Stabilizability of nonlinear systems by means of time-dependent switching rules

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Abstract

Given a family of linear systems which admit an asymptotically stable convex combination, the existence of stabilizing time-dependent switching rules can be proved by using the Baker-Campbell-Hausdorff formula for exponentials. The control laws obtained in this way are periodic, fast switching and independent of the initial state. We prove that under a similar assumption, the approach can be extended, to provide stabilizing time-dependent switching rules for families of nonlinear vector fields, as well. However, the resulting control law in general is not periodic, not fast switching and it may depend on the initial state.

Keywords: Switched systems, stabilizability, time-dependent switching rules.

1 Introduction

In modern technology, it is not rare to encounter systems which may exhibit different modes of evolution, each mode being characterized by different dynamical features. When the changes of mode can be thought of as sudden events, these systems are usually called *switched systems* in the engineering literature (see [8], [10] and the references therein). It turns out that a switched system can be conveniently modelled by assigning:

- a family $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ of smooth vector fields $f_n : \mathbf{R}^d \to \mathbf{R}^d$, where d is the dimension of the state space and $\mathcal{N}$ is a set of indices;
- a rule which determines the switching policy.

The interpretation is as follows. For each $n \in \mathcal{N}$, the vector field $f_n(x)$ defines a differential equation $\dot{x} = f_n(x)$. The task of this equation is to determine the evolution of the system over each interval I where no switching event occurs and where the switching rule assigns n as the active index. The vector field $f_n(x)$ will be also called the n-th *subsystem* or the n-th *component* of $\mathcal{F}$.

The task accomplished by the switching rule is to generate a sequence of switching times $\{t_i\}$ and a sequence of indices $\{n_i\}$. The sequence $\{t_i\}$ may be infinite or finite, possibly reduced to the initial instant t_0; if it is infinite, then it is required to be divergent. The index n_i specifies the index of the vector field active on the interval $[t_i, t_{i+1})$ (if the sequence of the switching times is finite, say $t_0, \ldots, t_i$, then we set $t_{i+1} = +\infty$). Roughly speaking, we can distinguish two types of switching rules. In this paper, we use the terms *state-dependent switching rule* and *closed-loop switched system* when the switching law is determined at each instant by the position of the state variable x at that instant. We use the terms *time-dependent switching rule* and *open-loop switched system* when the sequences $\{t_i\}$ and $\{n_i\}$ are programmed in advance and explicitly assigned as part of the data.

In many applications, it is crucial to investigate the stability of a switched system, under the assumption that all the vector fields $f_n(x)$ have a common equilibrium position (say the origin, without loss of generality). Now, it is well known that the stability properties of a switched system may look very different from those of each single subsystem. For instance, sometimes an unfortunate switching policy may generate divergent trajectories even if all the vector fields of $\mathcal{F}$ are asymptotically stable and, symmetrically, a suitable switching policy may generate trajectories which converge to the origin even if the origin is unstable for each single subsystem (see [8]). In the latter case, the switched system is usually said to be stabilizable.

Most of the results about stabilization of switched systems available in the literature concern the finite linear case, that is the case where $\mathcal{N} = \{1, \ldots, N\}$ for some integer $N > 1$ and $f_n(x) = A_n x$ where, for each n, A_n is a $d \times d$ real constant matrix. In particular, the existence of a Hurwitz (i.e., having all its eigenvalues in the open left half complex plane) convex combination $\bar{A}$ of the matrices $A_1, \ldots, A_N$, has been proven to be a sufficient condition for the existence of both state-dependent and time-dependent stabilizing switching rules. More precisely, in [13] (see also [8]) a stabilizing state-dependent switching rule is constructed using a sector decomposition of the state space, while in [11] (see also [10]) a stabilizing time-dependent switching rule is obtained using the Baker-Campbell-Hausdorff formula for matrices (see [12]). In spite of the common starting assumption, these two methods lead to switching rules with very different features: in general, they cannot be reduced to each other. The reader will find a more detailed discussion about this topic in the next section. For the moment we limit ourselves to notice that the stabilizing switching rule determined with the aid of the Baker-Campbell-Hausdorff formula turns out to be independent of the initial state, periodic and "fast switching" (in the sense that stable switched trajectories are generated provided that the period is sufficiently small).

Now we come to the contribution of this paper. We focus on families of nonlinear (analytic) vector fields. Our aim is to extend, as far as possible, the method for the construction of stabilizing time-dependent switching rules based on the Baker-Campbell-Hausdorff formula. While our basic assumption is the natural extension of the above mentioned one (existence of an asymptotically stable convex combination of the vector fields f_n) the switching law we obtain is no more periodic and may depend on the initial state. Moreover, we show by an example that stabilizing time-dependent "fast switching" rules need not to exist, while it is possible to construct stabilizing time-dependent switching rules such that the time elapsed between two consecutive switches becomes greater and greater as t goes to infinity.

The paper is organized as follows. Section 2 contains further motivations, basic definitions and preliminary material. The main result and its proof are given in Section 3. Section 4 is devoted to examples.

2 Motivations and definitions

As already mentioned, stabilizing state-dependent switching rules for the linear case are constructed in [13], in a closed-loop perspective, by exploiting the following assumption:

(L) *there exist* $\alpha_1, \ldots, \alpha_N$ *(*$\alpha_n > 0$, $\sum_{n=1}^{N} \alpha_n = 1$*) such that the origin is asymptotically stable for the system*

$$\dot{x} = \bar{f}(x) = \bar{A}x = \sum_{n=1}^{N} \alpha_n A_n x \ . \tag{1}$$

The method of construction in [13] consists of a sector decomposition of the state space induced by a Lyapunov function of $\bar{A}$ and leads to a strong form of stability, called *quadratic stability* (when $N = 2$, condition **(L)** has been recognized to be necessary for quadratic stabilization, as well: see [8]). With this construction, the switches arise when a trajectory intersects the boundary of a sector.

Note that a state-dependent switching rule is basically equivalent to a discontinuous feedback. This causes some mathematical problems about existence and continuability of solutions. In particular, behaviors like chattering (sliding motions approximated by fast switches) and Zeno phenomenon (accumulation of switching events in finite time) may arise when trying to convert a state-dependent switching rule in a time-dependent one. Although these behaviors have been widely studied (for instance in the context of variable structure control theory), they are undesirable and can be hardly recognized as originated by a genuine switching signal. These drawbacks are overcome in [13, 8] by introducing hysteresis in the definition of the switching law, and in [3] by resorting to the notion of Krasowski solution.

Note also that even if a state-dependent switching rule can be unambiguously converted in a time-dependent one, the resulting control law is different for each initial state, in general.

Finally, we emphasize that both the sector decomposition and the possibility to convert a state-dependent switching rule in a time-dependent one, strictly rest on the linear nature of the system. For all these reasons, since we are interested in general families of nonlinear vector fields, we focus directly on time-dependent switching rules.

The main purpose of this section is to introduce a formal definition of open-loop switched system, and the main notions of stability and stabilizability we are interested in.

Let $\mathcal{N} = \{1, \dots, N\}$, where $N > 1$ is a fixed integer. Throughout this paper, we denote by $\mathcal{F} = \{f_n\}_{n \in \mathcal{N}}$ a family of complete and analytic vector fields of $\mathbf{R}^d$. Since we are interested in stability of a common equilibrium position, we assume that $f_n(0) = 0$ for each $n \in \mathcal{N}$.

By a *switching signal* we mean any right continuous, piecewise constant map $\sigma : [0, +\infty) \to \mathcal{N}$. We denote by $\mathcal{U}_{\mathcal{N}}$ the set of all the switching signals.

By a *(open-loop) switched system* we mean a pair $(\mathcal{F}, \Sigma)$, where $\Sigma : \mathbf{R}^d \to \mathcal{U}_{\mathcal{N}}$ is a (single valued) map which assigns a switching signal $\sigma(t) = \Sigma_{x_0}(t)$ to each initial state x_0. A switched system for which Σ is constant i.e., the same switching signal $\sigma(t)$ is applied for each initial state x_0, will be simply denoted by $(\mathcal{F}, \sigma)$.

For each $n \in \mathcal{N}$ we denote by $\varphi_n(t, x)$ the flow generated by the vector field $f_n(x)$. A *switched solution* of $(\mathcal{F}, \Sigma)$, corresponding to the initial state x_0, is the continuous curve $\chi(t, x_0) : [0, +\infty) \to \mathbf{R}^d$ satisfying the condition $\chi(0, x_0) = x_0$, and the following property: for each pair a, b (with $b > a \geq 0$), if $\Sigma_{x_0}(t) = n$ for $a \leq t < b$, then $\chi(t, x_0) = \varphi_n(t, \chi(a, x_0))$, $\forall t \in (a, b)$. Clearly, for each $(\mathcal{F}, \Sigma)$ and each x_0 the switched solution $\chi(t, x_0)$ exists on $[0, +\infty)$ and it is unique.

Definition 1 *We say that the origin is* stable *for the switched system* $(\mathcal{F}, \Sigma)$ *if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|x_0| < \delta \Longrightarrow |\chi(t, x_0)| < \varepsilon \ , \quad \forall t \geq 0 \ .$$

We say that the origin is locally attractive *for the switched system* $(\mathcal{F}, \Sigma)$ *if there exists* $\delta_0 > 0$ *such that*

$$|x_0| < \delta_0 \Longrightarrow \lim_{t \to +\infty} \chi(t, x_0) = 0 \ .$$

Finally, we say that the origin is locally asymptotically stable *for the switched system* $(\mathcal{F}, \Sigma)$ *if it is stable and attractive.*

Since we are interested only in local stability, from now on we shall drop the word local.

Definition 2 *We say that a family* $\mathcal{F}$ *is* stabilizable *if there exists a map* $\Sigma : \mathbf{R}^d \to \mathcal{U}_{\mathcal{N}}$ *such that the origin is asymptotically stable for the switched system* $(\mathcal{F}, \Sigma)$.

Note that according to Definition 2, stabilizability means that all the initial states in a neighborhood of the origin can be eventually driven toward the origin, but different initial states could require different switching rules..

Remark 1 A more general definition of switched system can be given by allowing multivalued maps $\Sigma : \mathbf{R}^d \to \mathcal{V}$, where $\mathcal{V}$ is some preassigned subset of $\mathcal{U}_{\mathcal{N}}$. In this case, we expect that for each x_0 there is more than one trajectory. The stability properties of these systems are usually studied by means of single or multiple Lyapunov functions (see for instance [4] and the references therein).

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Remark 2 Following the use of switched system literature, the term "stabilizability" has been adopted in Definition 2. However, in accordance with classical control theory literature, the term "asymptotic controllability" would be more appropriate.

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3 The main result

For linear switched systems, assumption **(L)** can be exploited in an open-loop setting, in order to construct a time-dependent switching rule, as well ([8]). The idea is as follows. The vector field $\bar{f}(x) = \bar{A}x$ can be seen as a selection of the differential inclusion

$$\dot{x} \in \mathrm{co}\{A_n x, n = 1, \dots, N\}$$

where co$E$ denotes the convex hull of a set E. Then, in analogy with Wazeŵski relaxation theorem ([1]), one can try to approximate the integral curves of $\bar{f}(x)$ by those of the subsystems $f_n(x) = A_n x$. However, Wazeŵski relaxation theorem guarantees only that the integral curves of $\bar{f}(x)$ can be approximated by absolutely continuous curves which are solutions of the differential inclusion

$$\dot{x} \in \{A_n x, n = 1, \dots, N\} .$$

It is not obvious that the approximation can be actually realized by means of solutions generated by a time-dependent switching rule. Moreover, there may be further problems related to the unboundedness of the interval (see [7]). The situation is similar with other well known approximation theorems available in the literature (for instance [5], p. 78), which guarantee uniform convergence only on bounded intervals.

The construction of stabilizing time-dependent (fast) switching rules for linear switched systems is performed in [11] (see also [10]) under assumption **(L)**, by a completely different and more direct method. Here, the idea is to use the Baker-Campbell-Hausdorff formula for matrices (see [12]) in order to approximate the vector field $\bar{f}(x) = \bar{A}x$. The resulting switching rule is the same for each initial state and it is periodic. The Baker-Campbell-Hausdorff formula suggests the partition of the periodicity interval, in order to obtain a convenient approximation of the trajectories of $\bar{A}$.

Now we come to the nonlinear case. The natural extension of assumption **(L)** is

(NL) *there exist* $\alpha_1, \dots, \alpha_N$ ($\alpha_n > 0$, $\sum_{n=1}^{N} \alpha_n = 1$) *such that the origin is asymptotically stable for the system*

$$\dot{x} = \bar{f}(x) = \sum_{n=1}^{N} \alpha_n f_n(x) \tag{2}$$

Let $r > 0$ and let $\mathcal{B}_r = \{x \in \mathbf{R}^d : ||x|| < r\}$. Let $\mathcal{H}$ be the space of all the bounded, analytic vector fields on $\mathcal{B}_r$. We endow $\mathcal{H}$ with a norm, so that it becomes a Banach space.

Let $f_1, f_2 \in \mathcal{H}$ and let α, β be positive numbers such that $\alpha + \beta = 1$. Let h be defined as the sum of the series

$$h = (\alpha f_1 + \beta f_2)T + \frac{\alpha\beta}{2} T^2 [f_1, f_2] + \frac{T^3}{12} \left(\alpha^2\beta[f_1, [f_1, f_2]] + \alpha\beta^2[f_2, [f_2, f_1]]\right) + \dots \tag{3}$$

where $[f_1, f_2]$ denotes the Lie product between $f_1, f_2 \in \mathcal{H}$, that is $[f_1, f_2] = (Df_1)f_2 - (Df_2)f_1$, and Df denotes the jacobian matrix of a vector field f.

Lemma 1 *There exists* $\bar{T} > 0$ *such that for all* $T < \bar{T}$ *and for all* $x \in \mathcal{B}_r$, *the series (3) converges, so that* h *is well defined. Moreover,* $\varphi_1(\alpha T, \varphi_2(\beta T, x)) = \varphi_h(1, x)$, *where* φ_h *is the flow generated by the vector field* h.

Lemma 1 is a consequence of Baker-Campbell-Hausdorff formula for vector fields (see [12]). The following theorem is our main result of this paper.

Theorem 1 *Let* $\mathcal{F} = \{f_n\}_{n \in \mathcal{N}}$ *be any family of vector fields of* $\mathbf{R}^d$, *such that* $f_n(0) = 0$ *for each* $n = 1, \dots, N$. *Assume that condition* **(NL)** *holds. Then,* $\mathcal{F}$ *is stabilizable.*

Proof For the sake of simplicity, we prove the theorem for a family formed by two vector fields $\{f_1, f_2\}$. Then, we give appropriate suggestions to indicate how to remove this restriction.

By hypothesis, there exist $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta = 1$ and $\alpha f_1 + \beta f_2$ is asymptotically stable. Let V be a C^1 Lyapunov function for $\alpha f_1 + \beta f_2$. For any $c > 0$, we denote by L_c the connected component of the level set $\{x : V(x) < c\}$ containing the origin. Let us consider $c > 0$ such that $\overline{L}_c$ is bounded and it is contained in the basin of attraction of the origin. Finally, let r be such that $\overline{L}_c \subset \mathcal{B}_r$.

Now, we consider $T_0 > 0$ such that (3) converges for any $T \in [0, T_0]$ and for any $x \in \overline{L}_c$. Then:

$$\begin{aligned} \nabla V(x) \cdot h(x) &= \nabla V(x) \cdot (\alpha f_1(x) + \beta f_2(x))T + \nabla V(x) \cdot \left\{ \tfrac{\alpha\beta}{2} T^2 [f_1, f_2](x) + \dots \right\} \\ &= \nabla V(x) \cdot (\alpha f_1(x) + \beta f_2(x))T + T^2 \nabla V(x) \cdot \left\{ \tfrac{\alpha\beta}{2} [f_1, f_2](x) + \dots \right\} \\ &= TU_0(x) + T^2 R(x), \end{aligned} \tag{4}$$

where $U_0(x) < 0$ for any $x \in \overline{L}_c \setminus \{0\}$, $U_0(0) = 0$, and $R(x)$ is a continuous function.

Let us consider the closed set

$$\mathcal{V}_1 = \overline{L}_c \setminus L_{\frac{c}{2}}.$$

We set:

$$\begin{array}{rcl} m & = & \max_{x\in\mathcal{V}_1} U_0(x) < 0 \\ M & = & \max_{x\in\mathcal{V}_1} |R(x)|. \end{array}$$

Let us consider $T' > 0$ such that $m + T'M < 0$ and $T_1 = \min\{T', T_0\}$. Then:

$$\nabla V(x)\cdot h(x) = TU_0(x) + T^2R(x) \leq T(m+TM) < 0, \qquad \forall T \in (0,T_1], \forall x \in \mathcal{V}_1\ . \tag{5}$$

We take now K as any integer such that

$$K > \frac{c}{2}\cdot\frac{1}{|(m+T_1M)T_1|}\ , \quad \text{and} \quad KT_1 \geq 1\ .$$

We shall prove that, for any $x_0 \in \mathcal{V}_1$, $V(\varphi_h(K,x_0)) \leq \frac{c}{2}$, through the following steps:

1. $V(\varphi_h(T,x_0)) < c, \forall T \in (0,K]$.
2. $\exists \theta_1 \in (0,K]$ such that $V(\varphi_h(\theta_1,x_0)) = \frac{c}{2}$.
3. $V(\varphi_h(K,x_0)) \leq \frac{c}{2}$.

Proof of Step 1. Let us suppose, by contradiction, that there exists $\overline{\theta} \in (0,K]$ such that, $\forall T \in (0,\overline{\theta}]$, $\frac{c}{2} < V(\varphi_h(T,x_0)) < c$ and $V(\varphi_h(\overline{\theta},x_0)) = c$. Then:

$$\begin{array}{rcl} V(\varphi_h(\overline{\theta},x_0)) & = & V(x_0) + \displaystyle\int_0^{\overline{\theta}} \frac{d}{d\theta}V(\varphi_h(\theta,x_0))d\theta = V(x_0) + \int_0^{\overline{\theta}} \nabla V(\varphi_h(\theta,x_0)\cdot h(\varphi_h(\theta,x_0))d\theta \\ & \leq & V(x_0) + \overline{\theta}\,[T_1(m+T_1M)] \leq c + K\,[T_1(m+T_1M)] < c, \end{array}$$

a contradiction.

Proof of Step 2. Let us suppose by contradiction that, for any $T \in (0,K]$, $\frac{c}{2} < V(\varphi_h(T,x_0)) < c$. Then,

$$V(\varphi_h(K,x_0)) = V(x_0) + \int_0^K \frac{d}{d\theta}V(\varphi_h(\theta,x_0))d\theta \leq c + K\,[T_1(m+T_1M)] < \frac{c}{2},$$

a contradiction. Therefore, there exists $\theta_1 \in (0,K]$ such that $V(\varphi_h(\theta_1,x_0)) = \frac{c}{2}$.

Proof of Step 3. Let us suppose, by contradiction, that $V(\varphi_h(K,x_0)) > \frac{c}{2}$. Then there exists $\theta_2 \geq \theta_1$ such that $\frac{c}{2} \leq V(\varphi_h(T,x_0)) \leq c$, for all $T \in (\theta_2,K]$ and

$$V(\varphi_h(K,x_0)) = V(\varphi_h(\theta_2,x_0)) + \int_{\theta_2}^K \frac{d}{d\theta}V(\varphi_h(\theta,x_0))d\theta = \frac{c}{2} + (K-\theta_2)\,[T_1(m+T_1M)] < \frac{c}{2},$$

a contradiction.

Thus, we have proved that there exist $T_1 > 0$, $K \in \mathbf{N}$ such that, if we set $\Phi_{T_1}(x) = \varphi_2(\beta T_1, \varphi_1(\alpha T_1, x))$, we have

$$V(\Phi_{T_1}^K(x_0)) < \frac{c}{2}, \qquad \forall x_0 \in \mathcal{V}_1$$

where the superscript K denotes the iterated map $\Phi_{T_1}(\Phi_{T_1}(\ldots x)\ldots)$ (K times).
Equivalently, on the interval $[0,KT_1)$, we may define the switching signal:

$$\Sigma_{x_0}(t) = \begin{cases} 1, & t \in [iT_1, iT_1+\alpha T_1), \\ 2, & t \in [iT_1+\alpha T_1, (i+1)T_1), \end{cases} \qquad i = 0,\ldots,K-1.$$

If $\chi(t,x_0)$ is the trajectory associated to $\Sigma_{x_0}(t)$, we have:

$$V(\chi(KT_1,x_0) < \frac{c}{2}, \qquad \forall x_0 \in \mathcal{V}_1.$$

For notational consistency, rename $K = K_1$. If we repeat the same procedure in $\mathcal{V}_2 = \overline{L_{\frac{c}{2}}} \setminus L_{\frac{c}{4}}$, we obtain $K_2 \in \mathbf{N}$, $T_2 > 0$ and Φ_{T_2} such that

$$V(\Phi_{T_2}^{K_2}(x_0)) < \frac{c}{4}, \qquad \forall x_0 \in \mathcal{V}_2.$$

Equivalently, we may define a switching law in the following way:

$$\text{If } x_0 \in \mathcal{V}_1 : \qquad \tilde{\Sigma}_{x_0}(t) = \begin{cases} 1, & t \in [iT_1, iT_1 + \alpha T_1), \\ 2, & t \in [iT_1 + \alpha T_1, (i+1)T_1), \end{cases} \qquad i = 0, \dots, K_1 - 1$$

$$\text{If } x_0 \in \mathcal{V}_2 : \qquad \tilde{\Sigma}_{x_0}(t) = \begin{cases} \Sigma_{x_0}(t), & t \in [0, K_1 T_1) \\ 1, & t \in [K_1 T_1 + iT_2, K_1 T_1 + iT_2 + \alpha T_2), \\ 2, & t \in [K_1 T_1 + iT_2 + \alpha T_2, K_1 T_1 + (i+1)T_2), \end{cases} \qquad i = 0, \dots, K_2 - 1\ .$$

We may iterate the procedure to get the thesis.

As far as the case $N > 2$ is concerned, we first notice that Lemma 1 can be easily extended by a recursive procedure. For instance, if we have three vector fields f_1, f_2, f_3 and three constants α, β, γ, we look for a vector field $k \in \mathcal{H}$ such that $\varphi_k(1, x) = \varphi_1(\alpha T, \varphi_2(\beta T, \varphi_3(\gamma T, x))) = \varphi_h(1, \varphi_3(\gamma T, x)$. Using (3), it is not difficult to see by substitution that

$$k = (\alpha f_1 + \beta f_2 + \gamma f_3)T + \left(\frac{\alpha\beta}{2}[f_1, f_2] + \frac{\beta\gamma}{2}[f_2, f_3] - \frac{\gamma\alpha}{2}[f_3, f_1]\right) T^2 + \dots\ .$$

The remaining part of the proof requires only minor modifications.

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Remark 3 In the time-dependent switching rule constructed in the proof of Theorem 1 we distinguish two types of switching times. At the "principal" switching times

$$0, K_1T_1, K_1T_1 + K_2T_2, K_1T_1 + K_2T_2 + K_3T_3, \dots$$

(with $K_i T_i \geq 1$) the features of the switching rule are updated. In each interval determined by two consecutive principal switching times the signal switches periodically (with period T_i). The number T_i depends on the distance of the current state from the origin and may become greater and greater as the corresponding trajectory approaches the origin.

Remark 4 Many results about stabilization of switched systems available in the literature assume that all the vector fields of the given family are unstable. The results of [9] apply to pairs of vector fields with opposite stability properties. On the contrary, Theorem 1 above requires no assumption about the stability or instability of each single vector field of $\mathcal{F}$, but only condition **(NL)**. Hence, it can be used to characterize stability in certain applications where the system must necessarily switch among all its possible modes.

4 Examples

In this section we discuss three examples. The first shows that for certain families $\mathcal{F}$ containing nonlinear vector fields, it is impossible to find periodic stabilizing rules independent of the initial state. The second one shows that in order to stabilize families $\mathcal{F}$ containing nonlinear vector fields, we cannot limit ourselves to periodic switching. Finally, the last example is an application of Theorem 1: in particular it shows that in certain situations a stabilizing time-dependent switching rule can be easily obtained, while it is not clear how to construct a state-dependent one.

Example 1 Consider a family composed by three vector fields of $\mathbf{R}^2$

$$f_1(x_1, x_2) = \begin{pmatrix} 2x_1 \\ -x_2 \end{pmatrix}, \quad f_2(x_1, x_2) = \begin{pmatrix} -x_1 \\ 2x_2 \end{pmatrix}, \quad f_3(x_1, x_2) = -x_1^2 x_2^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where f_3 is nonlinear. The points lying on the x_2-axis can be driven to the origin using the vector field f_1, while the points lying on the x_1-axis can be driven to the origin using the vector field f_2. All the others points can be driven to the origin by the vector field f_3.

Consider now a point of the form $(0, x_2)$, and assume that there exists a periodic switching law driving this point to the origin. Clearly the rate of time spent on the vector field f_1 must be strictly greater than the rate of time spent on the vector field f_2. But, if we look at a point of the form $(x_1, 0)$, we arrive exactly at the opposite conclusion. Thus, a periodic stabilizing switching law independent of the initial state cannot exist.

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Example 2 Let us consider a family formed by two one-dimensional vector fields $f_1(x)$ and $f_2(x)$, such that:

- f_1 is a C^∞, odd function, such that:

  (*i*) for $x > 0$, $f_1(x) \leq 0$, and $f_1(x) = 0 \Longleftrightarrow x = \frac{1}{2^k}$, $(k \in \mathbf{N})$;

  (*ii*) for $x \in \left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)$, $|f_1(x)| \leq M_{k+1}$, $(k \in \mathbf{N})$;

- f_2 is a C^∞, odd function, such that:

  (*i*) for $x > 0$, $f_2(x) \leq 0$, and $f_2(x) = 0 \Longleftrightarrow x = \frac{3}{2 \cdot 2^k}$, $(k \in \mathbf{N} \setminus \{0\})$;

  (*ii*) for $x \in \left(\frac{3}{2 \cdot 2^{k+1}}, \frac{3}{2 \cdot 2^k}\right)$, $|f_2(x)| \leq M_k$, $(k \in \mathbf{N} \setminus \{0\})$,

where $M_k = \frac{1}{k 2^k}$, $k \in \mathbf{N} \setminus \{0\}$.

Clearly, neither f_1 nor f_2 are asymptotically stable, but condition (**NL**) is fulfilled. The natural way to construct a stabilizing switching law is to fix a decreasing sequence of points $\{x_k > 0\}$ such that $3/4 < x_0 < 1$, $1/2 < x_1 < 3/4$, $3/8 < x_2 < 1/2$, $1/4 < x_3 < 3/8$, ..., so that

$$\ldots < \frac{3}{2 \cdot 2^{h+2}} < x_{2h+2} < \frac{1}{2^{h+1}} < x_{2h+1} < \frac{3}{2 \cdot 2^{h+1}} < x_{2h} < \frac{1}{2^h} < \ldots$$

We choose a state dependent switching law σ such that

$$\sigma(x) = \begin{cases} 1 & \text{if } x_{2h+1} < x < x_{2h} \\ & \\ 2 & \text{if } x_{2h+2} < x < x_{2h+1} \end{cases} \qquad (k \in \mathbf{N})$$

We may work symmetrically for $x < 0$. The switching law we obtain can be converted in a time-dependent one, $t \mapsto \sigma(x(t))$.

Assume that for some choice of the points x_k the corresponding time-dependent switching law is periodic. Let T_1 and T_2 be the share of time respectively spent on f_1 and f_2, and let $T = T_1 + T_2 < +\infty$ be the period. For any $h \in \mathbf{N} \setminus \{0\}$, we have:

$$x_{2h} - x_{2h+2} > \frac{3}{2 \cdot 2^{h+1}} - \frac{1}{2^{h+1}} = \frac{1}{2 \cdot 2^{h+1}}$$

$$x_{2h} - x_{2h+2} < T\,(M_{h+1} + M_{h+1}) = T \cdot \frac{1}{(h+1) \cdot 2^h}.$$

Thus, we get $T > \frac{h+1}{4}$, a contradiction.

Example 3 Consider the family formed by the planar vector fields

$$f_1(x_1, x_2) = \begin{pmatrix} -2x_1^3 x_2 \\ -2x_2 \end{pmatrix}, \qquad f_2(x_1, x_2) = \begin{pmatrix} 0 \\ 2x_1^2 \end{pmatrix}.$$

Taking a convex combination with $\alpha = \beta = 1/2$, one obtains

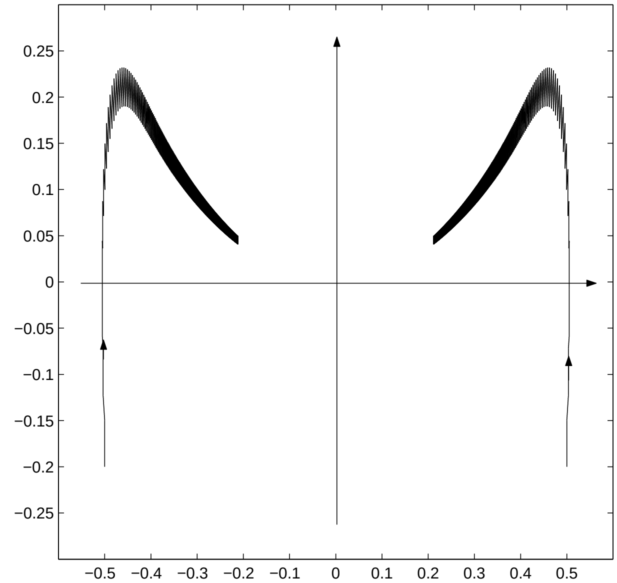

Figure 1: Switched trajectories for Example 3: $\alpha = \beta = 0.5$, $T = 0.1$

$$\bar{f}(x_1, x_2) = \begin{pmatrix} -x_1^3 x_2 \\ -x_2 + x_1^2 \end{pmatrix} .$$

Using the center manifold reduction method (see for instance [2]), it is easy to see that $\bar{f}$ is asymptotically stable at the origin. However, it does not seem obvious to write a Lyapunov function for $\bar{f}$, so that it is not clear how to determine a state-dependent switching rule based on a state space decomposition. On the contrary, as suggested by the proof of Theorem 1, one can construct trajectories which approach the origin (at least on a bounded interval) by switching with the same frequency between f_1 and f_2. A simulation is shown in Figure 1.

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