigenvalues of A. More precisely, when A is *Hurwitz* (i.e., all its eigenvalues lie on the open left half complex plane) all the solutions converge to the origin for $t \to +\infty$. Moreover, when A is *stable* (i.e., all its eigenvalues lie on the closed left half complex plane and the possible eigenvalues on the imaginary axis are simple[1]) all the solutions are bounded and the solutions issuing from a sufficient small neighborhood of the origin remain near the origin for all $t \geq 0$.

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[1] This means that their algebraic and geometric multiplicities coincide.

Stability analysis of linear systems can be performed by means of quadratic Liapunov functions i.e., functions of the form $V(x) = x^{\mathbf{t}}Px$. Indeed, it is possible to prove that A is Hurwitz [stable] if and only if there exists a symmetric, positive definite matrix P such that $2x^{\mathbf{t}}PAX = x^{\mathbf{t}}(PA + A^{\mathbf{t}}P)x$ is negative definite [negative semi-definite].

If A is Hurwitz and if $u : [0, +\infty) \to \mathbf{R}^m$ is a measurable input function such that $||u||_\infty < +\infty$, then the variation of constants formula can be used to prove that there exist positive numbers $\alpha, \gamma_1, \gamma_2$ such that

$$|\varphi(t; x_0, u(\cdot))| \leq \gamma_1 |x_0| e^{-\alpha t} + \gamma_2 ||u||_\infty \tag{6}$$

for each $x_0 \in \mathbf{R}^n$ and $t \geq 0$. Conversely, if (6) holds for each initial state, each admissible input and each $t \geq 0$, then the special choice $u \equiv 0$ shows that A must be Hurwitz.

Inequality (6) admits the following interpretation: for t large enough, the effect of the initial conditions is negligible, and the solutions are ultimately bounded by a term which is related to the input energy (measured by its L_∞ norm) by means of the constant "gain" γ_2. This reflects the distinction between transient and steady state in the classical engineering literature.

Beside (6), we are also interested in the following condition: there exist some constants $\gamma_1, \gamma_2 > 0$ for which

$$|\varphi(t; x_0, u(\cdot))| \leq \gamma_1 |x_0| + \gamma_2 ||u||_\infty \tag{7}$$

for each $t \geq 0$, each x_0 and each measurable, essentially bounded input $u(\cdot)$.

Of course, (7) is implied by (6) and (7) in turn implies that A is stable.

Inequality (7) can be interpreted by saying that when the input is bounded, then the solutions are bounded by a term which depends linearly on the energy initially stored in the system (measured by the norm of the initial state) and the energy due to the input supply.

Inequalities (6) and (7) are appropriate tools in order to describe the effect of inputs and initial conditions on the evolution of the linear system defined by (4). Sometimes, we will refer to (6) [respectively, (7)] by saying that (4) has the *strong* [respectively, *weak*] *finite gain property.*

From the previous discussion, we can single out in particular the following conclusion.

Proposition 1 *If A is Hurwitz, then (4) has the weak finite gain property.*

The meaning of Proposition 1 is often referred to in an informal way by the following paradigm:

internal stability implies external stability.

2.2 External stabilization

The converse of Proposition 1 is true only under additional assumptions; for instance, if (4) is completely controllable. On the other hand, as already noticed, if (7) is valid then A is a stable matrix. Again, the converse is false, but we can obtain a positive result making use of linear feedback.

Proposition 2 *Assume that the matrix A is stable. Then, there exists a linear feedback $u_c = Fx$ such that the closed loop system satisfies (7).*

In other words, Proposition 2 states that if A in not Hurwitz, but at least stable, then the system can be externally stabilized (in the sense of the weak finite gain property).

A sketch of the proof of Proposition 2 can be carried out according to the following argument. Assume without loss of generality that (4) has been put in Kalman's canonical form and let A_{11} and A_{22} be respectively the diagonal blocks of A corresponding to its controllable part and its uncontrollable part. Then, (7) holds if and only if A_{11} is Hurwitz and A_{22} is stable. Hence, it follows immediately that (7) can be recovered under feedback action if and only if A_{22} is stable. The proof is concluded by the trivial remark that if A is stable then A_{22} is stable, as well.

2.3 Internal stabilization

If A is not even stable, we still have some hope to achieve external stability by making a different use of feedback. Indeed, we may ask whether there exists a linear feedback law $u = Fx$ such that all the trajectories of the unforced associated closed loop system converge to zero (i.e., the matrix $A + BF$ is Hurwitz). If such a matrix F exists, we also say that (4) is *internally stabilizable.*

Notice that, according to Proposition 1, if the system it is internally stabilizable, then the closed loop system has the property (6) (in fact, it has the strong finite gain property). Hence, we can formulate an updated version of our paradigm:

internal stabilization implies external stabilization.

Many different (but equivalent) conditions for internal stabilizability of a linear systems can be found in the literature. We recall the following one, which is of some interest for nonlinear developments.

Proposition 3 *The linear system (4) is internally stabilizable if and only if there exists a real, positive definite, symmetric matrix P such that*

$$\{x \in \mathbf{R}^n : \ x^{\mathbf{t}} P A x \geq 0\} \cap \ker B^{\mathbf{t}} P = \{0\} \ . \tag{8}$$

Moreover, if (8) holds then there exists $\alpha_0 > 0$ such that the stabilizing feedback can be taken of the form

$$u_c = -\alpha B^{\mathbf{t}} P x \tag{9}$$

for any $\alpha \geq \alpha_0$.

The "only if" part of Proposition 3 is easy. As already recalled, if the system admits a linear stabilizer $u = Fx$, then there is a quadratic Liapunov function $V(x) = x^{\mathbf{t}} P x$ for the closed loop system. Hence,

$$2(x^{\mathbf{t}} P A x + x^{\mathbf{t}} P B F x) < 0$$

for each $x \neq 0$. When $x \in \ker B^{\mathbf{t}} P$ this clearly reduces to $x^{\mathbf{t}} P A x < 0$.

A simple proof of the "if" part can be given when the system is in the so-called *critical position.* This means that A is stable but not Hurwitz (in other words, the set of simple eigenvalues on the imaginary axis is nonempty). Systems in critical position arise in certain applications. For instance, the mathematical model of a satellite in a circular orbit in a gravitational central field has a linearization of this type.

We know that A is stable if and only if there exists a symmetric, positive definite matrix P such that

$$x^{\mathbf{t}}(PA + A^{\mathbf{t}} P)x \leq 0$$

for each $x \in \mathbf{R}^n$. Define the feedback law as indicated in Proposition 3 with $\alpha = \frac{1}{2}$ and take $V(x) = x^{\mathbf{t}} P x$ as a Liapunov function for the closed loop system. We have

$$\dot{V}(x) = 2x^{\mathbf{t}} P(A - \frac{1}{2} B B^{\mathbf{t}} P)x = x^{\mathbf{t}}(PA + A^{\mathbf{t}} P)x - x^{\mathbf{t}} P B B^{\mathbf{t}} P x \ .$$

According to (8), it follows that $\dot{V}(x)$ is negative definite, as required.

The last conclusion can be restated by saying that $\dot{V}(x) = -x^{\mathbf{t}} Q x$ for some symmetric positive definite matrix Q. In other words, (8) implies that for some $Q > 0$, P is a solution of

$$PA + A^{\mathbf{t}} P - P B B^{\mathbf{t}} P = -Q \ . \tag{10}$$

This is a form of the so-called *algebraic Riccati matrix equation.* In turn, the existence of a positive definite solution of (10) for some $Q > 0$, clearly implies (8).

3 Nonlinear systems: stability

From now on, we turn our attention to nonlinear systems. In particular, this section is devoted to internal and external notions of stability, Liapunov functions and the related theorems.

3.1 Internal notions

Definition 1 *We say that (1) is* internally (Liapunov) stable *at the origin (or that the origin is internally stable for (1)) if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each* x_0 *with* $|x_0| < \delta$ *and all the solutions* $\varphi(\cdot) \in \mathcal{S}_{x_0}$ *the following holds:* φ *is right continuable for* $t \in [0, +\infty)$ *and*

$$|\varphi(t)| < \varepsilon \quad \forall t \geq 0 \ .$$

Note that if the origin is internally stable, then it is an equilibrium position for (2) i.e., $f(0,0) = 0$.

Definition 2 *We say that (1) is* internally Lagrange stable *(or that it has the property of uniform boundedness of solutions) if for each* $R > 0$ *there exists* $S > 0$ *such that for* $|x_0| < R$ *and all the solutions* $\varphi(t) \in \mathcal{S}_{x_0}$ *one has that* φ *is right continuable for* $t \in [0, +\infty)$ *and*

$$|\varphi(t)| < S \ , \quad \forall t \geq 0 \ .$$

In the linear case Liapunov stability and Lagrange stability are equivalent; in general, it should be clear that they are distinct properties.

Definition 3 *We say that system (1) is* internally, (locally) asymptotically stable *at the origin (or that the origin is internally (locally) asymptotically stable for (1)) if it is internally stable at the origin and, in addition, the following condition holds: there exists* $\delta_0 > 0$ *such that*

$$\lim_{t \to +\infty} |\varphi(t)| = 0$$

for each x_0 *such that* $|x_0| < \delta_0$*, and all the solutions* $\varphi(\cdot) \in \mathcal{S}_{x_0}$*.*

The origin is said to be internally, globally asymptotically stable *if* δ_0 *can be taken as large as desired.*

The definitions above are nothing else that the classical[2] definitions of Liapunov stability, Lagrange stability and asymptotic stability referred to the unforced system (2) associated with (1). For each concept of stability there is a corresponding concept of Liapunov function.

Definition 4 *A* weak Liapunov function in the small *is a real map* $V(x)$ *which is defined on* $\mathcal{B}_h$ *for some* h*, and fulfills the following properties:*

(i) $V(0) = 0$

(ii) $V(x) > 0$ *for* $x \neq 0$

(iii) $V(x)$ *is of class* C^1 *on* $\mathcal{B}_h$

(iv) $\nabla V(x) \cdot f(x, 0) \leq 0$ for each $x \in \mathcal{B}_h$.

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[2] When dealing with systems without uniqueness, one should distinguish between weak and strong notions. The previous definitions are *strong* notions in the sense that the properties are required to hold for all the solutions, and not only for some of them.

Recall now that a real function $V(x)$ is said to be *radially unbounded* if

$$\lim_{|x| \to +\infty} V(x) = +\infty \ .$$

Definition 5 *A function $V(x)$ defined on $\mathcal{B}^h$ for some $h > 0$, which is radially unbounded and fulfills (iii) and (iv) of Definition 4 (with with $\mathcal{B}_h$ replaced by $\mathcal{B}^h$), will be called a* weak Liapunov function in the large*.*

Definition 6 *A* strict Liapunov function in the small *is a weak Liapunov function which satisfies, instead of (iv),*

(v) $\nabla V(x) \cdot f(x, 0) < 0$ for each $x \in \mathcal{B}_h$ $(x \neq 0)$.

A function $V(x)$ defined for all $x \in \mathbf{R}^n$, which is radially unbounded and fulfills the properties (i), (ii), (iii), (v) with $\mathcal{B}_h$ replaced by $\mathbf{R}^n$, will be called a global strict Liapunov function*.*

Stability properties can be checked by means of an appropriate Liapunov function, according to the following well known criteria.

Theorem 1 *(First Liapunov Theorem) If there exists a weak Liapunov function in the small, then (1) is internally (Liapunov) stable.*

Theorem 2 *(Yoshizawa, [20]) If there exists a weak Liapunov function in the large, then (1) is internally Lagrange stable.*

Theorem 3 *(Second Liapunov Theorem) If there exists a strict Liapunov function in the small [in the large], then (1) is internally, locally [globally] asymptotically stable.*

3.2 Converse theorems

In this section we want to discuss briefly the question of the invertibility of the previous theorems (for other information and a more complete list of references, we address the reader to the recent papers [8], [9], [10], [13], [2]). First of all, we report the following classical result.

Theorem 4 *(Kurzweil, [12]) If (1) is internally, locally [globally] asymptotically stable then there exists a strict Liapunov function in the small [in the large]. In addition, such a function can be taken with continuous partial derivatives of any order.*

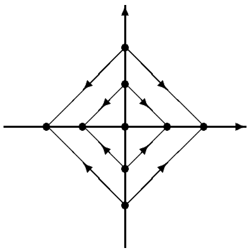
It is worth noticing that Kurzweil's Theorem provides a Liapunov function of class C^∞ in spite of f being only continuous. Unfortunately, such a strong result does not hold for Liapunov and Lagrange stability. Indeed, it is well known that there exist internally stable systems (even with C^∞ right hand side) which have no continuous Liapunov functions ([3]). The following example shows that the lack of continuity is not the only obstruction to the existence of smooth Liapunov functions.

Example. *A two-dimensional system stable at the origin which does not have a weak Liapunov function of class C^1.*

Let $\psi : \mathbf{R}^2 \to \mathbf{R}$ be a C^∞ function such that $\psi(x_1,x_2) > 0$ on $\{(x_1,x_2) : x_1x_2 \neq 0\}$, and $\psi(x_1,x_2)$ vanishes, together with its partial derivatives of any order on $\{(x_1,x_2) : x_1x_2 = 0\}$. Consider the vector field of $\mathbf{R}^2$

$$f(x_1,x_2) = \begin{cases} \begin{pmatrix} \psi(x_1,x_2) \\ -\psi(x_1,x_2) \end{pmatrix} & \text{for } x_1 \geq 0, x_2 \geq 0 \\ \begin{pmatrix} \psi(x_1,x_2) \\ \psi(x_1,x_2) \end{pmatrix} & \text{for } x_1 > 0, x_2 < 0 \\ \begin{pmatrix} -\psi(x_1,x_2) \\ -\psi(x_1,x_2) \end{pmatrix} & \text{for } x_1 < 0, x_2 > 0 \\ \begin{pmatrix} -\psi(x_1,x_2) \\ \psi(x_1,x_2) \end{pmatrix} & \text{for } x_1 \leq 0, x_2 \leq 0 \end{cases}$$

whose trajectories are shown in the figure.



It is clear that $f \in C^\infty$, and that the origin is (Liapunov and Lagrange) stable for the system

$$(\dot{x}_1, \dot{x}_2) = f(x_1,x_2) \ . \tag{11}$$

Assume now that there exists a C^1 weak Liapunov function $W(x_1,x_2)$ for (11). Then, for each $\epsilon > 0$ there must exist a number $\bar{x}_1$ with $0 < \bar{x}_1 < \epsilon$ such that

$$\frac{\partial W}{\partial x_1}(\bar{x}_1, 0) > 0 \ . \tag{12}$$

Otherwise, the function $x_1 \mapsto W(x_1,0)$ would be non-increasing and since $W(0,0) = 0$ the positive definiteness assumption is contradicted. Let us fix such a point $\bar{x}_1$. For any $x_2 > 0$, we have also

$$\frac{\partial W}{\partial x_1}(\bar{x}_1, x_2)\psi(\bar{x}_1, x_2) - \frac{\partial W}{\partial x_2}(\bar{x}_1, x_2)\psi(\bar{x}_1, x_2) \leq 0 \ .$$

Since $\psi \geq 0$, taking the limit for $x_2 \to 0$ we get

$$\frac{\partial W}{\partial x_1}(\bar{x}_1, 0) - \frac{\partial W}{\partial x_2}(\bar{x}_1, 0) \leq 0 \ . \tag{13}$$

On the other hand, for $x_2 < 0$ we have

$$\frac{\partial W}{\partial x_1}(\bar{x}_1, x_2)\psi(\bar{x}_1, x_2) + \frac{\partial W}{\partial x_2}(\bar{x}_1, x_2)\psi(\bar{x}_1, x_2) \leq 0 \ .$$

Arguing as before, this yields

$$\frac{\partial W}{\partial x_1}(\bar{x}_1, 0) + \frac{\partial W}{\partial x_2}(\bar{x}_1, 0) \leq 0 \ . \tag{14}$$

Comparing (13) and (14), we obtain

$$\frac{\partial W}{\partial x_1}(\bar{x}_1, 0) \leq 0$$

which is a contradiction to (12).

This example motivate the need for a more general definition of Liapunov function, at least for the "weak" versions. To this regard, we remark that property (iv) of Definition 4 is actually equivalent to say that:

(iv') for each solution $\varphi(t)$ of (2) defined on some interval I and lying in $\mathcal{B}_h$ for each $t \in I$, the composite map $V(\varphi(t))$ is non-increasing on I.

Of course, such a monotonicity condition can be stated without need of any differentiability (or even continuity) assumption about V. On the other hand, the reader should be warned that replacing (iv) by (iv'), has also an obvious disadvantage: indeed, in order to check the validity of (iv') one must solve explicitly a system of ordinary differential equations. Thus, if we are forced to use (iv'), a new problem arises: find appropriate tools which allows us to verify its validity. Such tools are the so-called generalized directional derivatives: they are typical objects of Nonsmooth Analysis ([5]).

If we agree to define weak Liapunov functions by requiring (iv') instead of (iv), it is immediate to verify that $V(x_1, x_2) = |x_1| + |x_2|$ is a continuous (actually, globally Lipschitz continuous) Liapunov function for (11).

Converse theorems which guarantee the existence of smooth Liapunov functions for Liapunov and Lagrange stability are known only in the one dimensional case and in the linear case. As far as the general case is concerned, we report the following result.

Theorem 5 *(Yorke, [19]) Assume that the uniqueness of solutions property holds for (2). Then, if (1) is internally, Liapunov [Lagrange] stable then there exists a semi-continuous weak Liapunov function in the small [in the large].*

The definition of weak Liapunov function can be further generalized.

Definition 7 *Let* $h > 0$. *A function* $V : \mathcal{B}_h \to \mathbf{R}$ *is called a* generalized Liapunov function in the small *if it satisfies (i), (iv') and, in addition, the following two properties:*

(ii') for some $\eta < h$ *and for each* $\sigma \in (0, \eta)$ *there exists* $\lambda > 0$ *such that* $V(x) > \lambda$ *when* $\sigma \leq |x| \leq \eta$

(iii') $V(x)$ *is continuous at* $x = 0$.

It is easy to see that (ii') is satisfied if $V(x)$ is, for instance, lower semi-continuous and positive definite. The definition of weak Liapunov function in the large is similar. The following theorem holds.

Theorem 6 *(Auslander-Seibert, [3]) System (1) is internally Liapunov [Lagrange] stable if and only if there exists a generalized Liapunov function in the small [in the large].*

In [3], it is also shown that the existence of a Liapunov function continuous on a whole neighborhood of the origin is equivalent to a strengthened form of stability, the so-called *absolute stability.*

3.3 External notions

In order to formalize the general notions of external stability, we first need to introduce certain classes of functions to be used as comparison or "gain" functions.

A function $\alpha : [0, r_1) \to [0, +\infty)$ is said to belong to the class $\mathcal{K}_0$ if it is continuous, strictly increasing and $\alpha(0) = 0$. Here, r_1 may be a positive number or $+\infty$ and may depend on α. When $\alpha \in \mathcal{K}_0$ and $r_1 = +\infty$, we shall say that α is of class $\mathcal{K}$.

Moreover, a function $\alpha : [0, +\infty) \to [0, +\infty)$ is said to belong to the class $\mathcal{L}$ if it is continuous, decreasing and satisfies $\lim_{r \to +\infty} \alpha(r) = 0$.

Further, a function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is said to belong to the class $\mathcal{LK}$ if it is of class $\mathcal{L}$ with respect to the first variable and of class $\mathcal{K}$ with respect to the second one.

Finally, we shall say that a function $\alpha : [r_2, +\infty) \to [0, +\infty)$ (with $r_2 \geq 0$) is of class $\mathcal{K}^\infty$ if it is continuous, strictly increasing and $\lim_{r \to +\infty} \alpha(r) = +\infty$.

We shall also set $\mathcal{K}_0^\infty = \mathcal{K}_0 \cap \mathcal{K}^\infty = \mathcal{K} \cap \mathcal{K}^\infty$.

The following are natural extensions of the weak and strong finite gain properties, respectively.

Definition 8 *We say that (1) is* UBIBS-stable *(i.e., uniformly bounded-input bounded-state stable) if for each* $R > 0$*, there exists* $S > 0$:

$$|x_0| < R \ , \ ||u(\cdot)||_\infty < R \quad \Longrightarrow |\varphi(t)| < S \ ,$$

$\forall t > 0$ *and for each solution* $\varphi(t) \in \mathcal{S}_{x_0, u(\cdot)}$.

An equivalent definition can be given in terms of comparison functions (see [1]).

It is obvious that if (1) is UBIBS-stable, then it is internally Lagrange stable, but the converse is false in general.

Definition 9 *We say that system (1) possesses the* input-to-state stability *(in short, ISS) property, or that it is ISS-stable, if there exist maps* $\beta \in \mathcal{LK}$, $\gamma \in \mathcal{K}$ *such that, for each initial state* x_0*, each measurable, essentially bounded input* $u : [0, +\infty) \to \mathbf{R}^m$ *and each* $t \geq 0$

$$|\varphi(t; x_0, u(\cdot))| \leq \beta(t, |x_0|) + \gamma(||u||_\infty) \ .$$

It is not difficult to see that the ISS property implies that:

1) the system is internally, globally asymptotically stable

2) the system is UBIBS-stable

On the contrary, there are examples of systems which are internally asymptotically stable and UBIBS-stable but not ISS-stable.

Even for UBIBS stability and the ISS property we can conceive appropriate Liapunov-like functions.

Definition 10 *A* UBIBS-Liapunov function *for (1) is a radially unbounded* C^1 *function* $V : \mathbf{R}^n \to \mathbf{R}$ *which enjoys the following property: for each* $R > 0$ *there exists* $\rho > 0$ *such that*

$$\dot{V}(x, u) = \nabla V(x) \cdot f(x, u) \leq 0 \tag{15}$$

for each $x \in \mathbf{R}^n$ *with* $|x| > \rho$ *and each* $u \in \mathbf{R}^m$ *with* $|u| < R$.

Definition 11 *A* ISS-Liapunov function *for (1) is a positive definite, radially unbounded* C^1 *function* $V : \mathbf{R}^n \to \mathbf{R}$ *which enjoys the following property: there exists a function* $\rho \in \mathcal{K}$ *such that for all* $x \in \mathbf{R}^n$ *(*$x \neq 0$*) and* $u \in \mathbf{R}^m$*, if* $|x| \geq \rho(|u|)$ *then*

$$\nabla V(x) \cdot f(x, u) < 0 \ .$$

The following results (Theorems 7 and 8) can be considered "external" versions of Theorems 2 3, respectively.

Theorem 7 *([6]) Assume that system (1) possesses a UBIBS-Liapunov function. Then, it is UBIBS-stable.*

A partial converse of the previous theorem has been recently given in [BM]: under the assumption that (1) satisfies a uniqueness assumption, it is proven that UBIBS-stability implies the existence of a radially unbounded, upper semi-continuous function $V(x)$ which satisfies, instead of (15), the following monotonicity property:

$\forall R > 0, \exists \rho > 0$ *such that for each admissible input* $u(t)$, *each solution* $\varphi(t)$ *defined on an interval* I *and corresponding to* $u(t)$ *and each* $t \in I$, *one has that* $V(\varphi(t))$ *is non-increasing on* I, *provided that* $|u(t)| < R$ *and* $|\varphi(t)| > \rho$ *for each* $t \in I$.

The problem of the invertibility of Theorem 7 is far from being trivial even in the linear case. Indeed, there are examples of linear systems with the weak finite gain property, for which there exists no quadratic UBIBS-Liapunov functions.

Theorem 8 *(Sontag, [14]) Assume that there exists a ISS-Liapunov function for (18); then the system is ISS-stable.*

In fact, under the assumption that $f(x, u)$ is locally Lipschitz continuous, also the converse of Theorem 8 holds (see [16], [17], where many other characterizations of the ISS property are given).

Concerning the ISS property, the situation looks to be better even in the linear case, where one expects to find quadratic Liapunov functions. Indeed, it is possible to prove that a linear system has the strong finite gain property if and only if there exists a quadratic ISS-Liapunov function.

4 Nonlinear systems: stabilization

As explained in Sect. 1, and as already illustrated for the linear case, if the system is not stable (in some sense) we can try to achieve the desired property by the implementation of a feedback connection.

4.1 Basic definitions

We are interested in the following notions.

Definition 12 *System (1) is said to be* internally, (locally or globally) asymptotically stabilizable *if there exists a feedback* $u_c = k(x)$ *such that the closed-loop system (3) is internally, (locally or globally) asymptotically stable.*

Definition 13 *We shall say that (1) is* ISS-[UBIBS-]stabilizable *if there exists a feedback* $u_c = k(x)$ *such that the closed-loop system (3) is ISS-[UBIBS-]stable.*

There is a point in the previous definitions which intentionally has been left unspecified, namely the regularity requirements about the feedback law. Obviously, in a smooth setting, one could be tempted to pretend that $k(x)$ is at least as regular as the right hand side of (4) is. Unfortunately, results of this type cannot be achieved in general (see [4]). This claim is illustrated, for instance, by the following celebrated example

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_2 - x_1^3 \ . \end{cases}$$

It cannot be stabilized by a feedback of class C^1, as a simple linearization argument shows, but there are explicit continuous local asymptotic stabilizers.

There are also simple examples of systems which are of class C^r for each preassigned integer value of $r > 0$, and which can be stabilized only by discontinuous feedback. Take for instance ($x \in \mathbf{R}$)

$$\dot{x} = x^{2r+1} - 2ux^{2r}|x| \ .$$

The feedback law $u = \operatorname{sgn} x$ renders the origin asymptotically stable, but if $u(x)$ is continuous, it is not difficult to see that there is an interval of the form $(0, \epsilon)$ or $(-\epsilon, 0)$ where the right hand side has the same sign than x.

Finally, combining Proposition 3.1 of [11] and Theorem 1 of [18], we obtain an example of an analytic, two-dimensional system which admits a discontinuous stabilizer, but not a continuous one.

The reader will notice that a serious problem now arises: if discontinuous feedback is allowed, then the right hand side of the closed loop system is discontinuous with respect to the state. Hence, the classical theory of ordinary differential equation is no more sufficient. In fact, also the notion of solution needs to be re-defined. A very popular notion of generalized solution for discontinuous differential equations is due to Filippov. For the purposes of this lecture, we can limit ourselves to a short sketch about Filippov's approach. A differential equation

$$\dot{x} = g(x)$$

with discontinuous right hand side is transformed in a differential inclusion

$$\dot{x} \in G(x)$$

where $G(x)$ is defined as the closed convex hull of all the limit of sequences of the form $\{g(x_i)\}$ for $x_i \to x$, except those limits which are attained only on some set of zero measure.

If not differently stated, when we deal with discontinuous equations, we always agree to adopt the notion of Filippov solution. However, it should be noticed that other different notions of generalized solutions have successfully been used in stabilization theory, as well.

We conclude this section by a variant of the notion of Liapunov function.

Definition 14 *We say that (1) satisfies a* control Liapunov condition in the small *(or that (1) has a control Liapunov function in the small) if there exist* $h > 0$ *and a radially unbounded, positive definite,* C^1 *function* $V(x)$ *with the following property: for each* x *with* $|x| < h$ *there exists* $u \in \mathbf{R}^m$ *such that*

$$\nabla V(x) f(x, u) < 0 \ . \tag{16}$$

If h *can be taken as large as we want, we shall say that (1) satisfies a* global control Liapunov condition*.*

We have already encountered an example of control Liapunov function dealing with linear systems. Indeed, it is not difficult to recognize, that $V(x) = x^{\mathbf{t}}Px$ is a global control Liapunov function for (4) if and only if (8) holds.

4.2 External stabilization

The claim that the external behavior is determined by the internal one is no more valid in the nonlinear case. This is shown for instance by the following simple example

$$\dot{x} = -x + ux^3 \ . \tag{17}$$

The unforced system is globally asymptotically stable but with $u \equiv 1$ there are unbounded solutions. However, some connections can be recovered provided that the use of feedback is allowed.

For the final part of this lecture, we limit ourselves to consider single input (i.e., with $m = 1$) *affine systems*, that is systems of the form

$$\dot{x} = f_0(x) + uf_1(x) \tag{18}$$

where $f_0(x)$ and $f_1(x)$ are continuous vector field of $\mathbf{R}^n$.

The first step on our way is the following theorem.

Theorem 9 *(Sontag, [14]) Assume that a global, strict Liapunov function* $V(x)$ *for the unforced system associated to (18) is known. Then, there exists a continuous feedback* $u_c = k(x)$ *such that the same* $V(x)$ *is a ISS-Liapunov function for the closed loop system*

$$\dot{x} = f_0(x) + [k(x) + u_e]f_1(x) \ . \tag{19}$$

The proof of the previous theorem is constructive. For instance, one can take $k(x) = -a(x) \cdot b(x)/2$ where $a(x) = -\nabla V(x) \cdot f_0(x)$ and $b(x) = \nabla V(x) \cdot f_1(x)$ (see [14] for more details). Note that under the assumption of Theorem 9 and by virtue of Theorem 8, the closed loop system (19) turns out to be the ISS-stable.

Note also that if (18) is internally, globally asymptotically stabilizable by a continuous feedback, then Kurzweil's Theorem guarantees the existence of a global, strict Liapunov function $V(x)$ of class C^∞ for the (unforced) closed loop system, so that a ISS stabilizer can be found with the same regularity as f_0 and f_1.

In conclusion, we arrive at the following important result.

Theorem 10 *(Sontag, [14]) Every globally, internally asymptotically stable (or continuously stabilizable) affine system of the form (18) is ISS-stabilizable.*

Next, we want to discuss the possibility of repeating, as far as possible, the same reasoning concerning the UBIBS property. Looking at Proposition 2, and recalling that in the linear case internal (Liapunov) stability is equivalent to Lagrange stability, it seems natural to identify Lagrange stability as the right internal property to be compared UBIBS-stability.

Next theorem is an analogous of Theorem 9 for UBIBS-stability.

Theorem 11 *([6]) Assume that a weak Liapunov function in the large $V(x)$ for the unforced system associated to (18) is known. Then, there exists a feedback law $u_c = k(x)$ such that the same $V(x)$ is a UBIBS-Liapunov function for the closed loop system (19).*

The feedback law mentioned in Theorem 11 can be taken, for instance, of the form

$$k(x) = -|x| \operatorname{sgn} \left(\nabla V(x) \cdot f_1(x) \right) .$$

It must be emphasized that this expression is not continuous in general. Fortunately, Theorem 7 remains valid even if the right hand side of the system possesses discontinuities with respect to x and the solutions are intended in Filippov's sense.

To complete the picture, one would need a converse theorem for Lagrange stability. However, we already know that it is possible to construct semi-continuous Liapunov functions for Lagrange stable systems, but further regularity cannot be guaranteed, in general. In conclusion, the better we can do is the following.

Theorem 12 *([6]) Assume that the unforced system associated to (18) is Lagrange stable and that it admits a weak Liapunov function in the large. Then, it is UBIBS-stabilizable.*

4.3 Internal stabilization

We want now to discuss the possibility of generalizing Proposition 3 to nonlinear systems.

Theorem 13 *(Artstein-Sontag, [15]) There exists a global control Liapunov function for system (18) (if and) only if the system is globally asymptotically stabilizable by a continuous feedback $u = k(x)$.*

If the vector fields f_0 and f_1 are of class C^q $(0 \leq q \leq +\infty)$ and a control Liapunov function of class C^r $(1 \leq r \leq +\infty)$ is known, the stabilizing feedback whose existence is ensured by Artstein-Sontag Theorem, can be explicitly constructed according to the following "universal" formula

$$k(x) = \begin{cases} 0 & \text{if } b(x) = 0 \\ \frac{a(x) - \sqrt{a^2(x) + b^4(x)}}{b(x)} & \text{if } b(x) \neq 0 \end{cases}$$

where, $a(x)$ and $b(x)$ are defined as before. We emphasize that such $k(x)$ is of class C^s (with $s = \min\{q, r-1\}$) in a punctured neighborhood of the origin. Under an additional (very reasonable) assumption, it can be proved that such a feedback law is continuous at $x = 0$. But further regularity at the origin can be obtained only in very special situations.

It is worth noticing that the universal formula above has a powerful regularizing property. Indeed, if a continuous stabilizer for (18) is known, then Kurzweil's Converse Theorem applies. Hence, the existence of a C^∞ strict Liapunov function $V(x)$ for the closed loop system is guaranteed. It is not difficult to see that the same $V(x)$ is a control Liapunov function for (18). But then, the universal formula can be applied with this $V(x)$, and we obtain a new stabilizing feedback with the same order of differentiability as f_0 and f_1 (at least for $x \neq 0$).

In any case, there is no hope that the feedback law could be represented by the simple expression

$$u = -\alpha \nabla V(x) \cdot f_1(x) \tag{20}$$

which is the natural extension of (9). See for instance the simple example

$$\dot{x} = x + ux \ . \tag{21}$$

Nevertheless, feedback laws of the form (20) have an important role to play in nonlinear stabilization. Apart from the fact that the feedback laws used in Theorems 9 and 12 are modifications of (20), the claim is supported by the following Theorem 14.

Assume that the origin is a stable equilibrium for the unforced system associated to (18), and assume that a C^1 Liapunov function $V(x)$ such that $\nabla V(x) \cdot f_0(x) \leq 0$ for each x has been found. The set $\{x : \nabla V(x) \cdot f_0(x) = 0\}$ will be called the *bad set*. In a similar manner, for any feedback law $u = k(x)$ we can consider the bad set of the closed loop system.

Theorem 14 *(Jurdjevic-Quinn) Assume that a weak Liapunov function of class C^1 for the unforced system associated to (18) is known, and let $u_c = k(x)$ be* *given by (20). Then, for any $\alpha > 0$, the bad set of the closed loop system is contained in the bad set of the unforced system associated to (18).*

The meaning of Jurdjevic-Quinn Theorem is that the stability performances of the closed loop system are in general better (or at least are not worse) than those of the unforced associated system. Moreover, under additional assumptions, the feedback law of the form (20) can actually stabilize the system at $x = 0$ (see [4]).

A final comment about Theorem 14 is appropriate. According to what already seen in Sect. 3.2, when the unforced system has a Liapunov stable equilibrium position at the origin the existence of a smooth weak Liapunov function in the small cannot be given for sure. If we have at disposal a Liapunov function which is at least locally Lipschitz continuous, then (20) can be defined almost everywhere. It can be proved that if $f_0(x)$ is continuous, then the conclusion of Jurdjevic-Quinn Theorem remain valid. Note that in these conditions, in general (20) turns out to be discontinuous.

5 Conclusions

We saw that in the linear case, information about the internal structure are useful in order to predict the external behavior and to design stabilizing feedback. However, in the nonlinear case the situation is more involved.

As far as we deal with "strict" notions such as asymptotic stability and the ISS property, a rather complete picture can be reconstructed, provided that the use of feedback is allowed. On the contrary, when we deal with "weak" notions (Liapunov and Lagrange stability, UBIBS stability), such a reconstruction cannot be performed in a satisfactory way in a classical smooth setting. However, a more complete theory seems to be possible in the framework of Nonsmooth Analysis. Among the reasons for which abandoning the smooth point of view seems to be advisable, we recall:

- the lack of regularity in converse Liapunov theorems
- the fact that in certain circumstances, discontinuous stabilizers cannot be avoided.

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