

L_2 -gain Stabilizability with respect to Filippov Solutions

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Abstract

We give a sufficient condition for discontinuous L_2 -gain stabilizability of a nonlinear affine system with respect to Filippov solutions. Our condition requires the existence of a viscosity supersolution of a suitable Hamilton-Jacobi equation.

1 Introduction

Since the work of J. Willems ([19]), the notion of dissipation with respect to a given supply rate function is playing an important role in nonlinear systems analysis. Depending on different specifications of the supply rate function, there have been several developments in the literature. Here, we focus on the so-called L_2 -gain stability (see later for the formal definition).

L_2 -gain stability is related to the H_∞ -control problem. It is well known that for nonlinear affine systems, it can be characterized by means of certain differential inequalities of the Hamilton-Jacobi type: see [5], and the more recent papers [17], [8]; see also the books [6], [18]. A common limitation of these works is the requirement concerning the existence of a smooth (i.e., at least C^1) storage function. Such a limitation is overcome in [7], [15], where it is proved that if the Hamilton-Jacobi inequality is intended in viscosity sense, then continuous or even merely lower semi-continuous storage function can be allowed.

The present note is a further contribution in this direction. Our main result (Theorem 2, Section 3) concerns L_2 -gain stabilizability: we prove that L_2 -gain stability can be achieved by means of a discontinuous feedback law, provided that a suitable Hamilton-Jacobi inequality admits a solution in viscosity sense. In order to explicitly define the feedback, we need to restrict ourselves to locally Lipschitz continuous storage functions. Moreover, since the closed loop system has a discontinuous right hand side, L_2 -gain stability will be referred to its Filippov solutions. A similar result has been recently obtained in [16], with respect to a different notion of solution (the so-called *sampling solutions*) and under different assumptions. We also believe that our

approach is technically simpler.

2 Basic material

We are interested in affine systems of the form

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=0}^m u_i g_i(x) \quad (1)$$

where $x \in \mathbf{R}^n$ denotes the state variable, and $u = (u_1, \dots, u_m) \in \mathbf{R}^m$ is the input variable. Throughout this paper, by $f(x), g_1(x), \dots, g_m(x)$ we denote vector fields of class C^1 on the whole of \mathbf{R}^n . The system description is completed by the observation map

$$y = c(x) \quad (2)$$

where $y \in \mathbf{R}^p$ is the output variable and $c(x)$ is continuous. The set \mathcal{U} of the admissible inputs is formed by all the piecewise continuous functions $u(t) : [0, +\infty) \rightarrow \mathbf{R}^m$ (without loss of generality, we may assume that they are right continuous). For each initial state x and each admissible input $u(\cdot) \in \mathcal{U}$, system (1) has a unique solution. It will be denoted by $\varphi(t; x, u(\cdot))$, or simply by $\varphi(t)$ when there is no possible confusion. We assume that all the solutions are right continuable for $t \geq 0$. The following definition is classical: it is reported for reader's convenience.

Definition 1 *We say that (1), (2) is L_2 -gain stable if there exist $\gamma > 0$ and a locally bounded function $V(x)$, called a storage function, such that $V(x) \geq 0$, $V(0) = 0$ and $\forall x \in \mathbf{R}^n, \forall u(\cdot) \in \mathcal{U}, \forall t \geq 0$,*

$$V(\varphi(t)) \leq V(x) + \int_0^t w(s) ds \quad (3)$$

where $w(s) = -\|c(\varphi(s))\|^2 + \frac{1}{2\gamma}\|u(s)\|^2$.

As mentioned in the Introduction, L_2 -gain stability is nothing else than dissipation with respect to the supply rate $W(y, u) = -\|y\|^2 + \frac{1}{2\gamma}\|u\|^2$. Note that when the system is initialized at $x = 0$, (3) implies

$$\int_0^t \|c(\varphi(s))\|^2 ds \leq \frac{1}{2\gamma} \int_0^t \|u(s)\|^2 ds \quad (4)$$

for each $t \geq 0$. This means that the energy of the output (measured by the L_2 norm, is majorized by the energy of the input. This explains why the property of Definition 1 is called L_2 -gain stability. Note that the gain coefficient in (4) is determined by γ .

Remark 1 In the literature, L_2 -gain stability is sometimes defined by means of inequality (4). Actually, (4) implies (3) under the additional assumptions that the system is completely controllable and the so-called *available energy* (see [5]) is finite at every point. In a smooth context, L_2 -gain stability is also defined by means of the differential version of (3).

Remark 2 When V and c vanish only at $x = 0$, L_2 -gain property can be reviewed as a particular instance of Sontag's ISS property ([12], [13], [14]). Moreover, in this case L_2 -gain stability implies that system (1) with $u = 0$ is globally asymptotically stable at the origin. Conversely, if the system (1) with $u = 0$ is globally asymptotically stable at the origin, then it can be stabilized in the ISS sense by means of a smooth feedback. In other words, the system can be rendered dissipative with respect to a suitable observation map and a suitable supply rate function (but in general, not L_2 -gain stable with respect to a given observation map).

L_2 -gain stability can be characterized by the aid of the map

$$M(x, p) = pf(x) + \frac{\gamma}{2} \|pG(x)\|^2 + \|c(x)\|^2 .$$

Theorem 1 ([7], [15]) *Assume that there exists a locally bounded, nonnegative supersolution $V(x)$ of the equation*

$$-M(x, \nabla V(x)) = 0 \quad (5)$$

with $V(0) = 0$. Then, system (1), (2) is L_2 -gain stable, and $V(x)$ is a storage function. Conversely, assume that system (1), (2) is L_2 -gain stable: then, there exists a lower semi-continuous supersolution $V(x)$ of equation (5).

Now we recall some useful tools from the calculus of nondifferentiable functions. Since we are going to consider only locally Lipschitz continuous functions, we can limit ourselves to Dini's generalized derivatives. The upper right Dini's derivative is defined as

$$\overline{D^+}V(x, v) = \limsup_{h \rightarrow 0^+} \frac{V(x + hv) - V(x)}{h} .$$

We leave to the reader the analogous definitions of $\underline{D^+}V(x, v)$, $\overline{D^-}V(x, v)$, $\underline{D^-}V(x, v)$. The subdifferential and the superdifferential of a map V at x are classically defined as extensions of the Fréchet differentials. Since V is locally Lipschitz continuous, we have the following characterization ([2] and the reference therein):

$$\begin{aligned} \underline{\partial}V(x) \\ = \{p : \overline{D^-}V(x, v) \leq pv \leq \underline{D^+}V(x, v), \forall v \in \mathbf{R}^n\} \end{aligned}$$

$$\begin{aligned} \overline{\partial}V(x) \\ = \{p : \overline{D^+}V(x, v) \leq pv \leq \underline{D^-}V(x, v), \forall v \in \mathbf{R}^n\} . \end{aligned}$$

Let finally $H(x, p) : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R}$ be continuous. We say that V is a viscosity supersolution of the equation $H(x, \nabla V(x)) = 0$ if for each $x \in \mathbf{R}^n$

$$H(x, p) \geq 0 \quad \forall p \in \underline{\partial}V(x) .$$

We say that V is a viscosity subsolution of the equation $H(x, \nabla V(x)) = 0$ if for each $x \in \mathbf{R}^n$

$$H(x, p) \leq 0 \quad \forall p \in \overline{\partial}V(x) .$$

3 L_2 -gain stabilizability

System (1), (2) is said to be L_2 -gain stabilizable if there exists a feedback law $k(x)$ such that the closed-loop system

$$\dot{x} = (f(x) + G(x)k(x)) + G(x)\tilde{u} \quad (6)$$

with the same observation map (2), is L_2 -gain stable with respect to the new input \tilde{u} . Note that (6) is obtained from (1) by the substitution $u = k(x) + \tilde{u}$. As we shall see later, L_2 -stabilizability is related to a modified Hamilton-Jacobi inequality. If we assume that this equation admits a solution of class C^1 , then it is natural to adopt a feedback law construction which involves the gradient of the candidate storage function $V(x)$ (see for instance [20]).

Our result concerns the case where $V(x)$ is merely locally Lipschitz continuous. Here, a problem arises. Indeed, we only have for sure that $\nabla V(x)$ exists almost

everywhere and that it is locally bounded. Thus, our feedback law $k(x)$ is not continuous, and the closed-loop system has a discontinuous right hand side even if the given vector fields f, g_1, \dots, g_m are smooth. As a consequence, solutions of (6) cannot be intended in classical sense. In this section, we adopt the notion of generalized solution due to Filippov ([4]). We write $\mathcal{S}_{x, \tilde{u}(\cdot)}$ to denote the set of Filippov solutions of (6) issuing from x and corresponding to an input $\tilde{u}(\cdot)$. We keep the assumption that all solutions are right continuable for $t \geq 0$.

In order to cover the case where the given vector field f (which we continue to assume of class C^1) is replaced, as in (6), by a vector field $\tilde{f}(x)$ which is a.e. defined and locally bounded, our first task is to restate the definition of L_2 -stability.

Definition 2 *We say that the system is L_2 -gain stable with respect to Filippov solutions if there exist $\gamma > 0$ and a locally bounded function $V(x)$ such that $V(x) \geq 0$, $V(0) = 0$ and $\forall x \in \mathbf{R}^n$, $\forall u(\cdot) \in \mathcal{U}$, $\forall \varphi \in \mathcal{S}_{x, u(\cdot)}$, $\forall t \geq 0$,*

$$V(\varphi(t)) \leq V(x) + \int_0^t w(s) ds \quad (7)$$

where, as above, $w(s) = -\|c(\varphi(s))\|^2 + \frac{1}{2\gamma}\|u(s)\|^2$.

Recall that if a function $V(x)$ is semi-concave then it is locally Lipschitz continuous, and its superdifferential and Clarke's gradient coincide at every point.

Theorem 2 *Let system (1), (2) be given, with the vector fields f, g_1, \dots, g_m of class C^1 . Let*

$$N(x, p) = pf(x) - \frac{\gamma}{2}\|pG(x)\|^2 + \|c(x)\|^2 .$$

Assume that there exists a semi-concave viscosity sub-solution of the equation

$$N(x, \nabla V(x)) = 0 \quad (8)$$

with $V(0) = 0$. Then, system (1), (2) can be L_2 -gain stabilized (in the sense of Definition 2) by means of the discontinuous feedback $u = -\gamma \nabla V(x)G(x) + \tilde{u}$.

Remark 3 Theorem 2 remains valid under the alternative assumption that there exists a locally Lipschitz continuous, C-regular viscosity supersolution of the equation $-N(x, \nabla V(x)) = 0$. In this case, the conclusion could be proved in a direct way, by using the approach in [11], [1].

Remark 4 Theorem 2 differs from Theorem 3 and 4 of [16], where the notion of sampling solution is employed, instead of Filippov solutions. Moreover, our approach requires fewer technicalities and somehow alternative assumptions.

4 Proof of Theorem 2

We start by two lemmas, whose proofs can be found in [9]. We denote by \mathbf{F}_x the Filippov's operator which transforms a differential equation with discontinuous right hand side in a differential inclusion (see [9] or [1] for its definition).

Lemma 1 *Let $h_1(x) : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be locally bounded, and let $h_2(t, x) : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ be continuous with respect to x and right continuous with respect to t . Then, for each pair (t, x) ,*

$$\mathbf{F}_x(h_1 + h_2)(t, x) \subset (\mathbf{F}_x h_1)(x) + h_2(t, x) .$$

Lemma 2 *Let V be locally Lipschitz continuous. Then, $\mathbf{F}_x(\nabla V(x)) = \partial_C V(x)$ for each $x \in \mathbf{R}^n$.*

We can now proceed to the proof of the theorem. Let $\varphi(t)$ be a Filippov's solution of (6), corresponding to the input $u(t) \in \mathcal{U}$. By virtue of Lemmas 1 and 2, there exists a set Z_1 of zero measure such that for each $t \notin Z_1$ there is a vector $q \in \partial_C V(\varphi(t))$ such that

$$\dot{\varphi}(t) = f(\varphi(t)) - \gamma G(\varphi(t))(qG(\varphi(t)))^\dagger + G(\varphi(t))u(t) \quad (9)$$

Since $V(x)$ is locally Lipschitz continuous and $\varphi(t)$ absolutely continuous, the derivative of the composite map $V(\varphi(t))$ exists for $t \notin Z_2$, where Z_2 is a set of zero measure. Using again the Lipschitz continuity of $V(x)$, we have for $t \notin Z_1 \cup Z_2$,

$$\begin{aligned} & \frac{d}{dt} V(\varphi(t)) \\ &= \overline{D^+} V(\varphi(t), \dot{\varphi}(t)) \\ &\leq pf(\varphi(t)) - \gamma(pG(\varphi(t)))(qG(\varphi(t))) + pG(\varphi(t))u(t) \end{aligned}$$

for every $p \in \overline{\partial V}(\varphi(t))$. Here, we used (9) and the definition of the superdifferential. Since V is semi-concave, we can choose $p = q$. So, in particular we obtain for $t \notin Z_1 \cup Z_2$

$$\begin{aligned}
& \frac{d}{dt}V(\varphi(t)) \\
\leq & qf(\varphi(t)) - \gamma\|qG(\varphi(t))\|^2 + qG(\varphi(t))u(t) \\
= & qf(\varphi(t)) - \gamma\|qG(\varphi(t))\|^2 - \frac{\|\gamma qG(\varphi(t)) - u(t)\|^2}{2\gamma} \\
& + \frac{\gamma}{2}\|qG(\varphi(t))\|^2 + \frac{\|u(t)\|^2}{2\gamma} \\
\leq & -\|c(\varphi(t))\|^2 + \frac{\|u(t)\|^2}{2\gamma} = w(t)
\end{aligned}$$

where we used the condition (8). The last inequality can be rewritten as

$$\frac{d}{dt}V(\varphi(t)) - w(t) \leq 0 \quad \text{a.e. } t \geq 0. \quad (10)$$

We emphasize that it holds for each admissible input $u(\cdot)$, each initial state x and each $\varphi \in \mathcal{S}_{x,u(\cdot)}$. From (10), it is not difficult to infer that

$$l(t) = V(\varphi(t)) - \int_0^t w(s) ds$$

is decreasing for $t \geq 0$, that is $l(0) \geq l(t)$. This is equivalent to (7). We have so proven that the closed loop system is L_2 -gain stable with respect to Filippov solutions.

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