Generalized solutions of differential inclusions and stability

A. Bacciotti
Dipartimento di Matematica del Politecnico di Torino
C.so Duca degli Abruzzi, 24 - 10129 Torino - Italy
bacciotti@polito.it

Abstract

This paper is a contribution to the development of a theory concerning stability of equilibrium positions with respect to Sentis solutions of ordinary differential equations with discontinuous right hand side.

Keywords: Ordinary differential equations with discontinuous right hand side, Generalized solutions, Lyapunov stability

1 Introduction

Roughly speaking, we may agree that a notion of solution for ordinary differential equations

$$\dot{x} = f(x) \ , \qquad x \in \mathbf{R}^n \tag{1}$$

is satisfactory if one can prove that there is at least one local solution for each initial state $\bar{x}$. For instance, *classical solutions*[1] represents a satisfactory notion, only if f is continuous. Neither the slightly more general notion of *Carathéodory solution*[2] is satisfactory in general, when f is not continuous.

A well known way to seek more satisfactory notions of solution of ordinary differential equations with a discontinuous right hand side is to replace (1) by a suitable differential inclusion

$$\dot{x} \in F(x) \ . \tag{2}$$

The notions of solution obtained in this way depend on the construction of the set valued map F. For instance, Filippov solutions of (1) are the ordinary solutions[3] of (2) where

$$F(x) = F_F(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\mathrm{co}} \, \{f(\mathcal{B}(x,\delta)\backslash N)\} \tag{3}$$

(here μ is the Lebesgue measure of $\mathbf{R}^n$, $\overline{\mathrm{co}}$ denotes the closure of the convex hull, and $\mathcal{B}(x,r)$ is the ball of radius r centered at x).

From now on, we always assume that f is measurable and locally bounded. Then the set valued map $F_F(x)$ is upper semicontinuous, compact and convex valued, and locally bounded, so that (1) has a Filippov solution for each initial state $\bar{x}$ ([12]). On the other hand, it is commonly recognized that Filippov solutions are "too many" for some applications (this is true in particular in the stabilization problem for nonlinear systems by means of discontinuous feedback, see [5], [6], [7]). To overcome this drawback, in [15] the author replaces (1) by a differential inclusion (2) where

$$F(x) = F_S(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\{f(\mathcal{B}(x,\delta)\backslash N)\}} \ . \tag{4}$$

[1] A function $\varphi(t) : I \to \mathbf{R}^n$ is a classical solution if it is differentiable and satisfies (1) for each $t \in I$

[2] A function $\varphi(t) : I \to \mathbf{R}^n$ is a Carathéodory solution if it is differentiable and satisfies (1) a.e. on the interval I

[3] By *ordinary solution* of the differential inclusion (2) we mean any function $\varphi(t) : I \to \mathbf{R}^n$ which is differentiable and satisfies $\dot{\varphi}(t) \in F(\varphi(t))$ a.e. on the interval I

The set valued map $F_S(x)$ turns out to be upper semicontinuous, locally bounded and compact (but in general not convex) valued. Unfortunately, the notion of ordinary solution is not satisfactory for general differential inclusions with nonconvex right hand side. Thus, in [15] a new class of solutions, called g-solutions, is introduced (their definition is reported in Appendix A, for the convenience of readers), and for differential inclusions with upper semicontinuous and locally bounded right hand side an existence theorem is proven. In [15] it is also proven that ordinary solutions of (2), when they exist, are g-solutions.

We propose to say that a function $\varphi(t)$ is a Sentis solution of (1) if it is a g-solution of (2) with $F(x) = F_S(x)$. From [15], we know that Sentis solutions are Filippov solutions, but the converse is false in general. Other studies about g-solutions are carried out in [13], especially for the time-dependent case.

One of the most important issue in the qualitative theory of systems is the stability analysis around equilibria. The stability problem for systems with discontinuous right hand side has been largely studied with respect to Filippov solutions (see [5] and the reference therein), but apparently it has not yet been addressed with respect to Sentis solutions. The present paper is a first contribution in this direction: we emphasize that we are interested in the so-called *strong* stability (the paper [14] concerns instead weak stability).

The theorem of Section 2 is the analogue of First Lyapunov Theorem for the case of Sentis solutions. The Lyapunov condition used in this paper includes a uniformity issue. In view of possible applications to control theory, it would be desirable to remove it. In Section 3 we deal with asymptotic stability.

Recently, in order to introduce satisfactory notions of solution more suitable for applications to control theory, new approaches (not involving differential inclusions) have been tried ([1], [2], [7], [8]). An interesting program for future research is to compare the notion of Sentis solution with the notion of sampling and forward Euler solution.

2 Stability for Sentis solutions

In order to illustrate that the stability behavior may look different if we change the notion of solution, we start by an example.

Example 1 Let us consider the discontinuous two-dimensional vector field defined for $x \geq 0$ by

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{cases} 0 & \text{if either } x = 0 \text{ or } y = 0 \\ \begin{pmatrix} -\frac{4y}{3} + \frac{\sqrt{3x^2+4y^2}}{3} \\ x \end{pmatrix} & \text{if } x > 0 \text{ and } y > 0 \\ \begin{pmatrix} \frac{4y}{3} + \frac{\sqrt{3x^2+4y^2}}{3} \\ -x \end{pmatrix} & \text{if } x > 0 \text{ and } y < 0 \end{cases}$$

The definition of f is completed in such a way that $f_1(-x,y) = -f_1(x,y)$ and $f_2(-x,y) = f_2(x,y)$. The phase portrait is plotted in Figure 1: trajectories are drawn in the counterclockwise sense in the quadrants where $xy > 0$, and in the clockwise sense in the quadrants where $xy < 0$.

The origin is not stable in the sense of Filippov solutions. Indeed, there are Filippov solutions which go to the infinity along the x-axis. According to Proposition 5 of [15], those are not Sentis solutions. In fact, the system is stable at the origin, with respect to Sentis solutions.

∎

The aim of this section is to state and prove an analogue of first Lyapunov Theorem for stability with respect to Sentis solutions. For a given initial state $\xi \in \mathbf{R}^n$, let us denote respectively by $\mathcal{S}_\xi$ and $\mathcal{F}_\xi$ the set of Sentis and Filippov solutions of (1). We recall that for each $\xi \in \mathbf{R}^n$, $\mathcal{S}_\xi$ is nonempty and that $\mathcal{S}_\xi \subseteq \mathcal{F}_\xi$. We make the standing assumption that $\varphi(t) \equiv 0$ is a Sentis solution for (1).

We say that the origin is (strongly) *S-stable* (respectively, *F-stable*) for (1) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\xi| < \delta$, then for all $\varphi \in \mathcal{S}_\xi$ (respectively, $\varphi \in \mathcal{F}_\xi$), $\varphi(t)$ is right continuable on $[0, +\infty)$ and

$$|\varphi(t)| < \varepsilon \ , \qquad \forall t \geq 0 \ .$$

Throughout the paper, $V(x)$ always denotes a *Lyapunov function*, that is a positive definite (i.e., such that $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$), locally Lipschitz continuous function $V : \mathbf{R}^n \to \mathbf{R}$. The symbols $\underline{D^+}V(x;v)$,

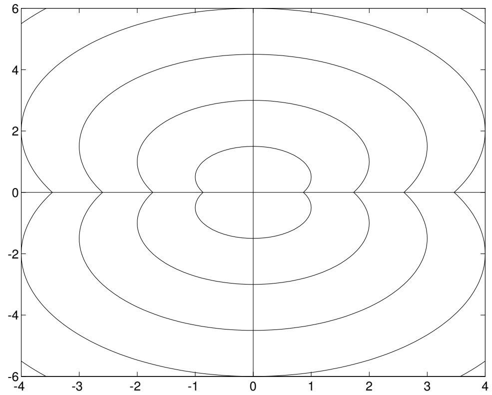

Figure 1: phase portrait of the system of Example 1

$\overline{D^+}V(x;v)$ respectively denote the lower and upper right directional Dini derivative of V at x with respect to v. It is well known that if there exists a Lyapunov function $V(x)$ such that the following condition

$$\underline{D^+}V(x;v) \leq 0 \tag{5}$$

is fulfilled for all $v \in F_F(x)$ and for all $x \in \mathbf{R}^n$, then the origin is F-stable (see [12], [11]): the proof of this fact can be carried out by simply exploiting a monotonicity condition like in [5] and [4]. Unfortunately, the same idea does not work for S-stability since Sentis solutions do not satisfy $\dot{\varphi}(t) \in F_S(\varphi(t))$, in general.

We say that (1) satisfies a *uniform weak Lyapunov S-condition* if there exists a Lyapunov function $V(x)$ such that the inequality

$$\overline{D^+}V(x;v) \leq 0 \tag{6}$$

holds for all $x \in \mathbf{R}^n$ and for all $v \in F_S(x)$, uniformly on compact sets. More precisely, we require that for each compact set $K \subset \mathbf{R}^n$ with $0 \notin K$, and for each $\varepsilon > 0$, there exists $\eta > 0$ such that

$$0 < \theta < \eta \quad \Longrightarrow \quad V(x + \theta v) - V(x) \leq \varepsilon\theta$$

for all $x \in K$, and for all $v \in F_S(x)$.

Theorem 1 *Let us consider system (1) with f measurable and locally bounded. Let us assume that a uniform weak Lyapunov S-condition is fulfilled. Then, the origin is S-stable.*

Proof Let us consider the differential inclusion

$$\dot{x} \in F_S(x) \ . \tag{7}$$

In the first part of the proof we show that if $\varphi(t)$ is a g-solution of (7) defined on some interval I, and such that $\varphi(t) \neq 0$ for each $t \in I$, then the function $t \mapsto V(\varphi(t))$ is nonincreasing for $t \in I$.

By contradiction, assume that there exists $a, b \in I$ with $a < b$ such that

$$\gamma = V(\varphi(b)) - V(\varphi(a)) > 0 \ . \tag{8}$$

Let K be a compact set ($0 \notin K$), such that the image of φ for $t \in [a, b]$ is contained in the interior of K. Let L be a Lipschitz constant for V on K. Let us fix a positive number σ in such a way that the following two conditions are satisfied:

(i) $|x - \varphi(b)| < \sigma \Longrightarrow |V(x) - V(\varphi(b))| < \frac{\gamma}{3}$ (this is possible, since V is continuous), and

(ii) $\cup_{t \in [a,b]} \mathcal{B}(\varphi(t), \sigma) \subset K$.

According to the uniform weak Lyapunov S-condition, we can find a number $\eta > 0$ such that

$$V(x+\theta v) - V(x) \leq \frac{\gamma}{6(b-a)}\theta \tag{9}$$

for each $x \in K$, each $v \in F_S(x)$, and each $\theta \in (0,\eta)$. By definition of g-solution, there exists a partition $a = t_0 < t_1 < \ldots < t_{k-1} < t_k = b$ and a right-continuous, piecewise affine function $\psi(t)$ such that

$$\begin{array}{rcl} x_0 & = & \psi(t_0) = \varphi(t_0) \\ x_1 & = & \psi(t_1) = \psi(t_0) + v_0(t_1 - t_0) + \varepsilon_0 \\ x_2 & = & \psi(t_2) = \psi(t_1) + v_1(t_2 - t_1) + \varepsilon_1 \\ & & \ldots\ldots\ldots \\ x_k & = & \psi(t_k) = \psi(t_{k-1}) + v_{k-1}(t_k - t_{k-1}) + \varepsilon_{k-1} \ . \end{array} \tag{10}$$

Moreover,

$$|\psi(t) - \varphi(t)| < \sigma \qquad \forall t \in [a,b] \tag{11}$$

and

$$\sum_{i=0}^{k-1} |\varepsilon_i| < \frac{\gamma}{6L} \ , \qquad |t_{i+1} - t_i| < \eta \ \ (i = 0, \ldots, k-1) \ . \tag{12}$$

From (11), (i) and (8) it follows

$$V(\psi(b)) = V(\psi(b)) - V(\varphi(b)) + V(\varphi(b)) \geq V(\varphi(b)) - \frac{\gamma}{3} = V(\varphi(a)) + \frac{2\gamma}{3} \ . \tag{13}$$

Now set

$$\begin{array}{rclcl} \bar{x}_1 & = & x_0 + v_0(t_1 - t_0) & & \text{that is} \ \ x_1 = \bar{x}_1 + \varepsilon_0 \\ & & \ldots\ldots\ldots & & \\ \bar{x}_k & = & x_{k-1} + v_{k-1}(t_k - t_{k-1}) & & \text{that is} \ \ x_k = \bar{x}_k + \varepsilon_{k-1} \ . \end{array}$$

We have

$$\begin{array}{rcl} V(\psi(b)) = V(\psi(t_k)) = V(x_k) & = & V(x_k) - V(\bar{x}_k) + V(\bar{x}_k) \\ & \leq & L|\varepsilon_{k-1}| + V(\bar{x}_k) \\ & \leq & V(x_{k-1}) + L|\varepsilon_{k-1}| + \dfrac{\gamma}{6(b-a)}(t_k - t_{k-1}) \end{array} \tag{14}$$

where the last inequality is a consequence of (9). Iterating the same argument, we obtain

$$V(\psi(b)) \leq V(\psi(a)) + L\sum_{i=0}^{k-1} |\varepsilon_i| + \frac{\gamma}{6} < V(\varphi(a)) + \frac{\gamma}{3} \tag{15}$$

a contradiction to (13). The second part of the proof is standard. Assume by contradiction that the origin is not S-stable for (7). Then, there exists $\bar{\varepsilon} > 0$ such that for each $\delta > 0$ there is a point x_0, an instant $T > 0$ and a g-solution $\varphi \in \mathcal{S}_{x_0}$ such that

$$|x_0| < \delta, \quad |\varphi(T)| = \bar{\varepsilon} \ .$$

Clearly, it is not restrictive to assume that $\varphi(t) \neq 0$ for each $t \in [0,T]$. Let $M = \min_{|x|=\bar{\varepsilon}} V(x) > 0$, so that $V(\varphi(T)) \geq M$. Since V is continuous, we can take δ in such a way that

$$|x| < \delta \Longrightarrow V(x) < M \ .$$

We should have $V(\varphi(0)) < M \leq V(\varphi(T))$ which is impossible by the first part of the proof.

∎

Coming back to Example 1, the uniform weak Lyapunov S-condition is satisfied with the Lyapunov function

$$V(x,y) = \frac{-|y| + \sqrt{3x^2 + 4y^2}}{3} \ .$$

3 Asymptotic stability

Once more we begin by an example, borrowed from control theory ([3]).

Example 2 Let us consider the two-dimensional system

$$\begin{cases} \dot{x} = u(x,y)(x^2 - y^2) \\ \\ \dot{y} = 2u(x,y)xy \end{cases} \qquad \text{where} \ \ u(x,y) = \begin{cases} 1 & \text{if } x \leq 0 \\ -1 & \text{if } x > 0 \end{cases} \ . \tag{16}$$

The system is asymptotically stable at the origin in the sense of Sentis solutions but not in the sense of Filippov solutions: indeed, according to Filippov's approach, there are equilibrium solutions along the y-axis.

∎

Asymptotic stability with respect to Filippov solutions has been recently characterized in terms of smooth (i.e., of class C^∞) Lyapunov functions (see [5], [9]). In this section we prove that asymptotic stability with respect to Sentis solutions is guaranteed when the following strict Lyapunov S-condition holds:

$$\underline{D^+}V(x;v) \leq -W(x) \ , \qquad \forall v \in F_S(x) \quad \forall x \in \mathbf{R}^n \tag{17}$$

where $W(x)$ is continuous and positive definite. More precisely, the following holds.

Theorem 2 *Assume that there exists a Lyapunov function $V(x)$ and a continuous, positive definite function $W(x)$ such that (17) holds. Then, the origin is asymptotically stable with respect to Sentis solutions of (1).*

Before proving the theorem, let us remark that under the assumption of Theorem 2 the origin is actually stable. Indeed, it would be not difficult to prove directly that (17) implies the uniform weak Lyapunov S-condition. In fact we have the following lemma.

Lemma 1 *Let $\varphi(t)$ be a Sentis solution on some compact interval I, and let $\varphi(t) \neq 0$ for each $t \in I$. Under the assumption (17), the function $t \mapsto V(\varphi(t))$ is strictly decreasing on I.*

Proof By contradiction, assume that there exist $a, b \in I$ such that $a < b$ and $V(\varphi(a)) \leq V(\varphi(b))$. Let

$$H = \cup_{t\in[a,b]} \mathcal{B}(\varphi(t), \sigma_0)$$

where $\sigma_0 > 0$ is chosen in such a way that $0 \notin \overline{H}$. Let $L > 0$ be a Lipschitz constant for V on $\overline{H}$, and let $\bar{w} = \min_{x\in\overline{H}} W(x) > 0$. Take $\sigma_1 > 0$ in such a way that

$$|x - \varphi(b)| < \sigma_1 \Longrightarrow |V(x) - V(\varphi(b))| < \frac{\bar{w}}{2}(b-a) \ . \tag{18}$$

Let finally $\sigma = \min\{\sigma_0, \sigma_1\}$. As in the proof of Theorem 1, we pick up a piecewise function ψ on the interval $[a,b]$ which satisfies (10), (11) and, instead of (12),

$$\sum_{i=0}^{k-1} |\varepsilon_i| < \frac{\bar{w}}{2L}(b-a) \ . \tag{19}$$

The image of $\psi(t)$ is inside H, and the map $t \mapsto V(\psi(t))$ is absolutely continuous on each interval of the form $t \in [t_i, t_{i+1})$. Hence, for each $i = 0, \ldots, k-1$ and for each $t \in [t_i, t_{i+1})$ we have

$$\begin{aligned} V(\psi(t)) &= V(\psi(t_i)) + \int_{t_i}^{t} [V(\psi(s))]' \, ds = \\ &= V(\psi(t_i)) + \int_{t_i}^{t} \underline{D^+} V(\psi(s), v_i) \, ds \leq \\ &\leq V(\psi(t_i)) - \bar{w}(t - t_i) \ . \end{aligned}$$

This implies

$$V(\bar{x}_{i+1}) \leq V(x_i) - \bar{w}(t_{i+1} - t_i) \ . \tag{20}$$

Taking into account (20), we can obtain an estimation of $V(\psi(b))$ stronger than (14):

$$V(\psi(b)) = V(\psi(t_k)) = V(x_k) = V(x_k) - V(\bar{x}_k) + V(\bar{x}_k) \leq L|\varepsilon_{k-1}| + V(\bar{x}_k) \leq L|\varepsilon_{k-1}| + V(x_{k-1}) - \bar{w}(t_k - t_{k-1}) \ .$$

Iteration of the argument finally leads to

$$V(\psi(b)) \leq V(\psi(a)) + L \sum_{i=0}^{k-1} |\varepsilon_i| - \bar{w}(b-a) \leq V(\varphi(a)) - \frac{\bar{w}}{2}(b-a) \tag{21}$$

where we also used (19).

On the other hand, from (18) we have

$$V(\psi(b)) = V(\psi(b)) + V(\varphi(b)) - V(\varphi(b)) > V(\varphi(b)) - \frac{\bar{w}}{2}(b-a) \geq V(\varphi(a)) - \frac{\bar{w}}{2}(b-a)$$

a contradiction to (21).

∎

We also need to exploit the concept of positive limit set of a g-solution φ. The definition is the same as in the case of classical solution:

$$\Omega^+(\varphi) = \{y : \exists t_k \to +\infty \text{ s.t. } \varphi(t_k) \to y\} \ .$$

We recall the main features of the limit set. If φ is bounded, then $\Omega^+(\varphi)$ is nonempty and compact. Moreover, the following weak invariance property can be proven as in [12], with some obvious modifications: for each $\bar{y} \in \Omega^+(\varphi)$ there exists a g-solution $\bar{\varphi}(t)$ such that $\bar{\varphi}(t) = y$ and $\bar{\varphi}(t) \in \Omega^+(\varphi)$ for each $t \in \mathbf{R}$.

Proof of Theorem 2 As already noticed, under the assumption of the theorem the origin is stable. This implies that for some $\delta_0 > 0$,

$$|x| < \delta_0 \Longrightarrow |\varphi(t)| < 1 \quad \forall t \geq 0$$

for each g-solution such that $\varphi(0) = x$. Let us fix z, with $|z| < \delta_0$, and let $\varphi(t)$ be any g-solution issuing from z. The positive limit set $\Omega^+(\varphi)$ is nonempty and compact. Moreover, the composite function $t \mapsto V(\varphi(t))$ is nonincreasing, so that

$$\lim_{t \to +\infty} V(\varphi(t)) = l \geq 0 \ .$$

Since $V(x)$ is continuous, it follows $V(y) = l$ for each $y \in \Omega^+(\varphi)$. Assume that there exists $\bar{y} \in \Omega^+(\varphi)$, $\bar{y} \neq 0$. By the weak invariance property, there exists a g-solution $\bar{\varphi}(t)$ lying in $\Omega^+(\varphi)$, and such that $\bar{\varphi}(0) = \bar{y}$. Of course, we can find $T > 0$ such that $\bar{\varphi}(t) \neq 0$ for $t \in [0, T]$. Thus, Lemma 1 applies and we conclude that

$$V(\bar{\varphi}(T)) < V(\bar{y})$$

a contradiction. Hence, we must have $\Omega^+(\varphi) = \{0\}$, or, equivalently,

$$\lim_{t \to +\infty} \varphi(t) = 0$$

as required.

∎

To complete Example 2, we remark that the strict Lyapunov S-condition is fulfilled with

$$V(x,y) = \sqrt{4x^2+3y^2} - |x| \qquad \text{and} \qquad W(x,y) = \frac{x}{\sqrt{4x^2+3y^2}} \left[4x^2+2y^2-|x|\sqrt{4x^2+3y^2}\right] + (\operatorname{sgn} x) y^2 \ .$$

4 Appendix A

In this appendix we recall some facts about g-solutions of differential inclusions of the form (2). Let the interval $I = [a,b]$ be given, and let $\bar{x} \in \mathbf{R}^n$. For each $m \in \mathbf{N}$, take a partition

$$a = t_{m,0} < t_{m,1} < \ldots < t_{m,k_m-1} < t_{m,k_m} = b$$

and let $l_m = \max\{|t_{m,i+1} - t_{m,i}|\ ,\ i = 0, \ldots, k_m - 1\}$. Finally, for each $i = 0, \ldots, k_m - 1$ choose $\varepsilon_{m,i} \in \mathbf{R}^n$ and construct a right-continuous, piecewise affine function $\psi_m(t)$ on the interval $[a,b]$ in such a way that $\psi_m(t_{m,0}) = \bar{x}$, and

$$\psi_m(t_{m,i+1}) = \psi_m(t_{m,i}) + v_{m,i}(t_{m,i+1} - t_{m,i}) + \varepsilon_{m,i}$$

where $v_{m,i}$ is any element in $F(\psi_m(t_{m,i}))$. According to [15], we call g-solution of (2) on the interval $[a,b]$ any function $\varphi(t)$ which is obtained as the uniform limit of a sequence of functions $\psi_m(t)$ of the form described above, under the additional conditions that l_m is decreasing, $\lim_m l_m = 0$ and

$$\lim_m \sum_{i=0}^{k_m-1} |\varepsilon^i_m| = 0 \ .$$

If $\varphi(t)$ is a g-solution on the interval $[a,b]$, then $\varphi(t)$ is locally Lipschitz continuous and

$$\dot{\varphi}(t) \in \operatorname{co} F(\varphi(t)) \ , \qquad \text{a.e. } t \in [0,T] \ .$$

However, in general, we do not have $\dot{\varphi}(t) \in F(\varphi(t))$ for a.e. $t \in [a,b]$: this is the main difference between g-solutions and ordinary solutions of differential inclusions.

In [15], it is proven that if $F(x)$ is upper semicontinuous, compact valued and locally bounded, then for each $\bar{x}$ there exist $T > 0$ and a g-solution $\varphi : [0,T] \to \mathbf{R}^n$ with $\varphi(0) = \bar{x}$. Ordinary solutions of (2) are g-solutions. Moreover, cutting and pasting g-solutions still give rise to g-solutions.

For other properties about g-solutions the reader is referred to [15] and [13].

References

[1] Ancona F. and Bressan A., *Patchy Vector Fields and Asymptotic Stabilization*, Esaim-Cocv, **4** (1999), pp. 445-472

[2] Bressan A., *Singularities of Stabilizing Feedbacks*, Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino, **56** (1998), pp. 87-104

[3] Artstein Z., *Stabilization with Relaxed Controls*, Nonlinear Analysis, Theory, Methods, and Applications, **7** (1983), pp. 1163-1173

[4] Bacciotti A., Ceragioli F., and Mazzi L., *Differential Inclusions and Monotonicity Conditions for Nonsmooth Liapunov Functions*, Set Valued Analysis, **8** (2000), pp. 209-309

[5] Bacciotti A., and Rosier L., *Liapunov Functions and Stability in Control Theory*, Springer Verlag, London, 2001

[6] Ceragioli F., *Some Remarks on Stabilization by means of Discontinuous Feedbacks*, Systems and Control Letters, **45** (2002), pp. 271-281

[7] Clarke F.H., Ledyaev Yu.S., Sontag E.D. and Subbotin A.I., *Asymptotic Controllability Implies Feeedback Stabilization*, IEEE Trans. Automat. Control, **42** (1997), pp. 1394-1407

[8] Clarke F.H., Ledyaev Yu.S., Stern R.J. and Wolenski P.R., *Nonsmooth Analysis and Control Theory*, Springer, New York, 1998

[9] Clarke F.H., Ledyaev Yu.S. and Stern R.J., *Asymptotic Stability and Smooth Lyapunov Functions*, Journal of Differential Equations, **149** (1998), pp. 69-114

[10] Evans L.C., and Gariepy R.F., *Measure Theory and Fine Properties of Functions*, CRC, Boca Raton, 1992

[11] Deimling K., *Multivalued Differential Equations*, de Gruyter, 1992

[12] Filippov A.F., *Differential Equations with Discontinuous Right Handsides*, Kluwer, Dordrecht, 1988

[13] Krbec P., *On Nonparasite Solutions*, in *Proceedings of Equadiff 6*, Ed.s J. Vosmansky and M. Zlamal, Lecture Notes in Mathematics 1192, Springer Verlag, Berlin 1986, Berlin, pp. 133-139

[14] Krbec P., *Parasitic and Nonparasitic Solutions of Differential Equations and Stability*, in *Qualitative Theory of Differential Equations, Coll. Mat. Soc. Janos Bolyai* 30, North Holland, Amsterdam, 1981, pp. 615-620

[15] Sentis R., *Equations différentielles à second membre mesurable*, Bollettino U.M.I., **15**-B (1978), pp. 724-742