Finite L_2-gain with Nondifferentiable Storage Functions

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Abstract

We consider affine control systems with the finite L_2-gain property in the case the storage function is nondifferentiable. We generalize some classical results concerning the connection of the finite L_2-gain property with the stability properties of the unforced system, the characterization of finite L_2-gain by means of partial differential inequalities of the Hamilton-Jacobi type and the problem of giving to a system the finite L_2-gain property by means of a feedback law. Moreover, we introduce and study the apparently new notion of exact storage function.

Keywords: L2-gain; nondifferentiable storage functions; Filippov solutions

1 Introduction

The notion of dissipation with respect to a given supply rate was introduced by J. Willems ([22]) at the beginning of the seventies and since then, it is playing an important role in nonlinear control systems analysis. For different specifications of the supply rate function, the class of dissipative systems includes passive systems and systems with finite L_2-gain. In this paper we focus on the so-called finite L_2-gain property (see later for the formal definition).

When one deals with finite L_2-gain, some of the issues which naturally come out are

(i) the connection of the finite L_2-gain property with the stability properties of the unforced system;

(ii) the characterization of finite L_2-gain by means of partial differential inequalities of the Hamilton-Jacobi type;

(iii) the problem of giving to a system the finite L_2-gain property by means of a feedback law.

These issues are classically treated in the case there exist continuously differentiable storage functions. However, in general, storage functions are not differentiable (see [12, 15]), so that it is interesting to generalize classical results to a nonsmooth context. The main effort which has already been done in order to generalize finite L_2-gain theory to the case of nondifferentiable storage functions concerns its characterization by means of partial differential inequalities of the Hamilton-Jacobi type (see [12, 17, 23]).

The present note is a contribution to the generalization of issues (i) and (iii). Moreover, we introduce and study the apparently new notion of exact storage function.

The paper is organized as follows. The basic material is collected in Section 2, where we precise our definition of finite L_2-gain and introduce some notations from nonsmooth analysis. In Section 3 we generalize some results concerning the connection between finite L_2-gain and stability of the unforced system to the case of nondifferentiable storage functions. In Section 4 we revisit the well known characterization of the finite L_2-gain property in terms of the associated Hamilton-Jacobi equation for the particular case of locally Lipschitz continuous storage functions. By means of suitable notions of solution of the same equation, we also give necessary and sufficient conditions for the existence of exact storage functions. In Section 5 we give our main results on finite L_2-gain by feedback. These are expressed in terms of Hamilton-Jacobi-like conditions and involve feedback laws that may be discontinuous. In order to deal with discontinuous feedback laws, we extend the definition of finite L_2-gain by making use of the notion of Filippov solution of differential equations with discontinuous righthand side. Finally, we point out an interesting connection between the solvability of a suitably defined optimal regulation problem and finite L_2-gain by feedback. The proofs are collected in Section 6.

2 Definitions and basic material

We are interested in affine systems of the form

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=0}^{m} u_i g_i(x) \tag{1}$$

where $x \in \mathbf{R}^n$ denotes the state variable, and $u = (u_1, \ldots, u_m) \in \mathbf{R}^m$ is the input variable. Throughout the paper $f(x), g_1(x), \ldots, g_m(x)$ are vector fields of class C^1 on the whole of $\mathbf{R}^n$ and $G(x)$ is the matrix whose columns are $g_1(x), \ldots, g_m(x)$. Moreover, we assume $f(0) = 0$. The system description is completed by the observation map

$$y = c(x) \tag{2}$$

where $y \in \mathbf{R}^p$ is the output variable and $c(x)$ is continuous, with $c(0) = 0$. The set $\mathcal{U}$ of the admissible inputs is formed by all measurable functions $u(t) : [0, +\infty) \to \mathbf{R}^m$. For each initial state x and each admissible input $u(\cdot) \in \mathcal{U}$, system (1) has a unique solution, which is denoted by $\varphi(t; x, u(\cdot))$, or simply by $\varphi(t)$ when there is no possible confusion. We assume that all the solutions are right continuable for $t \geq 0$.

We now introduce the classical definition of finite L_2-gain (however, we forewarn the reader that the developments of Section 5 demand a more general framework).

Definition 1 *We say that (1), (2) has* finite L_2-gain *if there exist $k \in \mathbf{R}$ and a locally bounded function $V : \mathbf{R}^n \to \mathbf{R}$, called* storage function*, such that $V(x) \geq 0$ for all $x \in \mathbf{R}^n$, $V(0) = 0$ and*

$$\forall x \in \mathbf{R}^n,\ \forall u(\cdot) \in \mathcal{U},\ \forall t \geq 0, \qquad V(\varphi(t; x, u(\cdot))) \leq V(x) + \int_0^t w(s)\, ds \tag{3}$$

where $w(s) = -\|c(\varphi(s; x, u(\cdot)))\|^2 + k^2\|u(s)\|^2$.

As mentioned in the Introduction, the finite L_2-gain property is nothing else than dissipation with respect to the supply rate $\tilde{w}(y, u) = -\|y\|^2 + k^2\|u\|^2$. Note that when the system is initialized at $x = 0$, (3) implies

$$\int_0^t \|c(\varphi(s; 0, u(\cdot)))\|^2\, ds \leq k^2 \int_0^t \|u(s)\|^2\, ds \tag{4}$$

for each $t \geq 0$. This means that the energy of the output (measured by the L_2 norm) is majorized by the energy of the input. This explains why the property of Definition 1 is called finite L_2-gain.

Remark 1 In the literature, the finite L_2-gain property is sometimes defined by means of inequality (4). Actually, (4) implies (3) under the additional assumptions that the system is completely controllable and the so-called *available energy* (see [9]) is finite at every point. In a smooth context, finite L_2-gain can also be defined by means of the differential version of (3).

The following lemma will be used later. The proof is straightforward.

Lemma 1 *Let $V : \mathbf{R}^n \to \mathbf{R}$ be locally Lipschitz continuous. With the notation above, assume that for each admissible input $u(\cdot)$ and each initial state x, one has*

$$\frac{d}{dt} V(\varphi(t)) \leq w(t) \quad for\ a.e.\ t \geq 0\ . \tag{5}$$

Then, the system has finite L_2-gain.

We now introduce some notations and definitions that we use throughout the paper in order to deal with nonsmooth storage functions (see [5, 6]). Let $V : \mathbf{R}^n \to \mathbf{R}$, $x, v \in \mathbf{R}^n$.

$D^+V(x,v)$ denotes the ordinary right directional derivative of V at x in the direction v;

$\overline{D}^+V(x,v)$ denotes Dini upper right derivative of V at x in the direction v;

$\partial_C V(x)$ denotes Clarke gradient at x;

$\partial^* V(x)$ denotes the limiting gradient at x;

$\overline{\partial} V(x)$ denotes the superdifferential at x;

$\underline{\partial} V(x)$ denotes the subdifferential at x.

We often exploit the notion of nonpathological function, due to Valadier ([19]). Recall that V is said to be *nonpathological* if it is locally Lipschitz continuous and for every absolutely continuous function $\varphi : \mathbf{R} \to \mathbf{R}^n$, for a.e. $t \in \mathbf{R}$, the set $\partial_C V(\varphi(t))$ is a subset of an affine subspace orthogonal to $\dot{\varphi}(t)$. The fundamental property of nonpathological functions is that for a.e. t the set $\{a \in \mathbf{R} : \ a = p \cdot \dot{\varphi}(t), \ p \in \partial_C V(\varphi(t))\}$ is reduced to the singleton $\{\frac{d}{dt}V(\varphi(t))\}$. More precisely, the following statement holds (see [3] for the case of Clarke regular functions; the case of nonpathological functions requires minor modifications).

Lemma 2 *Let V be nonpathological. Let $F : \mathbf{R} \times \mathbf{R}^n \to 2^{\mathbf{R}^n} \setminus \emptyset$ be upper semi-continuous with respect to x for a.e. t, measurable with respect to t for all x and with compact and convex values. Let $\varphi(t)$ be a solution of the differential inclusion $\dot{x} \in F(t,x)$. Then for a.e. t it holds*

$$\{\frac{d}{dt}V(\varphi(t))\} \subseteq \{a \in \mathbf{R} : \ \exists v \in F(t,\varphi(t)) \ s.t. \ p \cdot v = a \ \forall p \in \partial_C V(\varphi(t))\}.$$

Finally we introduce two other classes of locally Lipschitz continuous functions. We say that a function V is $\underline{C}$*-regular* if it is locally Lipschitz continuous and regular in Clarke's sense (see [7], page 39). This is equivalent to say that $\partial_C V(x) = \underline{\partial} V(x)$ at every x. We say that V is $\overline{C}$*-regular* if $-V$ is $\underline{C}$-regular. This is equivalent to say that $\partial_C V(x) = \overline{\partial} V(x)$ at every x.

We recall that semiconvex functions are $\underline{C}$-regular, while semiconcave functions are $\overline{C}$-regular. Both $\underline{C}$ and $\overline{C}$-regular functions are nonpathological (see [4]).

3 Internal stability

It is well known that if system (1), (2) has finite L_2-gain with a storage function of class (at least) C^1 and, additionally, it fulfills certain observability conditions, then the unforced system

$$\dot{x} = f(x) \tag{6}$$

is stable, or even asymptotically stable at the origin (see [9], [21], [11], [16]). The property of (1) being stable when u is set to be zero, is sometimes called *internal* *stability*. In this section we extend some of these results to the case where a storage function of class C^1 is not available. There are many notions of observability in the nonlinear systems literature. The notion we need in the following is taken from [16], page 45.

Let $Z_c = \{x : c(x) = 0\}$ and let K be the largest (in the sense of the union) positively invariant set for (6) contained in Z_c. Since c is continuous, Z_c is closed. It is not difficult to see that K is closed as well, and that $K \neq \emptyset$ (at least, $0 \in K$). We emphasize that Z_c is not required to be invariant for (6), so that K is not necessarily invariant. Moreover, K may be a proper subset of Z_c.

Definition 2 *We say that (1), (2) is* locally/globally zero state detectable *(in short, ZSD) if the unforced system (6) is locally/globally asymptotically stable conditionally to K.*

Roughly speaking, this means that only trajectories starting from points in K are required to remain near the origin and to converge to the origin when $t \to +\infty$ (we refer to [16] for the formal definition).

In order to prove stability, it is natural to take a storage function V as a candidate Lyapunov function. In doing this, one has to overcome two difficulties: the fact that V is only nonnegative definite and that it is nondifferentiable. To this purpose, we will use some old results in topological dynamics (see [13], and the more recent paper [10]). From now on, we will assume that the solutions of (6) are both right and left continuable on $(-\infty, +\infty)$. Moreover, we adopt the following notation:

• $\gamma^+(x)$ and $\gamma^-(x)$ denote respectively the positive and negative semiorbit of (6) issuing from x;

• $\Omega^+(x)$ and $\Omega^-(x)$ denote respectively the positive and negative limit set of x, with respect to (6).

Lemma 3 *Assume that (1), (2) is locally ZSD. Then the following holds:*

*(**O**) there exists a neighborhood U of the origin such that if $x \in U$ is such that $\gamma^-(x) \subset K \cap U$ then $x = 0$.*

Property (**O**) means that no negative semi-orbits (except the trivial one) are contained in U.

Theorem 1 *Assume that (1), (2) is locally ZSD and has finite L_2-gain with a storage function V such that $V(x) \geq 0$ for each $x \in \mathbf{R}^n$, $V(0) = 0$ and V is continuous at $x = 0$.*

Then, the origin of the unforced system (6) is stable.

In Theorem 1 the storage function is required to be continuous only at the origin. In order to obtain asymptotic stability, we need the storage function to be continuous on the whole of $\mathbf{R}^n$.

Theorem 2 *Assume that (1), (2) is locally ZSD and has finite L_2-gain with a storage function V such that $V(x) \geq 0$ for each $x \in \mathbf{R}^n$, $V(0) = 0$ and V is continuous on the whole of $\mathbf{R}^n$. Then, the origin of the unforced system (6) is locally asymptotically stable.*

If moreover (1), (2) is globally ZSD and V is radially unbounded then the origin of the unforced system (6) is globally asymptotically stable.

4 Hamilton-Jacobi characterization of finite L_2-gain

The main purpose of this section is to introduce the notion of exact storage function and to characterize it by means of the map

$$M(x,p) = pf(x) + \frac{1}{4k^2}\|pG(x)\|^2 + \|c(x)\|^2 \ .$$

We will refer to the following two equations:

$$M(x, \nabla V(x)) = 0 \tag{7}$$

$$-M(x, \nabla V(x)) = 0 \tag{8}$$

We begin by mentioning a known result (see [12, 17]). We refer to [5, 6] for the notions of viscosity solutions and solutions in the extended sense.

Theorem 3 *Assume that there exists a locally bounded viscosity supersolution of the equation (8) such that $V(x) \geq 0$ and $V(0) = 0$.*

Then system (1), (2) has finite L_2-gain and V is a storage function.

Vice versa assume that system (1), (2) has finite L_2-gain with a lower semicontinuous storage function V. Then V is a viscosity supersolution of (8).

From now on we restrict to locally Lipschitz continuous storage functions. We mention that Lipschitz continuity of viscosity solutions of Hamilton Jacobi equation is a subject treated in the literature (see, for example, [6]). In the following lemma the nonsmooth condition for finite L_2-gain in the case of a locally Lipschitz storage function is expressed in different equivalent forms.

Lemma 4 *If V is locally Lipschitz continuous the following three conditions are equivalent:*

(i) V *is a viscosity supersolution of* $-M(x, \nabla V(x)) = 0$;

(ii) V *is a viscosity subsolution of* $M(x, \nabla V(x)) = 0$;

(iii) for all x *and for all* $p \in \partial_C V(x) \quad M(x,p) \leq 0$.

Based on the previous lemma, we have the following proposition.

Proposition 1 *Let V be locally Lipschitz continuous and such that $V(x) \geq 0$ and $V(0) = 0$. V is a storage function for system (1), (2) if and only if*

$$\forall x \ \forall p \in \partial_C V(x) \quad M(x,p) \leq 0. \tag{9}$$

If moreover V is nonpathological then V is a storage function for system (1), (2) if and only if

$$\forall x \ \exists p_0 \in \partial_C V(x) \quad M(x,p_0) \leq 0. \tag{10}$$

Example 1 Let us consider the system

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -4x_2 + u \\ y = \sqrt{|x_1|}. \end{cases}$$

This system has finite L_2-gain with respect to the supply rate $\tilde{w}(y,u) = -|y|^2 + \frac{1}{4}|u|^2$. This can be proven by means of the storage function $V(x_1,x_2) = |x_1| + x_2^2$. Easy computations show that V satisfies all conditions (i), (ii) and (iii) of Lemma 4 (with $k = \frac{1}{2}$).

We now introduce a particular class of systems with the finite L_2-gain property. Let us recall that a dissipative system is said to be *inner* if the inequality in (3) becomes an equality for all inputs (see [20]).

Definition 3 *We say that system (1), (2) is* weakly inner *if it has finite L_2-gain and there exists a storage function V (that we call* exact storage function*) such that*

$$\forall x \ \exists u^*(\cdot): \ \forall t \geq 0 \quad V(\varphi(t;x,u^*(\cdot))) - V(x) = \int_0^t w(s)\,ds \tag{11}$$

where $w(s) = -\|c(\varphi(s;x,u^*(\cdot)))\|^2 + k^2\|u^*(s)\|^2$.

Even if not explicitly required in the definition of exact storage function, we just consider locally Lipschitz continuous exact storage functions. We now give necessary conditions for a system to have an exact storage function.

Theorem 4 *Assume that system (1), (2) admits a locally Lipschitz continuous exact storage function V such that $u^*(\cdot)$ in the definition of exact storage function is right continuous. Then V is a viscosity solution of equation (8) and a solution in the extended sense of equation (7).*

Moreover if V is $\overline{C}$-regular then

$$\forall x \ \forall p \in \partial_C V(x) \qquad M(x,p) = 0 \tag{12}$$

In the following theorem we give a sufficient condition for a system to have an exact storage function.

Theorem 5 *Assume that V is a locally Lipschitz continuous function such that $V(x) \geq 0$, $V(0) = 0$. If condition (12) holds then system (1), (2) is weakly inner and V is an exact storage function.*

Example 2 We consider the same system as in Example 1 with a different observation function: $y = \sqrt{|x_1| + 4x_2^2}$. The $\underline{C}$-regular function $V(x_1, x_2) = |x_1| + x_2^2$ satisfies condition (12) with $k = \frac{1}{2}$ then it is an exact storage function and the system is weakly inner.

Example 3 Let us consider the system

$$\begin{cases} \dot{x}_1 = -4x_1 + u \\ \dot{x}_2 = -x_2^3 \\ y = \sqrt{4x_1^2 + \frac{1}{2}x_2^4 + 2|x_1|x_2^4}. \end{cases}$$

The $\underline{C}$-regular function $V(x_1, x_2) = x_1^2 + x_2^2 + |x_1|x_2^2$ satisfies (9) then the system has finite L_2-gain and V is a storage function. V is not a solution in the extended sense of (7) so that it is not an exact storage function for the system.

5 Finite L_2-gain by feedback

In this section we deal with the problem of giving a system the finite L_2-gain property by means of a feedback law. More precisely, we look for a feedback law $K(x)$ such that the closed-loop system

$$\dot{x} = (f(x) + G(x)K(x)) + G(x)\tilde{u} \tag{13}$$

with the same observation map (2), has L_2-gain with respect to the new input $\tilde{u}$. Note that (13) is obtained from (1) by the substitution $u = K(x) + \tilde{u}$. In this section we give a result based on a modified viscosity-like inequality. In a natural way, the construction of the feedback law involves the gradient of the candidate storage function V. As in the smooth case we consider a feedback law in the damping form

$$K_\alpha(x) = -\alpha(\nabla V(x)G(x))^{\mathbf{t}}. \tag{14}$$

Since we assume that V is merely locally Lipschitz continuous, a problem arises. Indeed, we only have for sure that $\nabla V(x)$ exists almost everywhere and that it is locally bounded. Hence, our feedback law $K_\alpha(x)$ is not continuous in general, and the closed-loop system has a discontinuous right hand side even if the given vector fields $f, g_1, \ldots, g_m$ are smooth. As a consequence, solutions of (13) cannot always be intended in the classical sense. In this section, when the righthand side of (13) is discontinuous, we adopt the notion of generalized solution due to Filippov (see

[8]). We mention that the choice of Filippov solutions is not the only possible. In [18] the author treats a similar problem by means of the so-called sampling solutions. Another approach that would be interesting to be followed involves Carathéodory solutions. A possible future development of this work is to try to costruct a feedback law that defines a patchy feedback law as in [1].

In order to characterize finite L_2-gain by feedback we use the following map:

$$N(x,p) = pf(x) - \frac{1}{4k^2}\|pG(x)\|^2 + \|c(x)\|^2 \ .$$

If K_α is discontinuous, in order to cover the case where the given vector field f (which we continue to assume of class C^1) is replaced, as in (13), by a vector field $\tilde{f}(x) = f(x) + G(x)K_\alpha(x)$ which is a.e. defined and locally bounded, we need to restate of the definition of finite L_2-gain.

We denote by $\mathcal{S}_{x,\tilde{u}(\cdot)}$ the set of Filippov solutions of (13) issuing from x and corresponding to an input $\tilde{u}(\cdot)$. We keep the assumption that all solutions are right continuable for $t \geq 0$.

Definition 4 *We say that system (13) has* finite L_2-gain *if there exist $k \in \mathbf{R}$ and a function V, called a* storage function*, such that $V(x) \geq 0$, $V(0) = 0$ and*

$$\forall x \in \mathbf{R}^n, \forall \tilde{u}(\cdot) \in \mathcal{U}, \forall \varphi \in \mathcal{S}_{x,\tilde{u}(\cdot)}, \forall t \geq 0, \qquad V(\varphi(t)) \leq V(x) + \int_0^t w(s)\, ds \quad (15)$$

where $w(s) = -\|c(\varphi(s))\|^2 + k^2\|\tilde{u}(s)\|^2$.

Lemma 1 remains valid in this new setting.

Theorem 6 *Assume that V is a nonpathological function such that $V(x) \geq 0$, $V(0) = 0$ and*

$$\forall x \ \forall p \in \partial_C V(x) \qquad N(x,p) \leq 0. \quad (16)$$

Then system (13) with $K(x) = K_{\frac{1}{2k^2}}(x)$ has finite L_2-gain (in the sense of Definition 4).

Remark 2 The assumptions of Theorem 6 are satisfied in the following situations: there exists a $\underline{C}$-regular supersolution of the equation $-N(x, \nabla V(x)) = 0$ or there exists a $\overline{C}$-regular subsolution of the equation $N(x, \nabla V(x)) = 0$.

Example 4 Let us consider the system

$$\begin{cases} \dot{x}_1 = u(x_1 + x_2^2) \\ \dot{x}_2 = 2x_2 u \\ y = x_2^2. \end{cases}$$

The nonpathological function $V(x_1, x_2) = |x_1| + x_2^2$ satisfies condition (16) with $k = \frac{1}{2}$, then the system gets the finite L_2-gain property (in the sense of Definition 4) by means of the feedback law $K_2(x_1, x_2) = -|x_1| - (4 + \mathrm{sgn} x_1)x_2^2$ which is discontinuous.

Remark 3 In the particular case the following stronger condition is satisfied:

$$\forall x \ \forall p \in \partial_C V(x) \qquad N(x,p) = 0, \tag{17}$$

we have that the feedback law K_α admits a continuous extension (for a proof see [4]). In this case Filippov solutions of the implemented system coincide with classical (Carathéodory) solutions. Nevertheless the L_2-gain property that we obtain by means of Theorem 6 still has to be intended in the sense of Definition 4. In fact also in this case solutions of the implemented system are not unique in general.

Example 5 Let us consider the system

$$\begin{cases} \dot{x}_1 = u x_1 \\ \dot{x}_2 = u x_2 \\ y = \frac{7}{2}x_1^2 + \frac{13}{2}x_2^2 - 3\sqrt{3}|x_1|x_2. \end{cases}$$

The nonpathological function $V(x_1,x_2) = \frac{7}{4}x_1^2 + \frac{13}{4}x_2^2 - \frac{3\sqrt{3}}{2}|x_1|x_2$ satisfies condition (17) with $k = \frac{1}{2}$. Then the system gets the finite L_2-gain property when the feedback law $K_2(x_1,x_2) = -\frac{7}{2}x_1^2 - \frac{13}{2}x_2^2 + 3\sqrt{3}|x_1|x_2$ is implemented. Note that K_2 is continuous, so that solutions of the implemented system are classical solutions.

As a corollary of Theorem 6, we finally point out a relationship between optimal regulation and finite L_2-gain by feedback. Let us assume that

$$c(x) = 0 \text{ if and only if } x = 0.$$

Associated to (1), (2), let us consider the problem of minimizing for each $x \in \mathbf{R}^n$, the cost functional

$$J(x,u) = \int_0^{+\infty} \left(\|c(\varphi(t;x,u(\cdot)))\|^2 + k^2\|u(t)\|^2 \right) dt. \tag{18}$$

As admissible inputs $u(\cdot) \in \mathcal{U}$ we consider piecewise continuous and right continuous functions. Let V be the corresponding value function, i.e.

$$V(x) = \inf_{u(\cdot)\in\mathcal{U}} J(x,u).$$

We say that the minimization problem is solvable if for each x the infimum in the definition of V is a minimum. It is proved in [4] (Proposition 2) that if the optimal regulation problem is solvable and the value function V is $\underline{C}$-regular, then condition (17) is satisfied.

Corollary 1 *If the optimal regulation problem associated to (18) is solvable and if the value function V is $\underline{C}$-regular, then system (13) with $K(x) = K_{\frac{1}{2k^2}}(x)$ has finite L_2-gain (in the sense of Definition 4).*

We conclude by mentioning that in [4] it is proved that in the particular case the system is linear and $\|c(x)\|^2$ is convex the value function corresponding to the optimal regulation problem is $\underline{C}$-regular.

6 Proofs

6.1 Proofs of Section 3

Proof of Lemma 3

Assume that the property is false. Then, for each U there exists $y \neq 0$ such that

$$\gamma^-(y) \subset K \cap U \ . \tag{19}$$

Without loss of generality, we can assume that U is bounded, so that $\Omega^-(y)$ is compact and nonempty. Moreover, since K is closed and $\Omega^-(y)$ is invariant, we also have $\Omega^-(y) \subset K \cap \overline{U}$. Let us distinguish two cases.

1) Let $0 \notin \Omega^-(y)$, and let $z \in \Omega^-(y)$. Clearly, the trajectory issuing from z cannot be attracted to the origin, and this contradicts the assumption.

2) Let $0 \in \Omega^-(y)$. Then there exist a sequence $t_k \to -\infty$ such that $\lim_k \varphi(t_k; y, 0) = 0$. Because of (19), $\varphi(t_k; y, 0) \in K \cap U$ for each k. Hence, in any neighborhood of the origin we can find points from which it is possible to reach $y \neq 0$. This implies that the origin is not stable and the ZSD assumption is contradicted.

∎

Proof of Theorem 1

From the dissipation inequality (3) with $u \equiv 0$ we immediately deduce that the composite function $V(\varphi(t))$ is nonincreasing for each solution φ of (6). Let $Z_V = \{x : V(x) = 0\}$. Using again the dissipation inequality, from $x \in Z_V$ we deduce

$$V(\varphi(t; x, 0)) = 0 \ \text{ for each } t \geq 0 \tag{20}$$

and

$$\int_0^t ||c(\varphi(s; x, 0))||^2 \, ds = 0 \ \text{ for each } t \geq 0 \ . \tag{21}$$

In turn, (21) implies $c(\varphi(t; x, 0)) = 0$ for each $t \geq 0$, and in particular $c(x) = 0$. This means that $Z_V \subset Z_c$. On the other hand, (20) implies that Z_V is positively invariant for (6), so that we actually have $Z_V \subset K \subset Z_c$. According to Lemma 3, we conclude that there is a neighborhood U of the origin such that no negative trajectory of (6) is contained in $Z_V \cap U$. We are now in a position to apply Theorem 2.1 of [13], and this completes the argument.

∎

Proof of Theorem 2

We first prove local asymptotic stability. Theorem 1 tells us that the origin is stable. Let $\varepsilon > 0$ be such that $\overline{B_\varepsilon} \subset U$, where B_ε is the ball of radius ε centered at the origin and U is defined by property (**O**). Hence, there exists $\delta > 0$ such that $\gamma^+(x) \subset B_\varepsilon$ for each $x \in B_\delta$.

Let us fix $z \in B_\delta$. Since $\gamma^+(z)$ is bounded, the set $\Omega^+(z)$ is nonempty and compact. Moreover, it is invariant and it is contained in $\overline{B_\varepsilon} \subset U$. Let

$$N = \{x : V(\varphi(t;x,0)) = \text{constant } \forall t \in \mathbf{R}\} \ .$$

We claim that $\Omega^+(z) \subset N$. Indeed, $V(\varphi(t;z,0))$ being nonincreasing, we have

$$\lim_{t\to+\infty} V(\varphi(t,z,0)) = l \geq 0 \ .$$

Since V is continuous, this yields $V(y) = l$ for any $y \in \Omega^+(z)$. Fix any $y \in \Omega^+(z)$. By invariance, the trajectory issuing from such y is entirely contained in $\Omega^+(z)$, and therefore $V(\varphi(t;y,0)) = l$ =constant for each $t \in \mathbf{R}$. This implies $y \in N$, as required.

An argument based on the dissipation inequality, similar to that used in Theorem 1, shows that $N \subset Z_c$. Thus, a nonempty, compact, invariant set $\Omega^+(z)$ contained in $Z_c \cap U$ has been identified. In fact, by virtue of invariance, $\Omega^+(z) \subset K \cap U$.

In order to finish the proof, assume that there exists $y \neq 0$, $y \in \Omega^+(z)$. We have that $\gamma^-(y) \subset \Omega^+(z) \subset K \cap U$. According to Lemma 3, this is impossible.

Hence, $\Omega^+(z) = \{0\}$ and the origin is locally attractive.

In order to prove global asymptotic stability we limit ourselves to point out the main changes with respect to the the preceeding part of the proof. First of all, we remark that under the global ZSD assumption property (**O**) holds with $U = \mathbf{R}^n$. Now, let $r > 0$ and let $m = \max_{||x||\leq r} V(x)$. Since V is radially unbounded, we may assume $m > 0$; moreover, there exists $R > r$ such that $V(x) > m$ whenever $||x|| > R$. In summary we have

$$B_r \subset \{x : V(x) < m\} \subset B_R \ .$$

Since $V(\varphi(t;x,0))$ is nonincreasing for any x, the set $\{x : V(x) < m\}$ is positively invariant. Let $z \in \{x : V(x) < m\}$. By repeating the reasonning as in the first part of the proof, we see that

$$\Omega^+(z) \subset N \subset Z_c \ .$$

In addition, $\Omega^+(z) \subset B_R$ is bounded. The proof is easily carried over.

∎

6.2 Proofs of Section 4

Proof of Lemma 4

$(i) \Rightarrow (iii)$ The proof of this implication is analogous to that of Proposition 5.13, page 85, in [5].

From (i) it follows that $M(x,p) \leq 0$ for all $p \in \underline{\partial} V(x)$. Since in the points where V is differentiable $\underline{\partial} V(x) = \{\nabla V(x)\}$, we have that $M(x, \nabla V(x)) \leq 0$ for a.e. x and then $M(x,p) \leq 0$ for all x and for all $p \in \partial^* V(x)$. Let x and $p \in \partial_C V(x) = \mathrm{co}\, \partial^* V(x)$ be fixed arbitrarily. There exist $p_1, p_2 \in \partial^* V(x)$ and $\alpha_1, \alpha_2 \in [0,1]$ such that $\alpha_1 + \alpha_2 = 1$ and $p = \alpha_1 p_1 + \alpha_2 p_2$. Using the convexity of M in p we get that $M(x,p) \leq \alpha_1 M(x,p_1) + \alpha_2 M(x,p_2) \leq 0$.

$(iii) \Rightarrow (ii)$ follows from the inclusion $\overline{\partial} V(x) \subseteq \partial_C V(x)$.

$(ii) \Rightarrow (i)$ is a consequence of Proposition 5.4 page 79 in [5].

∎

Proof of Proposition 1

The first part of the Proposition is an immediate consequence of Lemma 4 and Theorem 3.

We have now to consider the case V is nonpathological. If V is a storage function for the system then condition (9) is satisfied and then also condition (10). Viceversa we now prove that if condition (10) is satisfied then V is a storage function for the system.

Let $\varphi(\cdot)$ be the (unique) solution of (1) corresponding to an initial state x and an admissible input $u(\cdot) \in \mathcal{U}$. Since V is nonpathological and $\varphi(\cdot)$ is absolutely continuous then for a.e. t the set $\{p \cdot \dot{\varphi}(t), p \in \partial_C V(\varphi(t))\}$ is reduced to the singleton $\{\frac{d}{dt} V(\varphi(t))\}$. Then in particular for a.e. t we have that $\{\frac{d}{dt} V(\varphi(t))\} = \{p_0 \cdot [f(\varphi(t)) + G(\varphi(t))u(t)]\}$, where $p_0 \in \partial_C V(\varphi(t))$ verifies condition (10). Still using condition (10) we get that for a.e. t

$$
\begin{aligned}
\frac{d}{dt} V(\varphi(t)) &\leq -\|c(\varphi(t))\|^2 - \frac{1}{4k^2} \|p_0 G(\varphi(t))\|^2 + p_0 G(\varphi(t)) u(t) = \\
&= -\|c(\varphi(t))\|^2 + k^2 \|u(t)\|^2 - \|\frac{1}{2k} p_0 G(\varphi(t)) - k u(t)\|^2 \leq \\
&\leq -\|c(\varphi(t))\|^2 + k^2 \|u(t)\|^2.
\end{aligned}
$$

The conclusion follows from Lemma 1.

∎

Proof of Theorem 4

Let $u^*(t)$ and $\varphi(t) = \varphi(t; x, u^*(\cdot))$ be as in the definition of exact storage function. From (11) we get that for all t

$$
\frac{V(\varphi(t+h)) - V(\varphi(t))}{h} = \frac{1}{h} \int_t^{t+h} -\|c(\varphi(s))\|^2 + k^2 \|u^*(s)\|^2 \, ds
$$

Since $u^*(\cdot)$ is right continuous, the limit as $h \to 0^+$ of the righthand side of the previous equality always exists. Then also the right limit of the lefthand side exists and, by using Lipschitz continuity of V, we get that

$$\begin{aligned}
\frac{d^+}{dt}V(\varphi(t)) &= \lim_{h\to 0^+}\frac{V(\varphi(t+h))-V(\varphi(t))}{h} = \lim_{h\to 0^+}\frac{V(\varphi(t)+h\dot{\varphi}(t))-V(\varphi(t))}{h} \\
&= \limsup_{h\to 0^+}\frac{V(\varphi(t)+h\dot{\varphi}(t))-V(\varphi(t))}{h} \\
&= \overline{D}^+V(\varphi(t), f(\varphi(t))+G(\varphi(t))u^*(t)) \\
&= -\|c(\varphi(t))\|^2 + k^2\|u^*(t)\|^2.
\end{aligned}$$

By writing the previous equality at $t = 0$ and using the characterization of the superdifferential by means of Dini derivatives, we get that for all $p \in \overline{\partial}V(x)$

$$p\cdot(f(x)+G(x)u(0)) \geq \overline{D}^+V(x, f(x)+G(x)u^*(0)) = -\|c(x)\|^2 + k^2\|u^*(0)\|^2.$$

From this it follows that for all $p \in \overline{\partial}V(x)$

$$-pf(x) - \frac{1}{4k^2}\|pG(x)\|^2 - \|c(x)\|^2 + \|\frac{1}{2k}pG(x) - ku^*(0)\|^2 \leq 0$$

and then

$$\forall x\ \forall p \in \overline{\partial}V(x)\ \ -M(x,p) \leq 0, \tag{22}$$

i.e. V is a viscosity subsolution of (8). Since V is a storage function from Theorem 3 we have that V is a viscosity supersolution of (8) and finally V is a viscosity solution of (8).

We now have to prove that V is a solution in the extended sense of (7). Since V is a storage function from Proposition 1 we have (9). Then we just need to prove that for all x there exists $p_0 \in \partial_C V(x)$ such that $M(x,p_0) = 0$. Let us consider $u^*(\cdot)$ and $\varphi(\cdot)$ as before. For all t we have that there exists $p_0 \in \partial_C V(\varphi(t))$ such that (see [4], Lemma 1)

$$\frac{d^+}{dt}V(\varphi(t)) = p_0\cdot\dot{\varphi}(t) = -\|c(\varphi(t))\|^2 + k^2\|u^*(t)\|^2.$$

By computing the previous equality at $t = 0$ we get that

$$p_0\cdot(f(x)+G(x)u(0)) = -\|c(x)\|^2 + k^2\|u^*(0)\|^2.$$

From this it follows that

$$(p_0\cdot f(x) + \frac{1}{4k^2}\|p_0G(x)\|^2 + \|c(x)\|^2) - \|\frac{1}{2k}p_0G(x) - ku^*(0)\|^2 = 0.$$

The quantity in the brackets is less or equal zero thanks to (9) then the only possibility to get zero in the previous sum is that

$$p_0\cdot f(x) + \frac{1}{4k^2}\|p_0G(x)\|^2 + \|c(x)\|^2 = 0.$$

Finally we have to prove condition (12) in the case V is $\overline{C}$-regular. On one hand, since V is $\overline{C}$-regular, $\partial_C V(x) = \overline{\partial} V(x)$ for all x and then from (22) it follows that $M(x,p) \geq 0$ for all x and for all $p \in \partial_C V(x)$. On the other hand, since V is a storage function, we have (9) and then (12).

∎

Proof of Theorem 5
The fact that V is a storage function follows from Proposition 1. We now have to find a control $u^*(\cdot)$ such that (11) holds. First of all we remark that if (12) is satisfied then for all x and for all $p,q \in \partial_C V(x)$ it holds $pG(x) = qG(x)$ (for the proof see [4], Theorem 3). Let us now consider the differential inclusion

$$\dot{x} \in f(x) + \frac{1}{2k^2} G(x)(G(x)\partial_C V(x))^{\mathbf{t}} \tag{23}$$

and let $\varphi(\cdot)$ be a solution of (23). Thanks to Filippov's Lemma (see [2], page 316) there exists a measurable function $q(t)$ such that for a.e. t one has $q(t) \in \partial_C V(\varphi(t))$ and

$$\dot{\varphi}(t) = f(\varphi(t)) + \frac{1}{2k^2} G(\varphi(t))(q(t)G(\varphi(t)))^{\mathbf{t}}.$$

Let us then consider $u^*(t) = \frac{1}{2k^2}(q(t)G(\varphi(t)))^{\mathbf{t}}$. For a.e. t we have that $\frac{d}{dt}V(\varphi(t))$ exists and moreover there exists $p(t) \in \partial_C V(\varphi(t))$ such that $\frac{d}{dt}V(\varphi(t)) = p(t)\cdot\dot{\varphi}(t)$. Using (12) and the fact that for all x and for all $p,q \in \partial_C V(x)$ $\;pG(x) = qG(x)$, we then get that

$$\begin{aligned} \frac{d}{dt}V(\varphi(t)) &= p(t) \cdot f(\varphi(t)) + \frac{1}{2k^2}(p(t)G(\varphi(t))) \cdot (q(t)G(\varphi(t))) \\ &= -\|c(\varphi(t))\|^2 + \frac{1}{4k^2}\|p(t)G(\varphi(t))\|^2 \\ &= -\|c(\varphi(t))\|^2 + k^2\|u(t)\|^2. \end{aligned}$$

The conclusion follows from Lemma 1.

∎

6.3 Proofs of Section 5

Proof of Theorem 6
We start by two lemmas. We denote by $\mathbf{F}_x$ the Filippov's operator which transforms a differential equation with discontinuous righthand side in a differential inclusion (see [14] or [3] for its definition).

Lemma 5 *Let $h_1(x) : \mathbf{R}^n \to \mathbf{R}^n$ be locally bounded, and let $h_2(t,x) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous with respect to x and right continuous with respect to t. Then, for each pair (t,x),*

$$\mathbf{F}_x(h_1 + h_2)(t,x) = (\mathbf{F}_x h_1)(x) + h_2(t,x)\ .$$

Lemma 6 *Let V be locally Lipschitz continuous. Then, $\mathbf{F}_x(\nabla V(x)) = \partial_C V(x)$ for each $x \in \mathbf{R}^n$.*

The proofs of Lemmas 5 and 6 can be found in [14].

We can now proceed to the proof of the Theorem 6. Let $\varphi(t)$ be a Filippov solution of (13) with $\alpha = -\frac{1}{2k^2}$ corresponding to the input $\tilde{u}(\cdot) \in \mathcal{U}$. By virtue of Lemmas 5 and 6 for a.e. t

$$\dot{\varphi}(t) \in f(\varphi(t)) - \frac{1}{2k^2} G(\varphi(t))(\partial_C V(\varphi(t)) G(\varphi(t)))^{\mathbf{t}} + G(\varphi(t))\tilde{u}(t). \qquad (24)$$

Since V is nonpathological, thanks to Lemma 2, for a.e. t we have

$$\begin{aligned} \frac{d}{dt} V(\varphi(t)) \quad \in \quad & \{a \in \mathbf{R} : \ \exists q \in \partial_C V(\varphi(t)) \text{ s.t. } \forall p \in \partial_C V(\varphi(t)) \\ & pf(\varphi(t)) - \frac{1}{2k^2}(pG(\varphi(t)))(qG(\varphi(t))) + pG(\varphi(t))\tilde{u}(t) = a\} \\ = \quad & \{a \in \mathbf{R} : \exists q \in \partial_C V(\varphi(t)) \text{ s.t. } \\ & qf(\varphi(t)) - \frac{1}{2k^2} \|qG(\varphi(t))\|^2 + qG(\varphi(t))\tilde{u}(t) = a\}. \end{aligned}$$

Then for a.e. t there exists $q \in \partial_C V(\varphi(t))$ such that

$$\begin{aligned} \frac{d}{dt} V(\varphi(t)) \quad &= \quad qf(\varphi(t)) - \frac{1}{2k^2} \|qG(\varphi(t))\|^2 + qG(\varphi(t))\tilde{u}(t) \\ &\leq \quad -\|c(\varphi(t))\|^2 + k^2 \|\tilde{u}(t)\|^2 - \|\frac{1}{2k} qG(\varphi(t)) - k\tilde{u}(t)\|^2 \\ &\leq \quad -\|c(\varphi(t))\|^2 + k^2 \|\tilde{u}(t)\|^2. \end{aligned}$$

The conclusion follows from Lemma 1.

∎

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