

Closed Loop Stabilization of Planar Bilinear Switched Systems

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Abstract

In this paper we address the closed loop switched stabilization problem for planar bilinear systems under the assumption that the control is one dimensional and takes only the values 0 and 1. We construct a class of state-static-memoryless stabilizing feedback laws which preserve the properties of open loop switching signals. In order to prove the stability of the implemented system, we use Lyapunov techniques for differential equations with discontinuous righthand side. Finally we point out some possible extensions of our result and compare it with related results previously proven by other authors.

Keywords: Switched systems, bilinear systems, stabilization, discontinuous feedback laws.

1 Introduction

In this paper we address the closed loop switched stabilization problem under the assumption that the input variable is constrained to take values in a finite set. This problem has been brought to the attention by the recent branch of the literature about switched and hybrid systems, which concerns single-input, bilinear systems of the form

$$\dot{x} = uAx + (1 - u)Bx \tag{1}$$

where $x \in \mathbf{R}^2$ and $u \in \{0, 1\}$ (see Xu and Antsaklis 2000, Boscain 2002, Hu *et al.* 2002 a,b, Bacciotti 2004); for more general references about switching control systems see Wicks *et al.* 1998, Liberzon and Morse 1999, Skafidas *et al.* 1999, DeCarlo *et al.* 2000, and the recent book Liberzon, 2004). In this perspective, the problem of steering the state to the equilibrium is usually approached by the construction of time-dependent and/or space-dependent appropriate switching rules.

Roughly speaking, a switching rule can be described as an open loop strategy, which limits the admissible inputs to the class of piecewise constant functions and, consequently, takes into consideration only solutions of the system which are piecewise of class C^1 . Indubitably, switching control is appealing, since it relies on simple and feasible procedures which can be easily implemented in computer simulations. However, from our point of view it is important to remark that the switching rules proposed in Wicks *et al.* 1998, Xu and Antsaklis, 2000, Hu *et al.* 2002 a,b, often cause hysteretic behaviors and are not realized by means of pure state-static-memoryless feedback laws.

On the other hand, since the set of possible values of the input variable is finite, a closed loop approach to the stabilization problem inevitably involves the use of discontinuous feedback laws. As a consequence, the closed loop system will have a discontinuous righthand side. As well known, in these conditions the classical definition of solution is no more adequate and one needs to resort to some notion of generalized solution. In this paper, we focus our attention on solutions in the sense of Carathéodory (their definitions are recalled in the Appendix, for reader's convenience). Carathéodory solutions seem to be well suited in certain stabilization problems (Ancona and Bressan 1999, 2002, 2004, Kim and Ha 2004, Rifford 2002, 2003), but they may exhibit the so-called Zeno phenomenon (accumulation of switching points) which is undesirable for practical applications. We are so led to select those Carathéodory solutions of the closed loop system which do not exhibit Zeno phenomenon and that we propose to

call *closed loop switched solutions*. The reader will find the formal definition in Section 2; here, we limit ourselves to emphasize that this notion is made up in order to meet a twofold goal: to be consistent with the conventional concept of feedback control, and to preserve the advantages of the switching control approach.

Finally, a feedback law will be called *s-stabilizing* if all Carathéodory solutions of the closed loop system are closed loop switched solutions and if it renders the closed loop system globally asymptotically stable with respect to all its closed loop switched solutions.

In the main result of this paper, stated in Section 3, we show that for certain systems of the form (1), it is possible to construct a (discontinuous) state-static-memoryless feedback $u = k(x) : \mathbf{R}^2 \rightarrow 0, 1$ which is s-stabilizing.

Typically, our construction applies when (1) represents a pair of coupled oscillators, but it works in many other cases, as well. The proof is given in Section 4. Section 5 contains some examples and Section 6 points out some possible extensions of our result and compares it with the results of the articles by Xu and Antsaklis 2000 and Boscain 2002.

2 Closed loop switched stabilization

Here, we formally introduce the notions of closed loop switched solution and s-stabilization used in this paper. Since they are not related to any special form of the righthand side, we find convenient to refer to a general nonlinear system

$$\dot{x} = f(x, u), \quad u \in U \tag{2}$$

where $x \in \mathbf{R}^n$, $u \in U \subset \mathbf{R}^m$, U finite, and $f : \mathbf{R}^{n+m} \rightarrow \mathbf{R}^n$.

Definition 1 *Let $u = k(x)$ be a given feedback law. We say that a curve $\varphi(t) : I \rightarrow \mathbf{R}^n$ is a closed loop switched solution of (2) if*

- (a) $u(t) = k(\varphi(t))$ is piecewise constant on the interval I ;
- (b) $\varphi(t)$ is piecewise of class C^1 on I ;
- (c) $\dot{\varphi}(t) = f(\varphi(t), k(\varphi(t)))$ for every $t \in I$ where $\dot{\varphi}(t)$ exists.

Here, *piecewise constant* on the interval I means that on every compact subinterval $I_0 \subseteq I$ there are at most finitely many points where the function is discontinuous; moreover, between two consecutive points where it is discontinuous, the function is constant. The value of the function at the discontinuity points does not matter. *Piecewise of class C^1* on I means that the function is continuous on I , and for each compact subinterval $I_0 \subseteq I$ there exists a finite set E such that $\varphi(t)$ has a continuous derivative at every point of $I_0 \setminus E$.

Results about the existence of closed loop switched solutions require restrictive assumptions about the points where $f(x, k(x))$ is discontinuous. This subject will not be discussed here. Actually, the particular form of the system we consider simplifies the matter a lot.

Given a point $\bar{x} \in \mathbf{R}^n$ and a feedback $k(x)$, let us denote by $\mathcal{S}_{\bar{x},k}$ the set of all Carathéodory solutions $\varphi : I \rightarrow \mathbf{R}^n$ of (2) such that $0 \in I$ and $\varphi(0) = \bar{x}$ and by $\tilde{\mathcal{S}}_{\bar{x},k}$ the set of all closed loop switched solutions $\tilde{\varphi} : I \rightarrow \mathbf{R}^n$ of (2) such that $0 \in I$ and $\tilde{\varphi}(0) = \bar{x}$. Let us remark that, in general, $\tilde{\mathcal{S}}_{\bar{x},k} \subseteq \mathcal{S}_{\bar{x},k}$; moreover, in the particular case f is continuous with respect to x and u , then $\tilde{\mathcal{S}}_{\bar{x},k} = \mathcal{S}_{\bar{x},k}$.

Definition 2 *We say that system (2) is s-stabilizable if there exists $k : \mathbf{R}^n \rightarrow U$ such that*

- (P0) for all $\bar{x} \in \mathbf{R}^n$, $\mathcal{S}_{\bar{x},k}$ is nonempty and $\mathcal{S}_{\bar{x},k} = \tilde{\mathcal{S}}_{\bar{x},k}$;
- (P1) for all $\bar{x} \in \mathbf{R}^n$ and all $\varphi(\cdot) \in \mathcal{S}_{\bar{x},k}$, the maximal interval of definition of $\varphi(t)$ contains $[0, +\infty)$;
- (P2) for all $\bar{x} \in \mathbf{R}^n$ and all $\varphi(\cdot) \in \mathcal{S}_{\bar{x},k}$, $\lim_{t \rightarrow +\infty} \varphi(t) = 0$;
- (P3) for all $\varepsilon > 0$ there exists $\eta > 0$ such that for all $\bar{x} \in \mathbf{R}^n$ and all $\varphi(\cdot) \in \mathcal{S}_{\bar{x},k}$,

$$|\bar{x}| < \eta \implies |\varphi(t)| < \varepsilon, \quad \forall t \geq 0.$$

3 Main result

Of course, the stability properties of (1) depend on the dynamic behavior of the two “subsystems”

$$\dot{x} = Ax, \quad \dot{x} = Bx. \quad (3)$$

It is well known that appropriate switching between the trajectories of the subsystems (3) may steer the state to zero, even if both matrices A and B are not Hurwitz (Utkin, 1977). The investigation carried out by Xu and Antsaklis in the framework of switching control, covers the following cases: 1) both A and B have complex eigenvalues with positive real part; 2) both A and B have real eigenvalues of opposite sign; 3) both A and B have real positive eigenvalues. As already pointed out, systems of the form (1) with $x \in \mathbf{R}^2$ are considered also by Xu and Antsaklis, 2000, Hu *et al.* 2002 a. However, we emphasize once more that the switching rules determined by Xu and Antsaklis 2000, Hu *et al.* 2002 a, do not correspond, in general, to state-static-memoryless feedback laws. In addition, our result covers a different class of systems: indeed, we prescribe the form of the matrix A , but we do not need any special form for the matrix B .

Systems of the form (1) are considered also by Bacciotti, 2004, with $x \in \mathbf{R}^n$. The main contribution of Bacciotti, 2004, is a sufficient condition (actually, a generalization of Wicks *et al.*, 1998) for the existence of a discontinuous feedback which provides stabilization with respect to Krasowski solutions and hence, a fortiori, in the sense of Filippov solutions. However, the feedback law constructed by Bacciotti, 2004, gives rise to a closed loop system whose Krasowski or Filippov solutions cannot be reproduced, in general, as open loop switched solutions. Moreover, the result by Bacciotti, 2004, does not apply to pairs of oscillators with opposite directions, a case covered by the results of the present paper.

In a different context, the hybrid stabilization problem for a pairs of oscillators is solved by Artstein, 1996, by the construction of a pair of piecewise linear feedback laws and a timed automaton.

Now we state the main result of this paper, whose proof will be exposed in the next section.

Theorem 1 *Consider system (1), and assume that*

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \quad (4)$$

where $\omega \neq 0$. Then, there exists a s -stabilizing discontinuous feedback law taking values in $\{0, 1\}$ if and only if the following condition holds:

(H) *there exists $x \in \mathbf{R}^2$ such that $\omega \cdot \det(Ax : Bx) > 0$.*

We report here some comments about Theorem 1.

- Theorem 1 can be actually applied under the more general assumption that A has a pair of conjugate imaginary eigenvalues $\pm i\omega$. Indeed, the form (4) can be always recovered by a preliminary change of coordinates associated to a matrix P with $\det P > 0$ (i.e., preserving the orientation of the plane), which transforms the pair of matrices A, B into the pair $P^{-1}AP, P^{-1}BP$. This can be done without affecting condition (H). Indeed, we point out that if $x = Py$, then

$$\omega \cdot \det(P^{-1}APy : P^{-1}BPpy) = \omega \cdot (\det P^{-1})(\det(APy : BPpy)) = \omega \cdot (\det P^{-1})(\det(Ax : Bx)).$$

Note that the phase portrait of the vector field Ax is an isochronous center: the trajectories are circumferences which are run counterclockwise if $\omega > 0$, and clockwise if $\omega < 0$.

- Condition (H) covers in particular the case $\omega \cdot \det(Ax : Bx) > 0$ for each $x \neq 0$. However, in this case the matrix B is Hurwitz and the solution of the problem is trivial. The most interesting case arises when the quadratic form $\det(Ax : Bx)$ is indefinite. However, we may have also the case where $\omega \cdot \det(Ax : Bx) \geq 0$ for each $x \neq 0$, but it does not vanish (see Example 3 in Section 5).

- By the same method used in the proof of Theorem 1, it is possible to address certain cases where the vector field Ax has an unstable equilibrium point (see Section 6).

4 Proof of Theorem 1

Before getting into the details of the proof we need to make some important remarks.

- In order to prove Theorem 1 we make use of the notion of Krasowski solution of a system with discontinuous righthand side (we recall this notion in the Appendix). Let us recall that Carathéodory solutions (and then also closed loop switching solutions), when they exist, are Krasowski solutions as well.

The proof consists in constructing a feedback law $k(x)$ that renders the (discontinuous) closed loop system globally asymptotically stable with respect to all its Krasowski solutions. This implies that the closed loop system is globally asymptotically stable also with respect to Carathéodory and closed loop switched solutions. Moreover our feedback law is such that all Carathéodory solutions of the implemented system are actually closed loop switched solutions (while it actually may give rise, in some cases, to Krasowski solutions which are not Carathéodory solutions).

The reason for using Krasowski solutions in the proof, is that an efficient Lyapunov theory is available for them. Theorem 1 may then be restated in terms of “Krasowski stabilizability”. We have not stated it in this form because Krasowski solutions may show pathological behaviours in the contest of switching systems, in the sense that they may not be reproduced by means of open loop switching rules (see Bacciotti 2004).

- As already mentioned, we use a Lyapunov function in the proof. This may be reviewed, according to the terminology of (Hu *et al.* 2002 a), as a *common weak Lyapunov-like function*.

Proof of the necessary part.

We need the following lemma. It concerns Krasowski solutions of discontinuous differential equations defined by pairs of continuous vector fields.

Lemma 1 *Let $f_1(x), f_2(x)$ be two continuous vector fields of \mathbf{R}^n . Let $f(x)$ be defined in such a way that for each $x \in \mathbf{R}^n$ either $f(x) = f_1(x)$ or $f(x) = f_2(x)$. Then,*

$$\mathbf{K}f(x) = \begin{cases} \{f(x)\} & \text{if } f(x) \text{ is continuous at } x \\ \text{co}\{f_1(x), f_2(x)\} & \text{otherwise.} \end{cases}$$

The proof of this lemma is easy and it is not reported. Now assume that (H) is false; in other words, assume that $\omega \cdot \det(Ax; Bx) \leq 0$ for each $x \in \mathbf{R}^2$. Let us consider the function

$$V(x_1, x_2) = \frac{x_1^2 + x_2^2}{2}. \quad (5)$$

It is clear that

$$\nabla V(x_1, x_2) \cdot Ax = 0 \quad \text{and} \quad \nabla V(x_1, x_2) \cdot Bx = -\frac{1}{\omega} \det(Ax; Bx) \geq 0$$

for each $x \in \mathbf{R}^2$. According to Lemma 1, for any Krasowski solution $\varphi(t)$ of the closed loop system we therefore have

$$\nabla V(\varphi(t)) \cdot \dot{\varphi}(t) \geq 0 \quad (6)$$

for a.e. $t \geq 0$. Assume that the system, with solutions intended in the Krasowski sense, is stabilized by the feedback law $u = k(x)$, and let $\varphi(t)$ a solution of the closed-loop system such that $\|\varphi(0)\| > 2$, which implies $V(\varphi(0)) > 2$. Then, for some $T > 0$ we should have $\|\varphi(T)\| < \sqrt{2}$, that is $V(\varphi(T)) < 1$, which is impossible because of (6). The proof of the necessary part is complete, since closed loop switched solutions, when they exist, are also Krasowski solutions.

Proof of the sufficient part.

The most intriguing aspect of this part of the proof consists in the construction of the feedback law. A crucial role is played by the identification of certain conic or semiconic regions of \mathbf{R}^2 .

Let $v \neq 0$ be a given vector of \mathbf{R}^2 . By a *ray* we mean the set of points of the form λv , where $\lambda \geq 0$. Given a pair of linearly independent vectors $v, w \in \mathbf{R}^2$, by a *semiconic region* (engendered by v, w) we mean the set of points of the form

$$\lambda v + \mu w \tag{7}$$

where λ, μ are arbitrary nonnegative numbers. By a *conic region* we finally means the set of points of the form (7) where λ, μ are both nonnegative or both nonpositive.

In other word, a ray can be viewed as a half straight line issuing from the origin, and a semiconic region as the convex region bounded by a pair of distinct rays. We need to distinguish between these two rays, in a way which depends on the vector field Ax . We agree to call the *right side* of a semiconic region of \mathbf{R}^2 , the one which can be superposed to the other by a counterclockwise rotation of an angle less than π if $\omega > 0$, or by a clockwise rotation of an angle less than π if $\omega < 0$. The other ray will be called the *left side*. Note that according to these definitions, semiconic and conic regions are closed and have a nonempty interior.

We are interested in the following lemma, relating linear vector fields and semiconic regions.

Lemma 2 *Let D be a semiconic region of \mathbf{R}^2 , and let E be a real 2×2 matrix. Assume that*

$$(Ex) \cdot (Ax) > 0 \tag{8}$$

for each $x \in D$ ($x \neq 0$). Let R and L be respectively the right and left side of D . Then, there exists $T > 0$ such that for each $\bar{x} \in R$ ($\bar{x} \neq 0$) the corresponding solution of the system

$$\dot{x} = Ex \tag{9}$$

reaches for the first time some point $\bar{y} \in L$ ($y \neq 0$) exactly at the time T . A similar result holds if the quadratic form $(Ex) \cdot (Ax)$ is strictly negative on $D \setminus \{0\}$, with $\bar{x} \in L$ and $\bar{y} \in R$.

Proof Let us denote by (ρ, θ) the polar coordinates of the plane and consider the equation

$$\dot{\theta} = \frac{1}{\rho^2}(Ex) \cdot \begin{pmatrix} -\omega x_2 \\ \omega x_1 \end{pmatrix} = \omega \left(E \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \right) \tag{10}$$

which is independent on ρ and describes the projection of the trajectories of system (9) on $S = \{x \in \mathbf{R}^2 : |x| = 1\}$. Condition (8) guarantees that (10) has no singular points on the compact arc $D \cap S$. Let p and q be respectively the intersection points of R and L with S . Then T is defined as the time needed to reach q starting from p along the solution of (10).

■

According to condition (H), the open set

$$\mathcal{O} = \{x \in \mathbf{R}^2 : \omega \cdot \det(Ax; Bx) > 0\}$$

is nonempty. Of course, if $x \in \mathcal{O}$ ($x \neq 0$) then $\lambda x \in \mathcal{O}$ for each $\lambda \neq 0$. Moreover, by continuity, if $x \in \mathcal{O}$ then \mathcal{O} contains a whole neighborhood of x . Hence, there exists a semicone $C \subset \mathcal{O} \cup \{0\}$. Let R and L be respectively the right and left side of C . Note that according to our agreements, if $x \in R$ ($x \neq 0$) the vector Ax points inward C . Moreover, we must have $Bx \neq 0$ for each $x \in \mathcal{O}$ and, hence, for each $x \in C$ ($x \neq 0$). Let us also remark that the reciprocal orientation of Ax and Bx is the same for every $x \in C$. This follows by the simple observation that $\det(Ax; Bx)$ represents the third component of the wedge product $W_1(x) \wedge W_2(x)$, where $W_1(x), W_2(x)$ are the vectors of \mathbf{R}^3 obtained by adding to Ax, Bx a third zero component. Let us distinguish two cases.

Case 1. For each $x \in R$ ($x \neq 0$), $(Ax) \cdot (Bx) > 0$ (Bx points inward C , as well as Ax).

Case 2. For each $x \in R$ ($x \neq 0$), $(Ax) \cdot (Bx) < 0$ (Bx points outward C).

Note that in force of our assumptions, only these two cases are actually possible.

Continuation of the proof for Case 1.

Clearly, we can take a ray $M \subset C$ ($M \neq R, M \neq L$) such that, denoting by $G \subset C$ the semicone delimited by R and M , we still have $(Ax) \cdot (Bx) > 0$ for any $x \in G$ ($x \neq 0$) (see figure 8). Let $\tilde{G} = R \cup \text{Int } G$. Let us define the feedback law

$$u = k(x) = \begin{cases} 1 & \text{if } x \in \mathbf{R}^2 \setminus \tilde{G} \\ 0 & \text{if } x \in \tilde{G} \end{cases} \quad (11)$$

The closed loop system is

$$\dot{x} = \begin{cases} Ax & \text{if } x \in \mathbf{R}^2 \setminus \tilde{G} \\ Bx & \text{if } x \in \tilde{G} \end{cases} \quad (12)$$

INSERT FIGURE 1 ABOUT HERE

The first goal is to show that for each initial state $p_0 \in \mathbf{R}^2$ there exists at least one closed loop switched solution issuing from p_0 .

If $p_0 \in \mathbf{R}^2 \setminus \tilde{G}$ ($p_0 \neq 0$), then the trajectory issuing from p_0 is an arc of circumference (centered at the origin), which intersects the ray R at some point $p_1 \neq 0$ after a finite duration T_0 . Re-starting from p_1 , the trajectory can be continued as an integral curve of the vector field Bx . Lemma 2 guarantees that after a finite duration T_B this trajectory intersects the ray M at some point $p_2 \neq 0$. Assuming p_2 as a new starting point, the trajectory is further continued according to the vector field Ax , and the ray R is reached again at some point $p_3 \neq 0$, after a finite duration T_A . The reasoning can be iterated. We have so constructed for each initial state $p_0 \in \mathbf{R}^2 \setminus \tilde{G}$ ($p_0 \neq 0$) a solution of the closed loop system. Such a solution can be also described in open loop terms. To this purpose, we can use a partition of the time axis determined by the points

$$0, \tau_0 = T_0, \tau_1 = T_0 + T_B, \tau_2 = T_0 + T_B + T_A, \tau_3 = T_0 + 2T_B + T_A, \dots$$

In order to complete our task, we have only to remark that the sequence $\{\tau_i\}$ does not have a finite limit (i.e., the sequence $\{\tau_i\}$ actually diverges and “zenoness” cannot occur). The argument is similar if $p_0 \in \tilde{G}$. Remark that all Carathéodory solutions are actually closed loop switched solutions. To finish the proof for the Case 1, it remains to show that the closed loop system (12) is asymptotically stable with respect to Krasowski solutions. We use the criterion recalled in the Appendix, with the Lyapunov function (5).

First of all, we point out that the Krasowski operator associated to (12) gives rise to the differential inclusion

$$\dot{x} \in \begin{cases} \{Ax\} & \text{if } x \in \mathbf{R}^2 \setminus G \\ \{\alpha Ax + (1 - \alpha)Bx : \alpha \in [0, 1]\} & \text{if } x \in R \cup M \\ \{Bx\} & \text{if } x \in \text{Int } G \end{cases}$$

Now, for any $x \in \mathbf{R}^2$ we have

$$\nabla V(x) \cdot Ax = 0$$

while for $x \in \text{Int } G$ we have

$$\nabla V(x) \cdot Bx = -\frac{1}{\omega} \det(Ax : Bx) < 0.$$

Hence, we easily infer that

$$\overleftarrow{V}(x) \subseteq (-\infty, 0] \quad \forall x \in \mathbf{R}^2 \quad (13)$$

and

$$Z = \{x : 0 \in \overleftarrow{V}(x)\} = \mathbf{R}^2 \setminus \text{Int } G.$$

From (13) we deduce that the origin is stable, and that each Krasowski solution is attracted by the maximal invariant set contained in Z . As already remarked, each trajectory issuing from a nonzero initial state $p_0 \in Z$ leaves Z in finite time. Hence, the maximal weakly invariant set contained in Z reduces to the origin. This completes the proof for the first case. By the way, we remark that in this case for each initial state there is a unique Krasowski solution, which is in fact a closed loop switched solution, as well.

Continuation of the proof for Case 2.

As in the previous case, we can take a ray $M \subset C$ ($M \neq R, M \neq L$) such that for each $x \in G$ ($x \neq 0$) (G being the semicone delimited by R and M), the inequality $Ax \cdot Bx < 0$ is still valid. Take a further ray $N \subset G$ ($N \neq R, N \neq M$); denote by K the semicone whose right and left sides are respectively given by R and N , and by H the semicone whose right and left sides are respectively given by N and M (see figure 8).

Next we construct two sequences of points $\{p_i\}, \{q_i\}$ ($i \in \mathbf{Z}$). Let p_0 be the intersection point between S and R , and let $\varphi_0(t)$ be the solution of the vector field Ax issuing from p_0 . According to Lemma 2, there exists a time $T_A > 0$ such that $\varphi_0(T_A)$ coincides with the intersection point between S and N , while $\varphi_0(t) \in K$ for each $t \in [0, T_A]$. Let $q_0 = \varphi_0(T_A)$ and $\Gamma_0 = \varphi_0((0, T_A])$. Then let $\psi_0(t)$ be the solution of the vector field Bx issuing from q_0 . Using again Lemma 2, we find a time $T_B > 0$ such that $\psi_0(T_B)$ coincides with the intersection point between S and R , while $\psi_0(t) \in K$ for each $t \in [0, T_B]$. Let $p_1 = \psi_0(T_B)$ and $\Delta_1 = \psi_0((0, T_B])$. The procedure can be iterated for any positive integer i , by considering at each step the solution $\varphi_i(t)$ of the vector field Ax issuing from p_i , defining $q_i = \varphi_i(T_A)$ and $\Gamma_i = \varphi_i((0, T_A])$, continuing with the solution $\psi_i(t)$ of the vector field Bx issuing from q_i and defining $p_{i+1} = \psi_i(T_B)$, $\Delta_i = \psi_i((0, T_B])$. The procedure can be repeated also for any negative integer i , following the trajectories of Ax and Bx backward in time.

Now we define a sequence

$$\dots, \tau_{-3} = -T_A - 2T_B, \tau_{-2} = -T_A - T_B, \tau_{-1} = -T_B, \tau_0 = 0, \tau_1 = T_A, \tau_2 = T_A + T_B, \tau_3 = 2T_A + T_B, \tau_4 = 2T_A + 2T_B, \dots$$

and a curve $\sigma(t)$ in the following way:

$$\sigma(t) = \begin{cases} \dots & \\ \psi_0(t - \tau_{-1}) & t \in [\tau_{-1}, \tau_0) \\ \varphi_0(t) & t \in [\tau_0, \tau_1) \\ \psi_1(t - \tau_1) & t \in [\tau_1, \tau_2) \\ \varphi_1(t - \tau_2) & t \in [\tau_2, \tau_3) \\ \dots & \end{cases}$$

Note that the image of $\sigma(t)$, denoted by Σ , is formed by the pairwise distinct union of the arcs Γ_i, Δ_i . We distinguish the sets

$$\Sigma_A = \cup \Gamma_i \quad \text{and} \quad \Sigma_B = \cup \Delta_i .$$

Claim The curve $\sigma(t)$ is unbounded for $t \rightarrow -\infty$; in addition, $\lim_{t \rightarrow +\infty} \sigma(t) = 0$.

We prove the second statement of the claim, the proof of the first one being similar. As already established in the proof of the first case, moving inside G along the solutions $\psi_i(t)$ of the vector field Bx the values of the function $V(x)$ decrease. As a consequence, the sequence of real numbers $\{|p_i|\}$ is decreasing. Assume that $\lim_{i \rightarrow +\infty} |p_i| = l > 0$. Then, the solutions $\psi_1(t), \psi_2(t), \dots$ uniformly converges for $t \in [0, T_B]$ to some function $\tilde{\varphi}(t)$. The image of this function must be contained in the level curve $\{x : V(x) = l^2/2\}$, which is of course an arc of a circumference. On the other hand, according to well known theorems about ordinary differential equations, $\tilde{\varphi}(t)$ must be also a solution of the vector field Bx . But this is impossible, since by virtue of (H), Bx is not tangent to a circumference.

We are now ready to define our feedback law. The set Σ divides K into two part. Let K_0 be the set of points of K bounded by Σ and the ray N . Let moreover $\Omega = \text{Int } K_0 \cup H \cup \Sigma_B$, $\Omega^c = \mathbf{R}^2 \setminus \Omega$, and

$$u = k(x) = \begin{cases} 1 & \text{if } x \in \Omega^c \\ 0 & \text{if } x \in \Omega \end{cases} . \quad (14)$$

The corresponding closed loop system is

$$\dot{x} = \begin{cases} Ax & \text{if } x \in \Omega^c \\ Bx & \text{if } x \in \Omega \end{cases} . \quad (15)$$

INSERT FIGURE 2 ABOUT HERE

As in the proof of Case 1, we first show that for each $p_0 \in \mathbf{R}^2$ there exists at least one closed loop switched solution. Assume first that $p_0 \in \Omega^c$; then the system evolves according to the vector field Ax until the set Σ_B is

reached in finite time. Therefore, the evolution is continued following the curve $\sigma(t)$. Instead, if $p_0 \in \Omega$, the system evolves according to the vector field Bx , until the set Σ_A is reached in finite time. Even in this case, the evolution is continued following the curve $\sigma(t)$. It is clear that in both cases the trajectory satisfies the required conditions.

Finally, we prove that the closed loop system is asymptotically stable with respect to Krasowski solutions. Again, we make use of the Lyapunov function (5). The Krasowski operator gives rise to the differential inclusion

$$\dot{x} \in \begin{cases} \{Ax\} & \text{if } x \in \text{Int } \Omega^c \\ \{\alpha Ax + (1 - \alpha)Bx : \alpha \in [0, 1]\} & \text{if } x \in \Sigma \cup M \\ \{Bx\} & \text{if } x \in \text{Int } \Omega . \end{cases}$$

This time we have $Z = \overline{\Omega^c}$. To conclude the proof, we have to check that for all $l > 0$, the origin is the maximal weakly invariant set contained in $Z \cap L_l$. Let us show first that there exists no bounded Krasowski solutions contained in $\Sigma \cup M$.

Recall that the vectors Ax and Bx are not parallel for $x \in \Sigma \cup M$ ($x \neq 0$). Hence, the closed loop system has not equilibrium solutions in the sense of Krasowski. If $p \in \Sigma$, the unique Krasowski solution issuing from p coincides with $\sigma(t)$ and we already know that this solution is unbounded. If $p \in M$, then we have uniqueness of backward solutions but not of forward solutions. Indeed, there exist three Krasowski (forward in time) solutions issuing from p . The first enters immediately into $\text{Int } \Omega$ and hence it is not contained in Z . The second enters immediately into $\text{Int } \Omega^c$ and the third one slides along M . However, the last one represents the unique Krasowski solution issuing from p backward in time. Note that the first two forward solutions are both closed loop switched solutions as well, while the third one is neither a Carathéodory solution nor a closed loop switched solution. In any case, no complete Krasowski solution issuing from $p \in M$ is contained in $Z \cap L_l$. Finally, we remark that by continuing backward in time any Krasowski solution of the closed loop system issuing from a point $p \in \text{Int } \Omega^c$, we hit the ray M in finite time. The proof is complete.

5 Examples

In this section we present a number of particular cases covered by Theorem 1.

We denote by λ_1^B, λ_2^B the eigenvalues of the matrix B . The trivial computation of the quantity $\det(Ax : Bx)$ shows that system (1) is always s-stabilizable in the following cases:

- λ_1^B, λ_2^B complex conjugate with null real part, $B \neq \alpha A$ with $\alpha \in \mathbf{R}$;
- $\lambda_1^B, \lambda_2^B \in \mathbf{R}$, $\lambda_1^B \lambda_2^B < 0$.

INSERT FIGURE 3 ABOUT HERE

Other particular situations covered by Theorem 1 are illustrated by the following examples.

Example 1 Let us consider system (1) with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & \alpha \\ -1 & 1 \end{pmatrix}.$$

If $\alpha > 0$ matrix B gives rise to an unstable focus. If $\alpha = 4$ condition (H) is satisfied then system (1) is s-stabilizable; if $\alpha = 1$ condition (H) is not satisfied then system (1) is not s-stabilizable.

Example 2 Let us consider system (1) with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & \alpha \\ 0 & 2 \end{pmatrix}.$$

Matrix B gives rise to an unstable node. If $\alpha = 3$ condition (H) is satisfied then system (1) is s-stabilizable; if $\alpha = 0$ condition (H) is not satisfied then system (1) is not s-stabilizable.

Example 3 Let us consider system (1) with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Matrix B has one null eigenvalues. Condition (H) is satisfied then system (1) is s-stabilizable. Note that in this case $\det(Ax \dot{=} Bx) \geq 0$ for all $x \in \mathbf{R}^2$.

INSERT FIGURE 4 ABOUT HERE

6 Extensions and comparisons

The aim of this section is to explicit all cases in which the s-stabilizability problem for planar bilinear systems is solved. In order to do this, we now list some situations which have been already considered in the literature. We refer in particular to the articles by Xu and Antsaklis, 2000, and by Boscain, 2002. By using a procedure analogue to the one developed in the proof of Theorem 1, it is possible to modify the control laws suggested by the authors in order to construct state-static-memoryless feedback laws.

In the following we denote by λ_1^A, λ_2^A and λ_1^B, λ_2^B the eigenvalues of the matrices A and B respectively. We remember that interesting cases are the ones in which both A and B have at least one eigenvalue with nonnegative real part.

In the article by Xu and Antsaklis, 2000, necessary and sufficient conditions for system (1) to be stable under appropriate switching are found in the following cases:

- $\lambda_1^k, \lambda_2^k \in \mathbf{R}, \lambda_1^k \lambda_2^k < 0, k = A, B;$
- $\lambda_1^k, \lambda_2^k \in \mathbf{R}, \lambda_1^k \lambda_2^k > 0, k = A, B;$
- λ_1^k, λ_2^k complex conjugates, $k = A, B.$

In these cases the same necessary and sufficient conditions hold for system (1) being s-stabilizable.

In (Boscain 2002) systems of the form

$$\dot{x} = u\tilde{A} + (1 - u)\tilde{B} \tag{16}$$

are considered in connection with the problem of stability under arbitrary switching. Admissible control laws are measurable functions $u : [0, \infty) \rightarrow [0, 1]$. Recall that system (16) is said to be *stable under arbitrary switching* if, for any implementation of an admissible control law, it is asymptotically stable. In the paper necessary and sufficient conditions in order to have stability under arbitrary switching are found. Of course a necessary condition for the system being asymptotically stable under arbitrary switching is that both matrices \tilde{A} and \tilde{B} have strictly negative eigenvalues. The necessary and sufficient conditions for stability under arbitrary switching of (16) are expressed in terms of the eigenvalues of the matrices \tilde{A} and \tilde{B} . Let us denote by $\tilde{\varphi}_{x,u}(t)$ the solution of (16) corresponding to the control law $u(t)$ and the initial condition x . Assume that the eigenvalues of \tilde{A} are complex conjugates with negative real parts and the eigenvalues of \tilde{B} are real and negative. It is proved in (Boscain, 2002) that a set of conditions doesn't hold if and only if there exists $x \in \mathbf{R}^2 \setminus \{0\}$ and $u(t)$ such that $\tilde{\varphi}_{x,u}(t) \rightarrow \infty$ as $t \rightarrow \infty$. Actually Boscain's conditions do not hold if and only if for any x there exists $u(t)$ such that $\tilde{\varphi}_{x,u}(t) \rightarrow \infty$ as $t \rightarrow \infty$ and moreover $u(t)$ can be chosen such that $u(t) \in \{0, 1\}$ and $u(t)$ has discontinuities only on a couple of lines through the origin.

Let us now consider our original system (1) with λ_1^A, λ_2^A complex conjugates with positive real parts and $\lambda_1^B, \lambda_2^B \in \mathbf{R}, \lambda_1^B \lambda_2^B > 0$ and system (16) with $\tilde{A} = -A$ and $\tilde{B} = -B$. Note that the eigenvalues of \tilde{A} and \tilde{B} are the opposites of the eigenvalues of A and B and that for any fixed $u(t)$ the trajectories of the two systems are the same except for their orientation. Denote by $\varphi_{x,u}(t)$ the solution of (1) corresponding to the initial condition x and the control law $u(t)$. We have that Boscain's conditions do not hold if and only if for all x there exists $u(t)$ such that $\tilde{\varphi}_{x,u}(t) \rightarrow \infty$ as $t \rightarrow \infty$, i.e. if and only if for all x there exists $u(t)$ such that $\varphi_{x,u}(t) \rightarrow 0$ as $t \rightarrow \infty$.

Finally in case

- λ_1^A, λ_2^A complex conjugates and $\lambda_1^B, \lambda_2^B \in \mathbf{R}$ with $\lambda_1^B \lambda_2^B > 0$,

if Boscain's conditions do not hold we get a time dependent switching law which can be modified in a s-stabilizing feedback by using a procedure analogue to the one used in the proof of Theorem 1.

Our Theorem 1 gives necessary and sufficient conditions for s-stabilizability in the cases

- λ_1^A, λ_2^A complex conjugates with null real part and any λ_1^B, λ_2^B .

Cases excluded by the previous enumeration are

- $\lambda_1^A, \lambda_2^A \in \mathbf{R}, \lambda_1^A \lambda_2^A < 0$ and $\lambda_1^B, \lambda_2^B \in \mathbf{R}, \lambda_1^B \lambda_2^B > 0$;
- $\lambda_1^A, \lambda_2^A \in \mathbf{R}, \lambda_1^A \lambda_2^A < 0$ and λ_1^B, λ_2^B complex conjugates;

and all cases in which either A or B have one or two null eigenvalues. These cases can be treated in an analogous way. In particular, in the case $\lambda_1^A, \lambda_2^A \in \mathbf{R}, \lambda_1^A \lambda_2^A < 0$ and λ_1^B, λ_2^B complex conjugates, s-stabilization can be always achieved by means of a state-static-memoryless feedback, in an analogous way to the one shown in the following example.

Example 4 Let us consider system (1) with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

We have that $\lambda_1^A = 1, \lambda_2^A = -1$, i.e. matrix A gives rise to a saddle point and $\lambda_1^B = 1 + i, \lambda_2^B = 1 - i$. i.e. matrix B gives rise to an unstable focus. Let $\mathcal{O} = \{x \in \mathbf{R}^2 : -\det(Ax : Bx) > 0\} = \{(x_1, x_2) \in \mathbf{R}^2 : \frac{x_2}{x_1} < 1 - \sqrt{2}, \frac{x_2}{x_1} > 1 + \sqrt{2}\}$. Let us consider the three rays $m : x_2 = 5x_1, r : x_2 = 4x_1$ and $n : x_2 = 3x_1, x_1 > 0$. Let $H \subset \mathcal{O}$ the semiconic region delimited by the rays m and r and $K \subset \mathcal{O}$ the semiconic region delimited by the rays r and n . Note that $Ax|_r \cdot Bx|_r < 0$ and $Ax|_n \cdot Bx|_n < 0$. By using a technique completely analogous to the one used in the proof of Theorem 1, Case 2, we can construct a curve $\Sigma \subset K$ by patching together pieces of trajectories of the systems associated to the matrices A and B respectively, and define the set Ω as the union of the cone H , the interior of the set of points of K bounded by the curve Σ and the ray r and the arcs of Σ which are trajectories associated to the matrix A . Finally the feedback law

$$u = k(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c \end{cases} \quad (17)$$

is a s-stabilizing feedback.

INSERT FIGURE 5 ABOUT HERE

7 Conclusions

As already remarked in the Introduction, the stabilizability problem for switched systems in the plane has been studied by several authors. In all papers that deal with this problem it is evident the central role played by “conic switching laws”. We have not given a precise definition of “conic switching law” here: very roughly speaking they can be seen as hybrid control laws whose discrete state space is the set $\{0, 1\}$ and whose discrete dynamics is determined by a couple of lines through the origin (see Xu and Antsaklis 2000). Such “conic switching laws” are not always expressed as state-static-memoryless feedbacks. In this paper we have considered the particular case of system (1) with the matrix A with imaginary eigenvalues and we have showed how a “conic switching law” can be modified into a state-static-memoryless feedback which preserves the switching nature of the system. Moreover we have proved rigorously that these feedbacks actually stabilize. In order to construct state-static-memoryless feedback laws, a procedure analogue to the one developed in the proof of Theorem 1 may be applied to different situations already considered in the literature.

8 Appendix

We recall here some facts about autonomous systems of ordinary differential equations

$$\dot{x} = f(x), \quad x \in \mathbf{R}^n \quad (18)$$

where $f(x)$ is in general Lebesgue measurable and locally bounded. Let I denote some interval of real numbers. A *Carathéodory solution* of (18) is a curve $\varphi(t) : I \rightarrow \mathbf{R}^n$ such that $\varphi(t)$ is absolutely continuous, and $\dot{\varphi}(t) = f(\varphi(t))$ for a.e. $t \in I$. A *Krasowski solution* of (18) is a curve $\varphi(t) : I \rightarrow \mathbf{R}^n$ such that $\varphi(t)$ is absolutely continuous, and

$$\dot{\varphi}(t) \in \mathbf{K}f(\varphi(t))$$

for a.e. $t \in I$, where the “operator” \mathbf{K} is defined by $\mathbf{K}f(x) = \bigcap_{\delta > 0} \overline{\text{co}} \{f(\mathcal{B}(x, \delta))\}$.

Under our assumption that $f(x)$ is Lebesgue measurable and locally bounded, the operators \mathbf{K} associates to $f(x)$ a set valued map which is upper semicontinuous, compact, convex valued. In particular, for each initial state \bar{x} there exists at least one Krasowski solution of (18). Existence of Carathéodory solutions in general requires more specific and restrictive assumptions (Pucci 1976, Bressan and Colombo 1990, Ancona and Bressan 1999).

In the proof of Theorem 1 we need the following invariance principle proven by the authors, 1999. It applies to differential inclusions with an upper semicontinuous, convex, compact valued righthand side of the form

$$\dot{x} \in F(x) \quad (19)$$

and hence, it can be used to derive stability results for Krasowski solutions of discontinuous differential equations, provided that a locally Lipschitz continuous, positive definite Lyapunov function $V(x) : \mathbf{R}^n \rightarrow \mathbf{R}$ is known. In this paper we do not need such a generality, since smooth Lyapunov functions suffice. Hence, we limit ourselves to state the result in a simplified form. Given $V(x) \in C^1$, let

$$\dot{\bar{V}}(x) = \{a \in \mathbf{R} : \exists v \in F(x) \text{ such that } \nabla V(x) \cdot v = a\}.$$

Theorem 2 *Let $F(x)$ be an upper semicontinuous set-valued map, with convex, compact values, and let $V : \mathbf{R}^n \rightarrow \mathbf{R}$ be a positive definite function of class C^1 . Let us assume that there exists $l > 0$ be such that the set $L_l = \{x \in \mathbf{R}^n : V(x) \leq l\}$ is connected and compact. Let $\bar{x} \in L_l$, and let $\varphi(t)$ be any solution of (19) issuing from \bar{x} . Suppose that $\dot{\bar{V}}(x) \subseteq (-\infty, 0]$ for each $x \in \mathbf{R}^n$. Let*

$$Z = \{x \in \mathbf{R}^n : 0 \in \dot{\bar{V}}(x)\}$$

and let M be the largest weakly invariant subset of $Z \cap L_l$. Then $\text{dist}(\varphi(t), M) \rightarrow 0$ as $t \rightarrow +\infty$.

Note that under the assumptions of Theorem 2 the set Z is closed.

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CAPTIONS TO FIGURES

Figure 1. The semicones C and G used in order to construct the stabilizing feedback law (case $\omega > 0$) when both Ax and Bx point inward C .

Figure 2. The semicones C , G , H and K (case $\omega > 0$) in order to construct the stabilizing feedback law (case $\omega > 0$) when Bx points outward C .

Figure 3. A trajectory of the implemented system in the case both matrices A and B have purely imaginary eigenvalues.

Figure 4. A trajectory of the implemented system in the case matrix B has one null eigenvalue.

Figure 5. A trajectory of the implemented system in the case matrix A has eigenvalues of opposite signs and matrix B has complex conjugates eigenvalues.

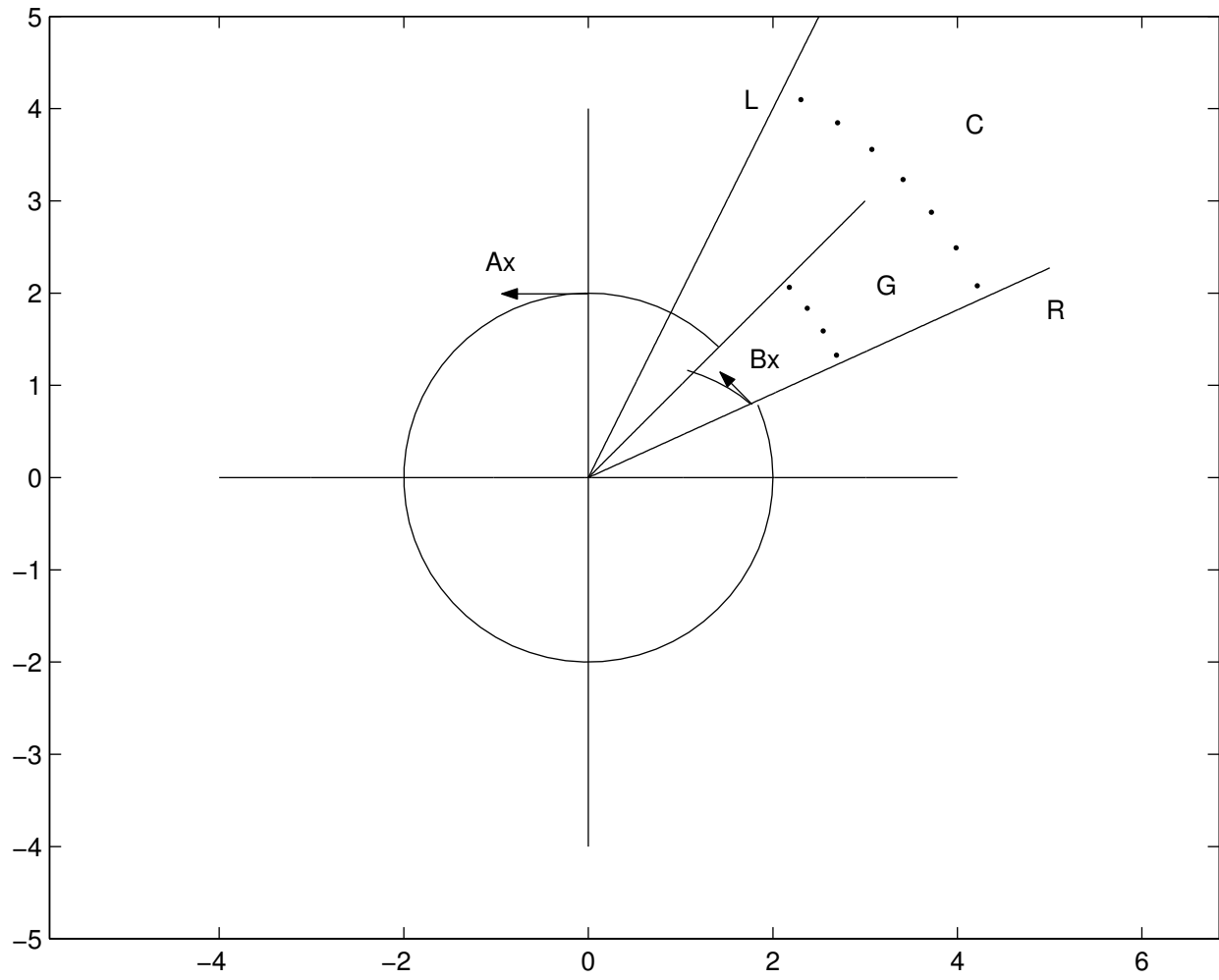


Figure 1.

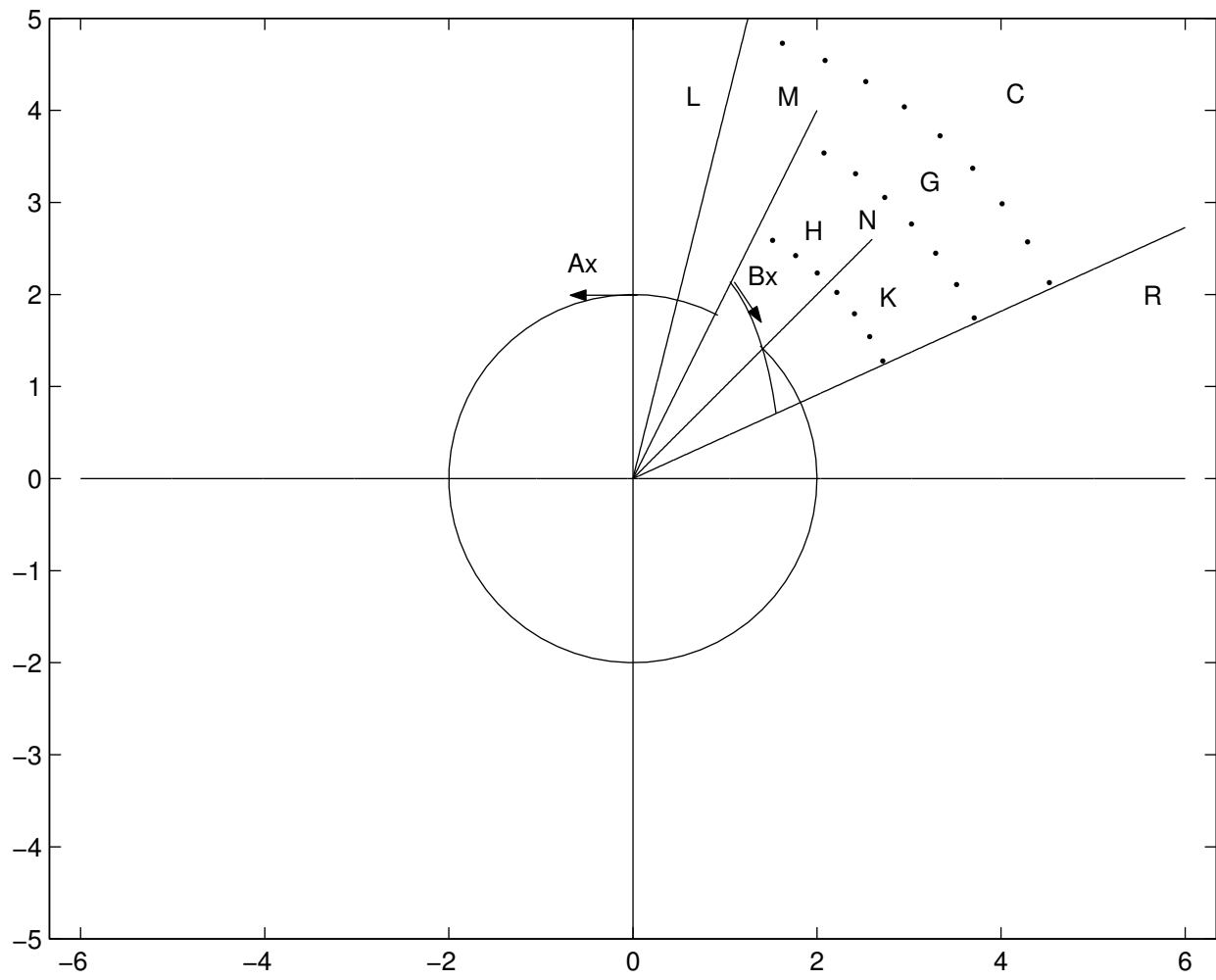


Figure 2.

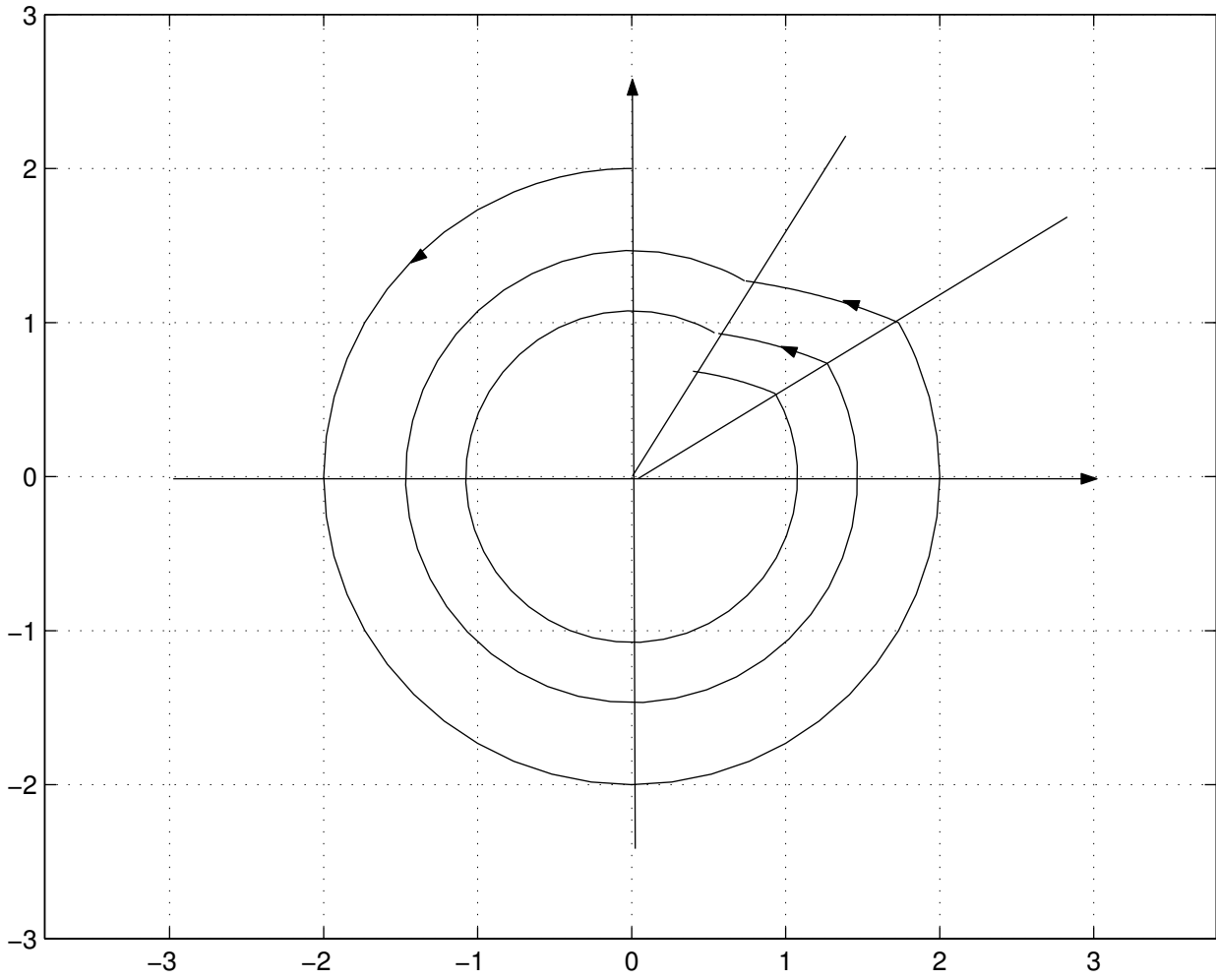


Figure 3.

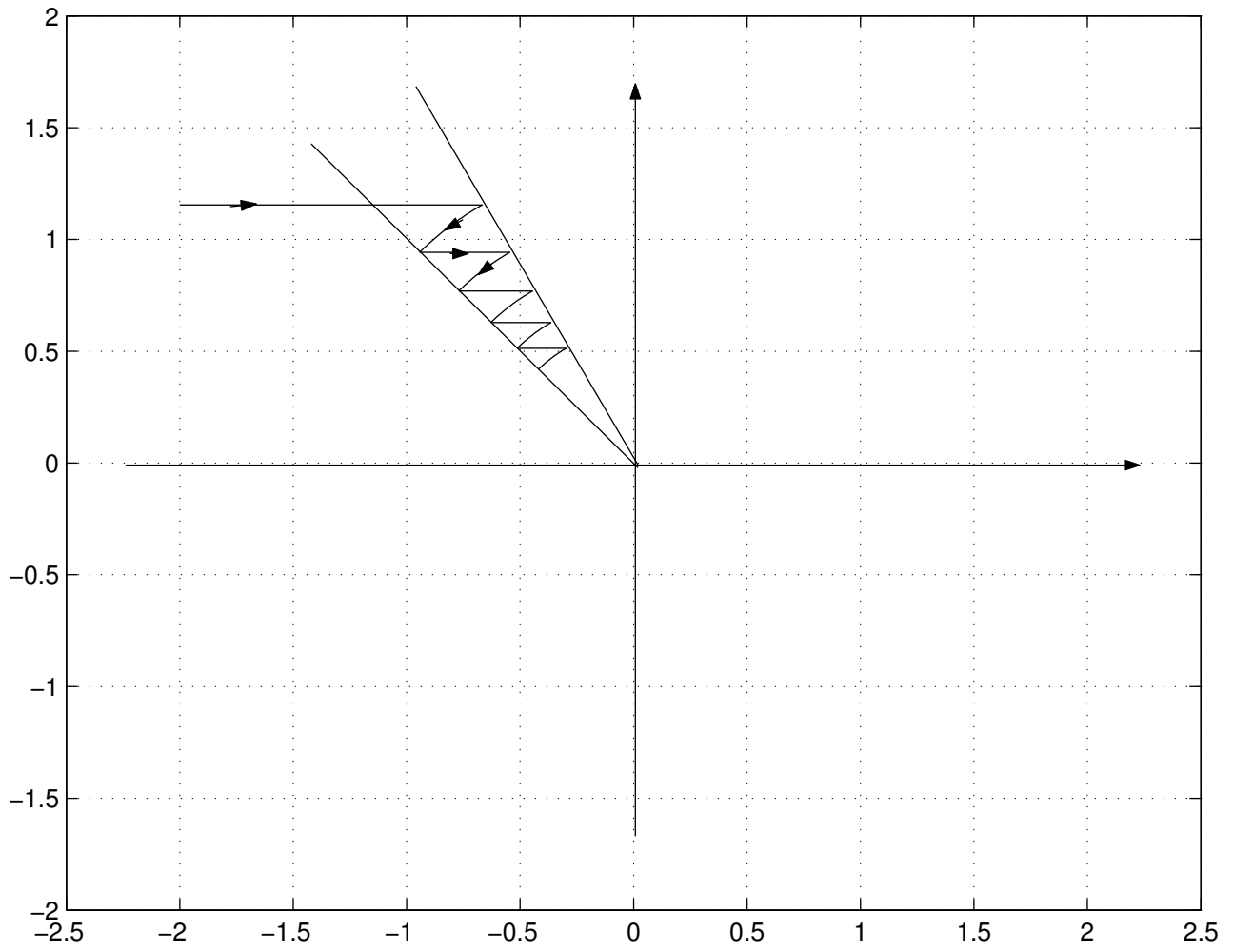


Figure 4.

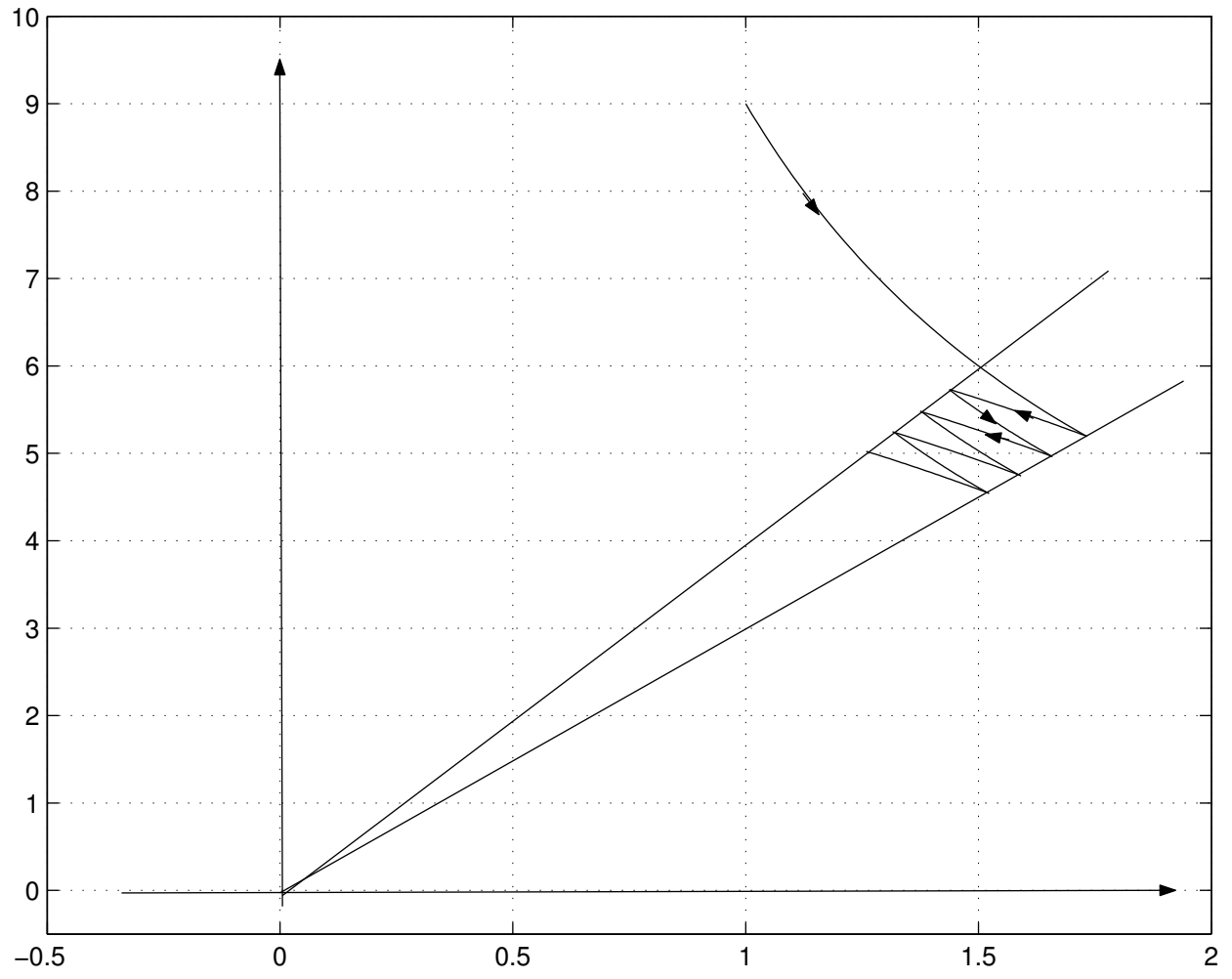


Figure 5.