Recurrent switching rules for pairs of linear planar vector fields

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Abstract: Given a switched system formed by a pair of (not asymptotically stable) linear vector fields of $\mathbf{R}^2$, we define a special class of hybrid feedback laws (called recurrent switching rules) whose structure depends on some conic regions with nonempty interior. We prove that if the system is asymptotically controllable and radially controllable, then it is robustly stabilized by any recurrent switching rule, provided that the conic regions are properly designed.

Keywords: Switched systems, robust stabilizability, state-dependent switching rules.

1. INTRODUCTION

In this note we are concerned with the state space interpretation and the related robustness properties of stabilizing eventually periodic switching signals for families of linear vector fields. Eventually periodic stabilizing signals were introduced in Bacciotti and Mazzi (2010, 2011). Their interest rests on two facts.

(1) Any family of linear vector fields asymptotically controllable and satisfying an independent mild finite time controllability condition can be stabilized by an eventually periodic switching signal (Bacciotti and Mazzi (2011)).
(2) For an eventually periodic switching signal, the dependence on the initial state is limited to an initial transient interval (whose length can be predicted): during the subsequent steady-state, the signal is periodic and hence can be implemented in automatic way.

Eventually periodic switching signals are essentially time-dependent control rules, but can be reformulated as state-dependent hybrid feedbacks (in the sense of Goebel et al. (2009)). However, since they typically exploit the stable subspaces of certain associated discrete time dynamical systems, the simulations often reveal a lack of robustness. This note is a first attempt to indicate how this drawback can be overcome.

We limit ourselves to families formed by two linear vector fields in the plane. We examine in particular the periodic steady state of an eventually periodic stabilizing switching signal. The stable subspaces of the associated discrete time systems are replaced by certain conic regions, which are going to play the role of *switching loci.* Since these conic regions have nonempty interior, more robustness with respect to state measurement errors is guaranteed.

The state-dependent switching rules defined in this way can be interpreted as hybrid feedbacks, as well; because of their special structure, they will be called here *recurrent switching rules.*

As usual, we complete this introduction by a short summary of the content of the following sections. In Section 2 we recall the basic notions and the terminology; we also illustrate the problem and report some results of Bacciotti and Mazzi (2011). The notions of conic neighborhood and recurrent switching rule are crucial for the developments of this note: they will be given in Sections 3 and 4. Section 4 contains also the main result, while Section 5 is devoted to the proof: for sake of brevity, we omit the proof of some preliminary lemmas, whose statements are intuitively clear.

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2. PRELIMINARIES AND ILLUSTRATION OF THE PROBLEM

For simplicity, we specialize the exposition to the case at hand (pairs of linear vector fields in the plane): the reader is referred to Bacciotti and Mazzi (2010, 2011) for further details, comments and examples in a more general setting.

2.1 Switching signals

By *switching signal* we mean a piecewise constant, right continuous function $\sigma : [0,+\infty) \to \{1,2\}$. The points where σ is discontinuous (if any) lie on $(0,+\infty)$; they are called the *switching times* of σ. We assume that they are indexed in such a way that $t_1 < t_2 < \ldots$. Moreover, we usually denote by $n_i \in \{1,2\}$ the value of σ on the interval $[t_{i-1}, t_i)$ (for notational consistency, we set $t_0 = 0$ and, in case of finitely many switching times $t_1, \ldots, t_{i_{max}}$, we agree that $t_{i_{max}+1} = +\infty$). The set of all the switched signals is denoted by $\mathcal{S}$.

A switching signal is said to be *periodic of period* $T > 0$ if $\sigma(t) = \sigma(t+T)$, for all $t \geq 0$. The set of all the periodic switching signals is denoted by $\mathcal{P}$.

A switching signal ρ is said to be *eventually periodic* if there exist a periodic switching signal σ and a time $\bar{t} > 0$

such that $\rho(t+\bar{t}) = \sigma(t)$, for all $t \geq 0$. We say that σ is the *periodic part* of ρ and that $\rho\big|_{[0,\bar{t})}$ is the *pre-periodic part* of ρ. We also say that $\bar{t}$ is the *pre-period.* We denote by $\mathcal{E}(\sigma)$ the set of all the eventually periodic switching signals having the same periodic part σ.

2.2 Switched systems

For the purposes of this note, a *linear switched system* in $\mathbf{R}^2$ is a pair $(\mathcal{F}, \Sigma)$ where:

- $\mathcal{F}$ denotes a pair of linear vector fields, that is $\mathcal{F} = \{f_1(x), f_2(x)\}$ where
$$\dot{x} = f_1(x) = A_1 x \ , \qquad \dot{x} = f_2(x) = A_2 x \qquad (1)$$
A_1, A_2 being square 2×2 real matrices, and $x \in \mathbf{R}^2$;
- Σ is a map assigning to each $x_0 \in \mathbf{R}^2$ an element of $\mathcal{S}$, denoted $\Sigma_{x_0}(t)$.

The map Σ is also called a *time-dependent switching rule.* Let the system $(\mathcal{F}, \Sigma)$ and the initial state $x_0 \in \mathbf{R}^2$ be given. Let $t_1, t_2, \dots$ and $n_1, n_2, \dots$ be respectively the sequence of the switching times and the sequence of the values of $\Sigma_{x_0}(t)$. The continuous curve
$$t \mapsto \Phi(t, x_0, \Sigma_{x_0}) : [0, +\infty) \to \mathbf{R}^2$$
such that for each $i = 0, 1, 2, \dots$ and $t \in [t_i, t_{i+1})$
$$\Phi(t, x_0, \Sigma_{x_0}) = e^{(t-t_i)A_{n_{i+1}}} \Phi(t_i, x_0, \Sigma_{x_0})$$
$$= e^{(t-t_i)A_{n_{i+1}}} e^{(t_i - t_{i-1})A_{n_i}} \dots e^{t_1 A_{n_1}} x_0 \ ,$$
is called the *switched trajectory* generated by $(\mathcal{F}, \Sigma)$ and corresponding to x_0.

2.3 Periodic switched systems

When we have a switched system with a constant $\Sigma_{x_0}(t) \equiv \sigma(t)$ i.e., when the same switching signal $\sigma(t)$ is assigned to each initial state x_0, we say that the systems is *consistent* (see Sun and Ge (2005)) and simply write $(\mathcal{F}, \sigma)$ instead of $(\mathcal{F}, \Sigma)$. A consistent switched system $(\mathcal{F}, \sigma)$ is said to be *periodic* if $\sigma(t)$ is periodic. A periodic switched system $(\mathcal{F}, \sigma)$ naturally defines a linear operator $\Phi = \Phi(T, \cdot, \sigma)$ (where T is a period of σ) and a discrete time dynamical system
$$x_{k+1} = \Phi x_k \ , \qquad k = 0, 1, 2, \dots \ . \qquad (2)$$

Φ and (2) will be called respectively the *linear operator* and the *discrete time dynamical system associated to* $(\mathcal{F}, \sigma)$. A linear subspace W of $\mathbf{R}^2$ is called a *stable subspace* of a periodic system $(\mathcal{F}, \sigma)$ if it is invariant for (2) and $\lim_{k\to\infty} \Phi^k x_0 = 0$ for every $x_0 \in W$. A stable subspace is *nontrivial* if it is not reduced to the origin.

2.4 Eventually periodic switched systems

A switching rule Σ is called *eventually periodic* if there exists a periodic switching signal σ such that $\Sigma_{x_0} \in \mathcal{E}(\sigma)$, for each $x_0 \in \mathbf{R}^2$. The switched system $(\mathcal{F}, \Sigma)$ is called *eventually periodic* if Σ is eventually periodic. Note that if $(\mathcal{F}, \Sigma)$ is near-periodic, the switching signals Σ_{x_0} have the same periodic part for all x_0, but the pre-period and the pre-periodic part may depend on x_0. The interest in eventually periodic switched systems is motivated by the application to stabilization (see Bacciotti and Mazzi (2011)).

Let $\mathcal{U} \subseteq \mathcal{S}$. The family $\mathcal{F}$ is said to be *asymptotically controllable relatively to* $\mathcal{U}$ if there exists $\Sigma : \mathbf{R}^2 \to \mathcal{U}$ such that for each x_0
$$\lim_{t\to+\infty} \Phi(t, x_0, \Sigma_{x_0}) = 0 \ .$$

Assuming that $(\mathcal{F}, \Sigma)$ is asymptotically controllable relatively to $\mathcal{S}$, it is interesting to identify possible proper subsets $\mathcal{U} \subset \mathcal{S}$ such that $\mathcal{F}$ is asymptotically controllable relatively to $\mathcal{U}$, as well. In particular, it is interesting to know if it is possible to choose $\mathcal{U}$ in such a way that some advantages are achieved.

This problem has been addressed in Bacciotti and Mazzi (2011). We report in particular the following result [1]: it concerns families $\mathcal{F}$ which satisfy the *radial controllability* assumption, meaning that for each pair of points x_0, x_1 ($x_0 \neq 0, x_1 \neq 0$) there exist $\bar{\sigma} \in \mathcal{S}$, $\bar{t} > 0$, and $\bar{c} > 0$ such that $\Phi(\bar{t}, x_0, \bar{\sigma}) = \bar{c} x_1$.

Proposition 1. If $\mathcal{F}$ is asymptotically controllable relatively to $\mathcal{S}$ and radially controllable, then there exists $\sigma \in \mathcal{P}$ such that the associated operator Φ has a real eigenvalue λ with $|\lambda| < 1$ and hence, it has a nontrivial stable subspace.

Based on this result, it is not difficult to construct stabilizing eventually periodic switching rules. In fact, the main result of Bacciotti and Mazzi (2011) states that under the assumptions of Proposition 1, $\mathcal{F}$ is asymptotically controllable relatively to $\mathcal{E}(\sigma)$.

3. CONIC NEIGHBORHOODS

Let (ρ, ϕ) be a system of polar coordinates fixed in the plane ($\rho > 0, \phi \in [0, 2\pi)$). A subset $N \subset \mathbf{R}^2$ is said to be a *conic set* if there exist a point $\bar{x} = (\bar{\rho}, \bar{\phi}) \in \mathbf{R}^2$, $\bar{x} \neq 0$, and numbers $\delta \in [0, \bar{\rho})$, $\varepsilon \in [0, \pi/2)$ such that
$$N = \{(\rho, \phi) : \bar{\rho} - \delta \leq \rho \leq \bar{\rho} + \delta, \ \ \bar{\phi} - \varepsilon \leq \phi \leq \bar{\phi} + \varepsilon\} \ .$$

Thus, by definition, any conic set is closed and does not contain the origin. Note that if $\delta = \varepsilon = 0$ then N reduces to the singleton $\{\bar{x}\}$. Instead, if $\delta > 0$ and $\varepsilon > 0$ then N contains an open neighborhood of $\bar{x}$: in such a case, we will say that N is a *conic neighborhood* of $\bar{x}$.

Lemma 1. Let $\bar{x} \in \mathbf{R}^2$, $\bar{x} \neq 0$. Let B be a linear nonsingular operator, and let N be a conic neighborhood of $\bar{x}$. Then, the set BN contains a conic neighborhood of $B\bar{x}$.

By *complete cone* we mean a nonempty subset Γ of $\mathbf{R}^2$ enjoying the following property: for each $x \in \Gamma$ and for each $c \in \mathbf{R}$ one has $cx \in \Gamma$. In particular, for each conic set N the set of all the points y for which
$$\exists c \in \mathbf{R}, \exists x \in N \text{ such that } y = cx$$
is a complete cone. It is denoted by $\ell(N)$ and it is called the *complete cone generated* by N. A complete cone generated

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[1] Proposition 1 is proved in Bacciotti and Mazzi (2011) without any restriction on the dimension of the state space and on the number of the members of $\mathcal{F}$

by a conic set is always a closed set. If $N = \{\bar{x}\}$, then $\ell(N)$ is simply the one-dimensional subspace generated by $\bar{x}$. In this case, we adopt the simplified notation $\ell(\bar{x})$. If N is a conic neighborhood of some point, then $\ell(N)$ has a nonempty interior. Moreover, by virtue of the condition $\varepsilon < \pi/2$, a complete cone generated by a conic set is always a proper subset of $\mathbf{R}^2$.

4. RECURRENT SWITCHING RULES

A switching rule is called *state-dependent* when the index of the current vector field is determined by the value of the state, rather than the time instant. This includes the classical concept of feedback, but also more general concepts, such as hybrid feedback (see Goebel et al. (2009)).

In this section we introduce a special type of state-dependent switching rule, whose structure is inspired by that of eventually periodic switching rule.

Let θ_1, θ_2 be fixed positive numbers. Consider the periodic switching signals of period $T = \theta_1 + \theta_2$ such that

$$\sigma_1(t) = \begin{cases} 1 & \text{if } t \in [0, \theta_1) \\ 2 & \text{if } t \in [\theta_1, \theta_1 + \theta_2) \end{cases}$$

$$\sigma_2(t) = \begin{cases} 2 & \text{if } t \in [0, \theta_2) \\ 1 & \text{if } t \in [\theta_2, \theta_1 + \theta_2) \end{cases} .$$

Note that $\sigma_2(t) = \sigma_1(t + \theta_1)$ for $t \geq 0$. From now on, we will be especially interested in the periodic switching systems $(\mathcal{F}, \sigma_1)$ and $(\mathcal{F}, \sigma_2)$. We re-denote

$$\Phi_1 = \Phi(\theta_1 + \theta_2, \cdot, \sigma_1) = e^{\theta_2 A_2} e^{\theta_1 A_1}$$

and

$$\Phi_2 = \Phi(\theta_1 + \theta_2, \cdot, \sigma_2) = e^{\theta_1 A_1} e^{\theta_2 A_2}$$

the linear operators associated to $(\mathcal{F}, \sigma_1)$ and $(\mathcal{F}, \sigma_2)$, respectively.

Assume that Φ_1 has a real eigenvalue λ. Note that Φ_1 is invertible, so that $\lambda \neq 0$. Let $w_1 \neq 0$ be a real eigenvector of Φ_1 corresponding to λ. Let $w_2 = e^{\theta_1 A_1} w_1$.

Lemma 2. λ is an eigenvalue of Φ_2, as well, and w_2 is an eigenvector of Φ_2 corresponding to λ.

Consider again the periodic switched system $(\mathcal{F}, \sigma_1)$ and assume now that the associated operator Φ_1 has a real eigenvalue λ with $|\lambda| < 1$. Let $w_1 \neq 0$ be a real eigenvector of Φ_1 corresponding to λ and let, as before, $w_2 = e^{\theta_1 A_1} w_1$. Then, $\ell(w_1)$ and $\ell(w_2)$ are invariant stable one-dimensional subspaces of Φ_1 and Φ_2, respectively. More precisely, for each $y \in \ell(w_h)$, $\Phi_h y = \lambda y \in w_h$ and $||\Phi_h y|| = |\lambda| \cdot ||y|| < ||y||$ $(h = 1, 2)$.

Remark 1. Assume that both A_1 and A_2 are not Hurwitz[2]*. Assume further that the following transversality condition*

$$\ell(w_1) \cap \ell(w_2) = \{0\} \tag{3}$$

holds. Then, without loss of generality, we can also assume that

$$0 < t < \theta_1 \Longrightarrow e^{tA_1} w_1 \notin \ell(w_2)$$

and that

$$0 < t < \theta_2 \Longrightarrow e^{tA_2} w_2 \notin \ell(w_1) .$$

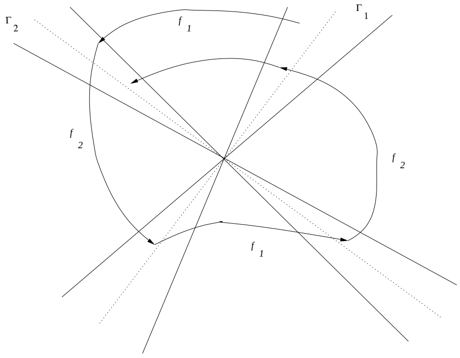

Fig. 1. Sketch of a recurrent switching rule in the plane

Indeed, let $e^{tA_1} w_1 = \mu w_2$, for some $\mu \in \mathbf{R}$. If $|\mu| < 1$ we can simply replace θ_1 by t. Otherwise, it is possible to prove that both $\ell(w_1)$ and $\ell(w_2)$ are invariant stable subspaces for the operator $e^{(\theta_1 - t)A_1}$. Taking into account (3), this in turn implies that A_1 is Hurwitz.

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Under the conditions stated in the previous Remark, the stabilizing action of σ_1 can be alternatively described in terms of state-dependence, according to the following simple rule: the system is required to switch on the vector field f_1 whenever the trajectory hits the subspace $\ell(w_1)$ and to switch on the vector field f_2 whenever the trajectory hits the subspace $\ell(w_2)$.

The intrinsic lack of robustness of such a rule is due to the fact that a stable subspace has empty interior in general. In order to overcome the drawback, it is natural to replace $\ell(w_1), \ell(w_2)$ by certain complete cones with nonempty interior.

Definition 1. Let Γ_1, Γ_2 be closed complete cones of $\mathbf{R}^2$ with nonempty interior, and assume that $\Gamma_1 \cap \Gamma_2 = \{0\}$. We say that Σ is a *recurrent switching rule* subject to Γ_1, Γ_2 if for every $x_1 \in \Gamma_1$ one has:

- (i) $\Phi^{-1}(\Gamma_1, x_1, \Sigma_{x_1}) = \{t \in [0, +\infty) : \Phi(t, x_1, \Sigma_{x_1}) \in \Gamma_1\}$ is the countable union of compact intervals $I_{1+2j} = [a_{1+2j}, b_{1+2j}]$ $(j = 0, 1, 2, \ldots)$;
- (ii) $\Phi^{-1}(\Gamma_2, x_1, \Sigma_{x_1}) = \{t \in [0, +\infty) : \Phi(t, x_1, \Sigma_{x_1}) \in \Gamma_2\}$ is the countable union of compact intervals $I_{2j} = [a_{2j}, b_{2j}]$ $(j = 0, 1, 2, \ldots)$;
- (iii) $t_0 = 0 = a_1 < b_1 < a_2 < b_2 < a_3 < b_3 \ldots$;
- (iv) $t_1 \in I_2, t_2 \in I_3, t_3 \in I_4, t_4 \in I_5, \ldots$ where $t_1, t_2, \ldots$ are the switching times of Σ_{x_1};
- (v) $\Sigma_{x_1}(t) = \begin{cases} 1 & \text{if } t \in [t_{2j}, t_{2j+1}) \\ 2 & \text{if } t \in [t_{2j+1}, t_{2j+2}) \end{cases}$ for each $j = 0, 1, 2 \ldots$.

In other words, a recurrent switched rule in the plane depends on a pair of complete cones, each of them associated with an index. When the system trajectory crosses one of the cones Γ_h, the corresponding index $n_h \in \{1, 2\}$ is selected: the vector field identified by the index n_h is activated and maintained until the system trajectory enters the other cone. Note that in this Definition, the switches do not occur when the system trajectory intersects exactly some prescribed point or the boundary of some region, but at an instant whatsoever, while the trajectory runs inside a set with nonempty interior. In practical applications,

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[2] Recall that a matrix is Hurwitz when its eigenvalues all have strictly negative real part

this allows to compensate small errors of the position sensor and/or of the actuator, and hence guarantees more robustness. Note that a recurrent switching rule is not necessarily periodic.

Theorem 1. Let θ_1, θ_2 be positive numbers such that the linear operator $\Phi_1 = e^{\theta_2 A_2} e^{\theta_1 A_1}$ has a real eigenvalue λ with $|\lambda| < 1$. Assume that both A_1 and A_2 are not Hurwitz, and that the transversality condition (3) holds. Then, there exist complete cones Γ_1, Γ_2 such that for each recurrent switching rule Σ subject to Γ_1, Γ_2 and for each $x_1 \in \Gamma_1$ one has

$$\lim_{t \to +\infty} \Phi(t, x_1, \Sigma_{x_1}) = 0 \ .$$

If the assumptions of Theorem 1 hold, and if in addition $\mathcal{F}$ is radially controllable, some point $x_1 \in \Gamma_1$ can be actually reached in finite time from any initial state $x_0 \in \mathbf{R}^2$, $x_0 \neq 0$. We can therefore conclude that $\mathcal{F}$ can be stabilized in a robust way by using recurrent switching rules.

5. PROOF OF THEOREM 1

Lemma 3. Under the transversality condition (3), $\ell(w_1)$ is a proper subspace neither of A_1 nor of A_2. The same conclusion holds about $\ell(w_2)$.

This implies in particular that $A_n y$ $(n = 1, 2)$ is transversal to y for each non zero $y \in \ell(w_2)$ and for each non zero $y \in \ell(w_1)$. The following lemma is a refinement of Lemma 2 of Bacciotti and Ceragioli (2006). Let us denote by (ρ_1, ϕ_1) and (ρ_2, ϕ_2) the polar coordinates of w_1, w_2, respectively.

Lemma 4. Assume that the transversality condition (3) holds. For each $\eta > 0$ and for each $h = 1, 2$, there exist two non zero vectors u_h, v_h, expressed in polar coordinates respectively by (p_h, μ_h) and (q_h, ψ_h), such that $\mu_h < \phi_h < \psi_h$, and the following holds: for $n = 1, 2$ there exists a time $\tau_{nh} \in \mathbf{R}$ such that $|\tau_{nh}| < \eta$ and $\ell(v_h) = e^{\tau_{nh} A_n} \ell(u_h)$.

Roughly speaking, Lemma 4 states that, for each combination of the values of the indeces n, m, the time needed to steer any point $y \in \ell(u_m)$ to same point of $\ell(v_m)$ along a trajectory of the vector field $f_n(x) = A_n x$ is independent of y (as far as $y \in \ell(u_m)$) and that such a time is small, provided that the angle formed by $\ell(u_m)$ and $\ell(v_m)$ is small. We denote $\tau = \max\{|\tau_{11}|, |\tau_{12}|, |\tau_{21}|, |\tau_{22}|\} < \eta \leq 1$.

The following facts will be also useful. For each $n = 1, 2$ there exists $\gamma_n > 0$ and $\alpha_n \in \mathbf{R}$ such that

$$||e^{tA_n}|| \leq \gamma_n e^{\alpha_n t}$$

for each $t \geq 0$. We denote $\gamma = \max\{\gamma_1, \gamma_2\}$ and $\alpha \geq \max\{\alpha_1, \alpha_2\}$. We may assume $\alpha > 0$ without loss of generality. Moreover, for each $n = 1, 2$ and each $t \in \mathbf{R}$, we can write

$$e^{tA_n} = I + tM_n(t)$$

where I is the identity matrix and $M_n(t)$ is the sum of a convergent power expansion of matrices. In particular, $M_n(t)$ is continuous as a function of t and if t is constrained to a compact interval J, then there exists a positive constant E_n such that

$$||M_n(t)|| < E_n, \quad \forall t \in J \ . \tag{4}$$

For later use, we chose $J = [-2, 2]$. Moreover, we denote $E = \max\{E_1, E_2\}$.

We are now in a position to undertake the construction of the regions Γ_1, Γ_2.

Fix a point $y_1 \in \ell(w_1)$, with $||y_1|| = 1$, and a number $\bar{\lambda}$ such that $|\lambda| < \bar{\lambda} < 1$. Define

$$y_2 = e^{\theta_1 A_1} y_1 \in \ell(w_2) \quad \text{and} \quad y_3 = e^{\theta_2 A_2} y_2 = \Phi_1 y_1 \in \ell(w_1) \ .$$

Recall that by construction,

$$||y_3|| = |\lambda| \cdot ||y_1|| \ . \tag{5}$$

Let N_3 be a conic neighborhood of y_3, with

$$N_3 \subset \mathcal{B}(y_3, \bar{\lambda} - |\lambda|) \ . \tag{6}$$

where $\mathcal{B}(z, r)$ denotes the open ball of radius r and center z. Since

$$||y|| \leq ||y - y_3|| + ||y_3||$$

from (5) and (6) we can be sure that $N_3 \subset \mathcal{B}(0, \bar{\lambda})$. Let N_2' be a conic neighborhood of y_2, contained in $e^{-\theta_2 A_2} N_3$. Let $\ell(N_2')$ be the generated complete cone.

Let $u_2 = (p_2, \mu_2), v_2 = (q_2, \psi_2)$ be vectors determined according to Lemma 4, applied with

$$\eta = \min\{1, \theta_1, \theta_2, \frac{\bar{\lambda} - |\lambda|}{8\gamma^2 E\tau(1 + e^{\alpha\tau}) e^{\alpha(\theta_1 + \theta_2)}}\} \ . \tag{7}$$

Without loss of generality, we can assume that $\ell(u_2)$ and $\ell(v_2)$ lie in $\ell(N_2')$. Define

$$N_2 = \{(\rho, \phi) \in N_2' : \mu_2 \leq \phi \leq \psi_2\}$$

and let Γ_2 be the complete cone generated by N_2. We proceed by taking a conic neighborhood N_1' of y_1, contained in $e^{-\theta_1 A_1} N_2 \cap \mathcal{B}(y_1, \chi)$, where

$$\chi = \frac{\bar{\lambda} - |\lambda|}{2\gamma^2 e^{\alpha(\theta_1 + \theta_2)}} \ . \tag{8}$$

Let $u_1 = (p_1, \mu_1), v_1 = (q_1, \psi_1)$ be the vectors provided by Lemma 4, for the same value of η indicated above. Again, we can assume that $\ell(u_1)$ and $\ell(v_1)$ lie in $\ell(N_1')$. Take a new conic neighborhood of y_1

$$N_1 = \{(\rho, \phi) \in N_1' : \mu_1 \leq \phi \leq \psi_1\} \ .$$

Since $||y_1|| = 1$, the condition $\delta < \bar{\rho}$ in the definition of conic set implies $N_1 \subset \mathcal{B}(0, 2)$.

Let Γ_1 be the complete cone generated by N_1. By construction, it is clear that $\Phi_1 N_1$ is contained in N_3. This implies that $||\Phi_1 y|| < \bar{\lambda}$ for each $y \in N_1$.

Let now $\hat{y}_1 \in N_1$ be fixed, so that in particular

$$||\hat{y}_1 - y_1|| \leq \chi \ . \tag{9}$$

Assume that $\hat{\theta}_1, \hat{\theta}_2$ are such that

$$\hat{y}_2 = e^{(\theta_1 + \hat{\theta}_1) A_1} \hat{y}_1 \in \Gamma_2$$

$$\hat{y}_3 = e^{(\theta_2 + \hat{\theta}_2) A_2} \hat{y}_2 \in \Gamma_1 \ .$$

According to Lemma 4 and (7), we have $|\hat{\theta}_n| \leq 2\tau \leq 2\eta \leq 2$, for $n = 1, 2$. Moreover, we also have $|\hat{\theta}_n| \leq \theta_n$, so that

$\theta_n + \hat{\theta}_n \geq 0$ $(n = 1, 2)$. It may be convenient to introduce the notation

$$\hat{\Phi}_1 = e^{(\theta_2+\hat{\theta}_2)A_2} e^{(\theta_1+\hat{\theta}_1)A_1}$$

so that $\hat{y}_3 = \hat{\Phi}_1 \hat{y}_1$. We need to estimate the difference

$$\begin{aligned}
& ||\hat{y}_3 - y_3|| \\
&= ||e^{(\theta_2+\hat{\theta}_2)A_2} e^{(\theta_1+\hat{\theta}_1)A_1} \hat{y}_1 - e^{\theta_2 A_2} e^{\theta_1 A_1} y_1|| \\
&\leq ||e^{\theta_2 A_2}|| \cdot ||e^{\hat{\theta}_2 A_2} e^{(\theta_1+\hat{\theta}_1)A_1} \hat{y}_1 - e^{\theta_1 A_1} y_1|| \\
&\leq \gamma e^{\theta_2 \alpha} ||(I + \hat{\theta}_2 M_2(\hat{\theta}_2)) e^{(\theta_1+\hat{\theta}_1)A_1} \hat{y}_1 - e^{\theta_1 A_1} y_1|| \\
&\leq \gamma e^{\theta_2 \alpha} \big(||e^{(\theta_1+\hat{\theta}_1)A_1} \hat{y}_1 - e^{\theta_1 A_1} y_1|| \\
&\quad + |\hat{\theta}_2| \cdot ||M_2(\hat{\theta}_2)|| \cdot ||e^{(\theta_1+\hat{\theta}_1)A_1}|| \cdot ||\hat{y}_1|| \big) \\
&\leq \gamma e^{\theta_2 \alpha} \big(||e^{\theta_1 A_1}|| \cdot ||e^{\hat{\theta}_1 A_1} \hat{y}_1 - y_1|| \\
&\quad + |\hat{\theta}_2| \cdot ||M_2(\hat{\theta}_2)|| \cdot ||e^{(\theta_1+\hat{\theta}_1)A_1}|| \cdot ||\hat{y}_1|| \big) \\
&\leq \gamma e^{\theta_2 \alpha} \big(\gamma e^{\theta_1 \alpha} ||(I + \hat{\theta}_1 M_1(\hat{\theta}_1)) \hat{y}_1 - y_1|| \\
&\quad + \gamma e^{(\theta_1+\hat{\theta}_1)\alpha} |\hat{\theta}_2| \cdot ||M_2(\hat{\theta}_2)|| \cdot ||\hat{y}_1|| \big) \ .
\end{aligned}$$

In the last inequality, we used the fact that $\theta_1 + \hat{\theta}_1 \geq 0$. Since $|\hat{\theta}_n| \leq 2$ for $n = 1, 2$, we may use the bounds (4) for $||M_n(t)||$. Moreover, we know that $||\hat{y}_1|| < 2$ and that $\tau < \eta$. Taking into account also (9), we obtain

$$\begin{aligned}
& ||\hat{y}_3 - y_3|| \\
&\leq \gamma e^{\theta_2 \alpha} \big(\gamma e^{\theta_1 \alpha} ||\hat{y}_1 - y_1|| + \gamma e^{\theta_1 \alpha} |\hat{\theta}_1| \cdot ||M_1(\hat{\theta}_1)|| \cdot ||\hat{y}_1|| \\
&\quad + \gamma e^{(\theta_1+\hat{\theta}_1)\alpha} |\hat{\theta}_2| \cdot ||M_2(\hat{\theta}_2)|| \cdot ||\hat{y}_1|| \big) \\
&\leq \gamma^2 e^{(\theta_1+\theta_2)\alpha} \chi + \gamma^2 e^{(\theta_1+\theta_2)\alpha} E |\hat{\theta}_1| \cdot ||\hat{y}_1|| \\
&\quad + \gamma^2 e^{(\theta_1+\theta_2)\alpha} E |\hat{\theta}_2| \cdot ||\hat{y}_1|| e^{\hat{\theta}_1 \alpha} \\
&\leq \gamma^2 e^{(\theta_1+\theta_2)\alpha} \chi + 2\gamma^2 e^{(\theta_1+\theta_2)\alpha} E (|\hat{\theta}_1| + e^{\hat{\theta}_1 \alpha} |\hat{\theta}_2|) \\
&\leq \gamma^2 e^{(\theta_1+\theta_2)\alpha} \chi + 4\gamma^2 e^{(\theta_1+\theta_2)\alpha} E \eta (1 + e^{\tau \alpha}) \ .
\end{aligned}$$

Using (8) and (7), finally we have

$$||\hat{y}_3 - y_3|| \leq \bar{\lambda} - \lambda$$

which in turn implies $||\hat{y}_3|| < \bar{\lambda}$.

Next, by virtue of homogeneity we are able to remove the restriction that the initial point belongs to N_1. Take a generic point $x_1 \in \Gamma_1$ $(x_1 \neq 0)$ and consider again the operator $\hat{\Phi}_1$, subject to the conditions

$$\begin{aligned}
x_2 &= e^{(\theta_1+\hat{\theta}_1)A_1} x_1 \in \Gamma_2 \\
x_3 &= e^{(\theta_2+\hat{\theta}_2)A_2} x_2 \in \Gamma_1 \ .
\end{aligned}$$

Since $x_1 \in \Gamma_1$, then either $x_1/||x_1|| \in N_1$ or $-x_1/||x_1|| \in N_1$. Without loss of generality, we can limit to the first case. Hence,

$$\hat{\Phi}_1(x_1)/||x_1|| = \hat{\Phi}_1(x_1/||x_1||) \leq \bar{\lambda}$$

which yields

$$\hat{\Phi}_1(x_1) \leq \bar{\lambda} ||x_1|| \ .$$

Finally recall that by assumption, $x_3 \in \Gamma_1$. Therefore, the previous reasoning can be iterated.

Formally, consider a recurrent switching rule subject to Γ_1, Γ_2, as defined above. For any $x_1 \in \Gamma_1$ $(x_1 \neq 0)$ the corresponding trajectory can be thought of as a composition of operators of the form

$$\Phi^{[k]} = e^{(\theta_2+\hat{\theta}_{2k})A_2} e^{(\theta_1+\hat{\theta}_{1k})A_1}$$

where the numbers $\hat{\theta}_{1k}, \hat{\theta}_{2k}$ are chosen in such a way that

$$\begin{gathered}
e^{(\theta_1+\hat{\theta}_{11})A_1} x_1 = x_2 \in \Gamma_2 \\
\Phi^{[1]} x_1 = x_3 \in \Gamma_1 \\
e^{(\theta_1+\hat{\theta}_{12})A_1} x_3 = x_4 \in \Gamma_2
\end{gathered}$$

and so on. We can therefore conclude that

$$||x_{2k+1}|| \leq \bar{\lambda}^k ||x_1||$$

and hence the sequence x_{2k+1} goes to zero as $k \to +\infty$. The conclusion can be now easily obtained by standard arguments.

Remark 2. The cones Γ_1, Γ_2 whose existence is stated in Theorem 1 can be actually constructed in such a way that the set of recurrent switching rules subject to Γ_1, Γ_2 is nonempty. This can be seen by introducing a simple modification in the proof, based on Lemma 4.

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