Some remarks about stability of nonlinear discrete-time control systems

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Abstract

[1] We study stability and stabilizability properties of nonlinear discrete-time control systems by means of Liapunov functions. We give a characterisation of the uniform bounded-input-bounded-state stability property of nonlinear systems in the framework of second Liapunov method. We find an explicit feedback law stabilizing a discrete-time control system affine-in-control under the Jurdjevic–Quinn approach.

1 Introduction.

It is a common opinion that many results obtained in the framework of the qualitative theory of continuous-time dynamical systems, can be applied, with some obvious modifications, to discrete-time systems, as well. In this paper we are interested in certain stability properties of systems with and without inputs. We consider in particular a number of results whose continuous-time version is well known. We prove that they can be actually extended to the discrete-time case, but the required modifications are not completely obvious and, in addition, their proof requires a different approach and the development of new tricks.

In the first two sections we consider finite dimensional, discrete-time, autonomous systems of the form

$$x_{k+1} = f(x_k) \tag{1}$$

where $k \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous (here, $\mathbb{N}$ is the set of natural numbers, zero included). As far as the authors know, in the continuous-time context the most general version of the so called First Liapunov Theorem is due to Auslander

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and Seibert (see [2]) and involves non-necessarily continuous Liapunov functions. We give a proof that the same result holds also in the discrete-time case. The main difficulty to be overcome is the lack of connectedness of trajectories. Moreover, we compare this result with more classical versions of First Liapunov Theorem available in the discrete-time literature, where the continuity of the Liapunov function playes an essential role. Note that relaxing the continuity assumption, makes possible to prove a converse theorem, so that the First Liapunov Theorem can be stated as a necessary and sufficient condition. Finally, we give an analogous characterization of the boundedness of solutions property (sometimes also called Lagrange stability).

The further two sections deal with discrete-time, autonomous input systems defined by

$$x_{k+1} = f(x_k, u_k) \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Equation (2) describes the behavior of the system in the sense that its evolution depends not only on an initial state x_0, but also on a sequence of input signal $\boldsymbol{u} = \{u_0, u_1, u_2, \dots\}$. We give an original characterization of a property of stability with respect to initial state and input disturbances (the so-called UBIBS stability property) in the framework of Liapunov method. Finally, we consider the feedback stabilization problem for nonlinear affine systems. Our result looks like a very natural extension to discrete-time systems of the Jurdjevic-Quinn approach to the construction of continuous-time stabilizing feedback laws.

We conclude this introduction by some words about notation.

The norm $|x|$ of a vector $x \in \mathbb{R}^n$ is the usual euclidean norm; we denote by $\mathcal{B}(\alpha)$ the open sphere of positive radius α centered at the origin. The notation $\varphi(k; x_0, \boldsymbol{u})$ represents the trajectory of the discrete-time system (2) corresponding to the initial condition x_0 and the input sequence $\boldsymbol{u} = \{u_0, u_1, u_2, \dots\}$. For a system of the form (1) we simply write $\varphi(k; x_0)$ to denote the trajectories.

If V is a C^2 real-valued function defined on the whole space $\mathbb{R}^n$, we denote by ∇V its gradient (row vector) and by $HV \in \mathbb{R}^{n \times n}$ the hessian of V. We write A^t for the transpose of matrix A, $\|A\|$ for its norm as a continuous linear operator and $r(A)$ for its spectral radius (i.e., the maximum modulus of eigenvalues). We recall that $r(A) \leq \|A\|$.

2 Local stability.

Consider the finite-dimensional, discrete-time autonomous system (1) where $k \in \mathbb{N}$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and $f(0) = 0$, so that the origin is a fixed point. Recall that this fixed point is *stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all x_0 with $| x_0 | < \delta$ and for all integer k then $| \varphi(k; x_0) | < \epsilon$.

Definition 2.1. A *Liapunov function* for the system (1) is a map $V : H \longrightarrow \mathbb{R}$, where H is a positively invariant neighborhood of the origin, satisfying the following properties:

(i) $V(0) = 0$ and V is continuous at $x = 0$;

(ii) there exists $\eta_1 > 0$ such that for all $\epsilon \in (0, \eta_1)$ there is $\lambda > 0$ such that

$$\text{if } \epsilon \leq |x| \leq \eta_1 \text{ then } V(x) \geq \lambda;$$

(iii) there exists $\eta_2 > 0$ such that

$$\text{if } |x| \leq \eta_2 \text{ then } V(f(x)) \leq V(x).$$

A *strict* Liapunov function for the system (1) is a map with the properties (i), (ii) and

(iii') there exists $\eta_2 > 0$ such that

$$\text{if } |x| \leq \eta_2 \text{ and } x \neq 0 \text{ then } V(f(x)) < V(x).$$

As we pointed out in the introduction, the main result of this section is the following characterisation of stability.

Theorem 2.1. *The origin is a stable fixed point for* (1) *if and only if there exists a Liapunov function for* (1).

Proof. Sufficiency. Let $\eta = \min\{\eta_1, \eta_2\}$ so that we can assume that both (i) and (ii) hold with, respectively, η_1 and η_2 replaced by η.
Since f is continuous and $f(0) = 0$, the set $f^{-1}(\mathcal{B}(\eta))$ contains a neighborhood of the origin. Let $\eta' \in (0, \eta)$ be such that

$$\mathcal{B}(\eta') \subset f^{-1}(\mathcal{B}(\eta)) \tag{3}$$

Let now $\epsilon \in (0, \eta')$. According to (ii) there exists $\lambda > 0$ such that $V(x) \geq \lambda$ for $\epsilon \leq |x| \leq \eta$. Since V is continuous at $x = 0$ and $V(0) = 0$, there exists $\delta \in (0, \epsilon)$ such that if $|x| < \delta$ then $V(x) < \lambda$. Now, assume that there exist $x_0 \in \mathcal{B}(\delta)$ and some index $i > 0$ such that $|x_i| \geq \epsilon$, where $x_k = \varphi(k; x_0)$ is the trajectory of (1) starting from x_0. Let $K > 0$ the smallest index for which this happens, so that we have

$$|x_i| < \epsilon < \eta \ \text{ for } i = 0, 1, \ldots, K-1 \text{ and } |x_K| \geq \epsilon. \tag{4}$$

In fact, we claim that it must be $|x_K| < \eta$, as well. Indeed, $|x_{K-1}| < \epsilon < \eta'$, and, since $x_K = f(x_{K-1})$, the claim follows by virtue of (3).
We are now ready to get the conclusion. According to (4), property (iii) can be applied recursively for $K - 1$ steps and we obtain

$$V(x_K) \leq V(x_{K-1}) \leq \ldots \leq V(x_0) < \lambda. \tag{5}$$

On the other hand, since $\epsilon \leq |x_K| < \eta$, (ii) applies as well and we get $V(x_K) \geq \lambda$, a contradiction to (5).

Necessity. Let $\epsilon = 1$. According to the stability assumption, there is a $\delta > 0$ such that if $|x_0| < \delta$ then $|x_k| < 1$, where $x_k = \varphi(k; x_0)$ is the trajectory of (1) starting from x_0. Let

$$H := \{x \in \mathbb{R}^n \text{ such that } \exists k \in \mathbb{N}, \exists \xi \in \mathcal{B}(\delta) \text{ for which } x = f^k(\xi)\}.$$

By construction, $H \subseteq \mathcal{B}(1)$. Moreover, H is positively invariant; indeed, if $x \in H$ and h is a positive integer, then also

$$f^h(x) = f^{h+k}(\xi) \text{ for some } k \in \mathbb{N} \text{ and } \xi \in \mathcal{B}(\delta).$$

We define the function V on H taking, for every $x \in H$

$$V(x) = \sup_{k \in \mathbb{N}} |f^k(x)|. \tag{6}$$

We remark that V is well defined on H; in fact, for $x \in H$ we have $V(x) \leq 1$. Moreover, $V(0) = 0$ and $V(x) \geq |x|$ for $x \neq 0$, which implies property (ii). As far as property (iii) is concerned, we note that for each $x \in H$ by construction

$$V(f(x)) = \sup_{k \geq 1} |f^k(x)|,$$

hence obviously $V(f(x)) \leq V(x)$.
We now prove that V is continuous at $x = 0$. Let $\sigma \in (0, 1)$: again by the stability assumption we can find $\rho > 0$ such that if $|x| < \rho$ then $|f^k(x)| < \sigma$ for each $k = 0, 1, 2, \ldots$. According to (6), we have therefore

$$V(x) = \sup_{k \in \mathbb{N}} |f^k(x)| \leq \sigma$$

and the assertion easily follows. □

Remark 2.1.

(1) The previous theorem is the analogous of a result proved by [2] for continuous-time systems. However, we note that the extension of the proof is not straightforward: indeed, in the continuous-time case the usual proof is based on the fact that trajectories are connected sets. Here we found a trick which allows us to overcome the difficulty of discrete-time case trajectories, which are non-connected sets. The necessity part is just the time-invariant version of a result of [6].

(2) In the discrete-time systems literature (see for instance [9]) the sufficiency statement usually presents the additional assumption that the Liapunov function V is everywhere continuous and the proof is quite different.

(3) Note that the existence of a Liapunov function, *continuous* in a whole neighborhood of the origin cannot be guaranteed for a stable (not asymptotically) system. This can be seen by the discrete-time version of the well-known center-focus example presented in [2]. The existence of a continuous Liapunov function will be the subject of a following section.

(4) Assumption (ii) is verified if, for instance, V is strictly positive for each $x \neq 0$ and V is lower semi-continuous.

(5) By the way, we remark that literature about converse theorems on asymptotic stability for discrete-time systems seems to be neither very systematic nor complete. Although not explicity stated, it follows easily from the proof of [6] that every asymptotically stable system of the form (1) has a time-invariant strict Liapunov function V, which turns out to be locally Lipschitz continuous, whenever f is locally Lipschitz continuous.

On the other hand, it can be seen that Theorem 10 of [2] is valid also for discrete-time system (see [3] for modifications needed in the proof). This theorem states that, when f is continuous, the existence of a *continuous*, strict Liapunov function is a necessary and sufficient condition for asymptotic stability.

According to [8], it is in fact reasonable to conjecture that a C^∞ Liapunov function exists, whenever f is continuos. However, as far as the author know, the validity of the proof for discrete-time case has never been checked.

3 Boundedness of solution.

For continuous-time systems, boundedness of solutions was deeply studied by Yoshizawa [12] (see also [2]). A sufficient condition which applies to discrete-time systems is given in [9, page 9]; it makes use of continuous Liapunov functions. Here we give a more general condition, which turns out to be necessary, as well.

Our result can be viewed as an *in large* version of the local stability result given in Section 2. As in that Section, the proof requires some triks to overcome the fact that trajectories of discrete-time systems are non-connected sets (see Remark 2.1).

Definition 3.1. For a *neighborhood of infinity* we mean any subset of $\mathbb{R}^n$ which contains a set of the form

$$\{x \in \mathbb{R}^n \text{ such that } |x| \geq \rho\}$$

for some $\rho > 0$.

Definition 3.2. Let H be a positively invariant neighborhood of infinity. A *Liapunov function in the large* for the system (1) is a function $V : H \longrightarrow \mathbb{R}$ such that

(iv) V is radially unbounded, i.e., $\lim_{|x|\longrightarrow+\infty} V(x) = +\infty$;

(v) there exists $\theta > 0$ whit the following property: for all $r > \theta$ we can find $m > 0$ such that if $\theta \le |x| \le r$ then $V(x) \le m$;

(vi) there exists $\eta > 0$ such that if $|x| \ge \eta$ then $V(f(x)) \le V(x)$.

Definition 3.3. System (1) is *Lagrange stable* if for each $R > 0$ there exists $S > 0$ such that if $|x| \le R$ then $|f^k(x)| \le S$ for $k \in \mathbb{N}$.

Theorem 3.1. *System* (1) *is Lagrange stable if and only if there exists a Liapunov function in the large for* (1).

Proof. Sufficiency. Let $\rho = \max\{\theta, \eta\}$ so that V is defined for $|x| \ge \rho$ and we can assume that (v) and (vi) hold with θ and η replaced, respectively, by ρ.
Let $A = f(\overline{\mathcal{B}(\rho)})$: since f is continuous, A is compact. Without loss of generality we can choose $R > 0$ so large that

$$\overline{\mathcal{B}(\rho)} \cup A \subset \mathcal{B}(R). \tag{7}$$

According to (v), we can find $m > 0$ such that if $\rho \le |x| \le R$ then $V(x) \le m$. Finally, using (iv), we can find $S > R$ such that if $|x| > S$ then $V(x) > m$.
We are now able to prove that for each $x_0 \in \overline{\mathcal{B}(R)}$ and for each $i \in \mathbb{N}$ the inequality

$$|f^i(x_0)| \le S \text{ holds.}$$

Assume the contrary, and let $K > 0$ be an integer such that $|f^K(x_0)| > S$. Let $q < K$ be the greatest integer such that

$$|f^q(x_0)| \le R \text{ while } |f^i(x_0)| > R \text{ for } i = q+1, \dots, K. \tag{8}$$

We claim that $|f^q(x_0)| > \rho$, as well. Indeed, in the opposite case, because of (7), we should have

$$|f^{q+1}(x_0)| = |f(f^q(x_0))| < R,$$

a contradiction to (8).
We can conclude the proof: on one hand since $\rho \le |f^q(x_0)| \le R$ and $|f^K(x_0)| > S$ then (v) and (vi) imply, as noted before,

$$V(f^q(x_0)) \le m \text{ and } V(f^K(x_0)) > m. \tag{9}$$

On the other hand, since $|f^i(x_0)| > \rho$ for $i = q, q+1, \dots, K-1$, property (vi) applies recursively and we get

$$V(f^K(x_0)) \le V(f^q(x_0)),$$

a contradiction to (9).

Necessity. Assume that the system (1) is Lagrange stable. We can define V on the whole space $\mathbb{R}^n$ by taking, as in Theorem 2.1,

$$V(x) := \sup_{k\in\mathbb{N}} |f^k(x)|.$$

We note that V is well defined. Indeed, for each fixed x, we can take $R = 2|x|$ and find S such that $|f^k(x)| < S$ for $k \in \mathbb{N}$, hence $\sup_{k\in\mathbb{N}} |f^k(x)| < +\infty$.
Clearly, $V(x) \geq |x|$, which implies (iv), and the monotonicity property (vi) follows as in Theorem 2.1.
As long as property (v) is concerned, take $\theta = 0$ and let r be an arbitrary positive number. Choosing $R = r$, we can find S such that if $|x| \leq R$ then $|f^k(x)| \leq S$ for $k \in \mathbb{N}$. By definition of V we have also $V(x) \leq S$, so that (v) holds with $S = m$. □

Remark 3.1. Property (v) holds if, for instance, V is upper semi-continuous.

4 Bounded–Input–Bounded–State Stability

In this section we consider systems with inputs (2). The notions of stability, Lagrange stability and asymptotic stability studied in previous sections are appropriate ways to describe evolution of a system without external inputs. However, they become meaningless when the behavior of the system is not only a consequence of the energy initially stored, but it is also affected by the energy supplied at each step through the input channel.

We emphasize that, in this section, $\boldsymbol{u}$ is viewed as an exogenous signal or a disturbance, and not as a control. A way to state that system (2) is *stable* with respect to such disturbances is given by the following

Definition 4.1. System (2) is said to be *uniformly bounded–input–bounded–state stable* (UBIBS in short) if for each $R > 0$ there exists $S > 0$ such that for each initial condition x_0 and each bounded input sequence $\boldsymbol{u} = \{u_0, u_1, \dots\}$ we have the property

$$\text{if } |x_0| \leq R \text{ and } \|\boldsymbol{u}\|_\infty \leq R \text{ then } |\varphi(k; x_0, \boldsymbol{u})| \leq S \text{ for all } k \in \mathbb{N},$$

where $\|\boldsymbol{u}\|_\infty$ is the l^∞ norm for a sequence:

$$\|\boldsymbol{u}\|_\infty = \sup_{k\in\mathbb{N}} |u_k|.$$

We are able to give a necessary and sufficient condition for UBIBS stability by means of appropriate Liapunov-like functions.

Definition 4.2. A *UBIBS–Liapunov function* for the system (2) is a function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

(iv) V is radially unbounded, i.e., $\lim_{|x|\longrightarrow+\infty} V(x) = +\infty$;

(v') V is locally bounded, i.e., for each $\tau > 0$ there exists $m > 0$ such that if $|x| \leq \tau$ then $V(x) \leq m$;

(vi') for each $R > 0$ there exists $\rho > 0$ such that if $|x| \geq \rho$ and $|u| \leq R$ then $V(f(x,u)) \leq V(x)$.

Theorem 4.1. *The system* (2) *is UBIBS–stable if and only if there exists a UBIBS–Liapunov function for it.*

Proof. Sufficiency. Suppose that there is a UBIBS–Liapunov function for the system (2); assume by contradiction that there exists some $R > 0$ for which the requirements of Definition 4.1 are not fulfilled. According to Definition 4.2, property (vi'), we can find a $\rho > R$ corresponding to such $R > 0$.
Let $A = f(\overline{\mathcal{B}(\rho)} \times \overline{\mathcal{B}(R)})$. Since f is continuous, A is a compact set. Let $\tau > \rho$ such that

$$A \subset \mathcal{B}(\tau).$$

By properties (iv) and (v') of the UBIBS–Liapunov function V, we can find $m > 0$ and $S > 0$ such that

$$\text{if } |x| \leq \tau \text{ then } V(x) \leq m \qquad\qquad \text{if } |x| > S \text{ then } V(x) > m; \tag{10}$$

it is not restrictive to assume $S > \tau$.
Recall that we are arguing by contradiction: hence, for this value S it would be possible to find an initial state x_0, a sequence of inputs $\boldsymbol{u} = \{u_0, u_1, \dots\}$ and an integer $K > 0$ such that

$$|x_0| \leq R, \|\boldsymbol{u}\|_\infty \leq R \text{ but } |\varphi(K; x_0, \boldsymbol{u})| > S.$$

Let $q < K$ be the greatest index for which

$$|\varphi(q; x_0, \boldsymbol{u})| \leq \tau;$$

then we have $|\varphi(i; x_0, \boldsymbol{u})| > \tau > \rho$ for $i = q+1, \dots, K$.
We claim that also $|\varphi(q; x_0, \boldsymbol{u})| > \rho$. In the opposite case, by construction we shold have

$$\varphi(q+1; x_0, \boldsymbol{u}) = f(\varphi(k; x_0, \boldsymbol{u}), u_q) \in A,$$

and this implies $|\varphi(q+1; x_0, \boldsymbol{u})| \leq \tau$, which is not the case. Therefore it is $|\varphi(i; x_0\boldsymbol{u})| > \rho$ for each $i = q, q+1, \dots, K$ and property (vi') applies recursively. This yields

$$V(\varphi(K; x_0, \boldsymbol{u})) \leq V(\varphi(q; x_0, \boldsymbol{u})). \tag{11}$$

On the other hand, using (10) we see that

$$V(\varphi(q;x_0,\boldsymbol{u})) \leq m < V(\varphi(K;x_0,\boldsymbol{u})),$$

a contradiction to (11).
Necessity. Assume that the system (2) is UBIBS stable. We recall that by $\boldsymbol{u}$ we denote a sequence of inputs $\boldsymbol{u} = \{u_0, u_1, \dots\}$. First of all, we define two sets of admissible controls and, respectively, of possible initial states as follow. For every $\xi \in \mathbb{R}^n$ we denote

$$U(\xi) = \{\boldsymbol{u} \text{ such that } |u_i| \leq \min_{0\leq j\leq i} |\varphi(j;\xi,\boldsymbol{u})| \text{ for all } i \in \mathbb{N}\} \tag{12}$$

and, for every $x \in \mathbb{R}^n$,

$$X(x) = \{\xi \in \mathbb{R}^n \text{ such that } \exists \boldsymbol{u} \in U(\xi), \exists K \geq 0 \text{ for which } x = \varphi(K;\xi,\boldsymbol{u})\}. \tag{13}$$

We remark that the previous sets are not empty for all ξ and x, because $\boldsymbol{u} = \{0, 0, \dots\} \in U(\xi)$ and $x \in X(x)$.
Define V as

$$V(x) = \inf_{\xi\in X(x)} |\xi|.$$

We note that V is nonnegative and, since $x \in X(x)$, that $V(x) \leq |x|$, then V is well defined and locally bounded (property (v') actually holds with $m = \tau$).
We claim that V is radially unbounded. Assume the contrary: there is an $R > 0$ such that for each $S > 0$ we can find x_S with the property that $|x_S| > S$ and $V(x_S) \leq R/2$. According to UBIBS stability, we can fix S in such a way that if $|y| \leq R$ and $\|\boldsymbol{u}\|_\infty \leq R$ then $|\varphi(k;y,\boldsymbol{u})| \leq S$ for all $k \in \mathbb{N}$. With this choice of R and S, we can now find, by (12) and (13), $\xi \in X(x_S)$, $\boldsymbol{u} \in U(\xi)$ and an integer $K > 0$ with the properties that $\varphi(K;\xi,\boldsymbol{u}) = x_S$

$$\|\boldsymbol{u}\|_\infty \leq \min_{0\leq j\leq k} \mid \varphi(j;\xi,\boldsymbol{u}) \mid \leq \mid \xi \mid < R$$

and $|x_S| > S$. This is clearly impossible because of UBIBS stability hypothesis.
As far as property (vi') is concerned, we will prove that for each $R > 0$ there is $\rho > 0$ such that if $|x| \geq \rho$ and $|w| \leq R$ then $X(x) \subset X(f(x,w))$. The inequality $V(x) \leq V(f(x,w))$ will easily follow. Fix $R > 0$ and get $S > R$ according to the UBIBS stability hypothesis. Choose $\rho > S > R$, $|x| \geq \rho$, $|w| \leq R$ and $\xi \in X(x)$. We can find an integer K and a control $\boldsymbol{u} \in U(\xi)$ such that $x = \varphi(K;\xi,\boldsymbol{u})$. Let $\tilde{\boldsymbol{u}} = \{\tilde{u}_0, \tilde{u}_1, \dots\}$ where

$$\begin{aligned} \tilde{u}_i &= u_i \text{ for } i = 0, 1, 2, \dots, K-1, \\ \tilde{u}_K &= w \\ \tilde{u}_j &= 0 \text{ for } j > K. \end{aligned}$$

Remark that

- $\varphi(i;\xi,\boldsymbol{u}) = \varphi(i;\xi,\tilde{\boldsymbol{u}})$ for each $i = 0, 1, 2, \dots, K$;
- $\varphi(K+1;\xi,\tilde{\boldsymbol{u}}) = \varphi(1;\varphi(K;\xi,\tilde{\boldsymbol{u}}),\boldsymbol{w}) = \varphi(1;x,\boldsymbol{w}) = f(x,w)$, where $\boldsymbol{w} = \{w, 0, 0, \dots\}$.

We prove now that $\tilde{\boldsymbol{u}} \in U(\xi)$. Indeed, property

$$|\tilde{u}_i| \le \min_{0\le j\le i} |\varphi(j;\xi,\tilde{\boldsymbol{u}})|$$

is obviously fulfilled for each $i \ne K$. We claim that

$$|w| \le | \varphi(j;\xi,\boldsymbol{u}) | \quad \text{for each } j = 0, 1, \dots, K. \tag{14}$$

This is verified for $j = K$, since

$$|w| \le R \le \rho \le |x| = |\varphi(K;\xi,\boldsymbol{u})|.$$

In order to show that (14) holds for each $j = 0, 1, \dots, K-1$ it sufficient to show that

$$|w| \le R \le |\varphi(j;\xi,\boldsymbol{u})| \text{ for every } j = 0, 1, \dots, K-1.$$

Assume the contrary: we can find an integer $\nu \in \{0, 1, \dots, K-1\}$ such that $|\varphi(\nu;\xi,\boldsymbol{u})| < R$. Let $\zeta = \varphi(\nu;\xi,\boldsymbol{u})$: then we have $x = \varphi(K-\nu;\zeta,\boldsymbol{v})$, where

$$\boldsymbol{v} = \{u_\nu, u_{\nu+1} \dots, u_{K-1}, 0, \dots\}.$$

Since $\boldsymbol{u} \in U(\xi)$, we have

$$|u_{\nu+i}| \le \min_{0\le j\le \nu+i} |\varphi(j;\xi,\boldsymbol{u})| \le |\zeta| < R.$$

Thus $\|\boldsymbol{v}\|_\infty \le R$, $|\zeta| \le R$ and

$$|\varphi(K-\nu;\zeta,\boldsymbol{v})| = |x| \ge \rho > S,$$

a contradiction to the UBIBS stability hypothesis.
We showed that there exist an integer $K+1$ and a control $\tilde{\boldsymbol{u}} \in U(\xi)$ such that $f(x,w) = \varphi(K+1;\xi,\tilde{\boldsymbol{u}})$, i.e., $\xi \in X(f(x,w))$, as desired. □

5 Asymptotic stabilization.

The local stability property considered in Section 2 is, in some sense, a critical property. Indeed, it may be lost when the system undergoes small perturbations. A property which, at least in certain circumstances, guarantees more robustness and which is more suitable for applications, is the so-called *asymptotic stability*.

In this section we address the following stabilization problem: let us be given a function $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, describing the behavior of a system with inputs. Assume that $f(0,0) = 0$, but that the origin is not an asymptotically stable equilibrium for the unforced system $x_{k+1} = f(x_k, 0)$.

The problem amounts to find a function $u : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $u(0) = 0$, if any, such that the closed-loop system

$$x_{k+1} = f(x_k, u(x_k)) \tag{15}$$

is (possibly, globally) asymptotically stable at the origin.

For the continuous-time case, one of the most celebrated approaches is the so-called *Jurdjevic and Quinn* ([7]) method, which applies when the unforced system associated to (2) is locally (but not asymptotically) stable.

Many stabilization results for discrete-time systems, inspired by [7] are been published (see, for instance, [5], [4], [11]). The following theorem is a further, new discrete-time extension of the Jurdjevic and Quinn method. The difference between our theorem and the results prevously appeared in literature will be discussed below.

We shall make use of discrete-time versions of LaSalle's invariance principle and of Second Liapunov Theorem. For short we do not report here these theorems, since the classical statements of [9] are sufficient for our pourposes.

Definition 5.1. A discrete-time control system is said to be *affine–in–control* when it is of the form

$$x_{k+1} = f(x_k) + g(x_k)u_k \tag{16}$$

where $g : \mathbb{R}^n \longrightarrow \mathbb{R}^{n,m}$ is a matrix of functions with n rows and m columns. The entries of g, as the entries of the vector field f, are assumed to be continuous.

Theorem 5.1. *Let us consider the system* (16)*: if*

(i) *there exists a function* $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ *positive definite,* $V \in C^2(\mathbb{R}^n)$*, such that*

$$V(f(x)) - V(x) \leq 0,$$

(ii) *the intersection of the sets*

$$A = \{x \in \mathbb{R}^n \text{ such that } V(f^i(x)) = V(x) \text{ per } i = 0, 1, \dots\}$$
$$B = \{x \in \mathbb{R}^n \text{ such that } \nabla V(f^{i+1}(x))g(f^i(x)) = 0 \text{ per } i = 0, 1, \dots\}$$

is reduced to the origin,

(iii) *there exists a constant* $M_1 > 0$ *such that* $\|g(x)\| \leq M_1$ *for all* $x \in \mathbb{R}^n$*,*

(iv) *there exists a constant* $M_2 > 0$ *such that* $\|HV(x)\| \leq M_2$ *for all* $x \in \mathbb{R}^n$

then there exists a real number $\overline{\alpha} > 0$ with the property that, for all $0 < \alpha < \overline{\alpha}$ the continuous feedback

$$u(x) = -\alpha[\nabla V(f(x))g(x)]^t,$$

stabilizes the system (16).
If moreover

(v) $\lim_{\|x\| \longrightarrow +\infty} V(\|x\|) = +\infty$,

then the origin for the closed-loop system is globally asymptotically stable.

Remark 5.1. We can replace the set A by

$$\tilde{A} = \{x \in \mathbb{R}^n \text{ such that } V(f(x)) = V(x)\};$$

indeed, $A \subset \tilde{A}$ then $\tilde{A} \cap B = \{0\}$ implies $A \cap B = \{0\}$.

Proof. We will show that the origin is stable for the closed-loop system using V as a Liapunov function. V satisfies properties (i) and (ii) of Theorem 2.1 by continuity and positive definiteness; by the above assumption (i) we can write

$$\begin{aligned} V(f(x) + g(x)u(x)) - V(x) = \\ = [V(f(x) + g(x)u(x)) - V(f(x))] + [V(f(x)) - V(x)]. \end{aligned} \tag{17}$$

The previous expression is non-positive if $V(f(x)+g(x)u(x))-V(f(x)) \leq 0$ for all $x \in \mathbb{R}^n$. We expand V according to the second order Taylor's formula in Lagrange's form

$$\begin{aligned} & V(f(x) + g(x)u(x)) - V(f(x)) = \\ & = \nabla V(f(x))g(x)u(x) + \frac{1}{2}[g(x)u(x)]^t HV(\tilde{y})[g(x)u(x)] = \\ & = -\alpha[\nabla V(f(x))g(x)][\nabla V(f(x))g(x)]^t + \\ & + \frac{1}{2}\alpha^2[\nabla V(f(x))g(x)]g(x)^t HV(\tilde{y})g(x)[\nabla V(f(x))g(x)]^t \\ & = \alpha \nabla V(f(x))g(x)[-I_m + \frac{1}{2}\alpha\, g(x)^t HV(\tilde{y})g(x)][\nabla V(f(x))g(x)]^t \end{aligned} \tag{18}$$

where $\tilde{y} = f(x) + \theta g(x)u(x)$ for some $\theta \in (0,1)$ depending on x and I_m is the $m \times m$ identity matrix . We note that

- $\frac{1}{2}\alpha g(x)^t HV(\tilde{y})g(x)$ is a symmetric matrix for all $x \in \mathbb{R}^n$, then it has only real eigenvalues;
- $-I_m + \frac{1}{2}\alpha\, g(x)^t HV(\tilde{y})g(x)$ is a symmetric matrix whose eigenvalues have the form $\lambda - 1$, where λ is some eigenvalue of the matrix

$$\frac{1}{2}\alpha\, g(x)^t HV(\tilde{y})g(x);$$

- for all $x \in \mathbb{R}^n$

$$\|g(x)^t HV(\tilde{y}) g(x)\| \leq \|HV(\tilde{y})\| \, \|g(x)\|^2 \leq M_1^2 M_2 = K;$$

- the matrix $\frac{1}{2}\alpha \, g(x)^t HV(\tilde{y}) g(x)$, has radius $r \leq \frac{1}{2}\alpha \, K$ (see, for instance, [9]).

We can conclude that if $0 < \alpha < \frac{1}{K}$ then

$$[-I_m + \frac{1}{2}\alpha \, g(x)^t HV(\tilde{y}) g(x)] < 0$$

as a quadratic form for all $x \in \mathbb{R}^n$, and the origin is stable for the closed-loop system. Moreover, taking into account of (17), (18) and (i), we remark that

$$\begin{aligned} E &= \{x \in \mathbb{R}^n \, / \, V(f(x) + g(x)u(x)) - V(x) = 0\} = \\ &= \{x \in \mathbb{R}^n \, / \, V(f(x)) = V(x)\} \cap \{x \in \mathbb{R}^n \, / \, \nabla V(f(x)) g(x) = 0\}. \end{aligned}$$

If $x_0 \in E$, the closed-loop solution starting by x_0 lies in E if and only if $V(f^{i+1}(x_0)) = V(f^i(x_0))$ and $\nabla V(f^{i+1}(x_0)) g(f^i(x_0)) = 0$. According to (ii), this happens only if $x_0 = 0$. We get asymptotic stability for the origin in the closed-loop system by LaSalle's theorem.
As long as *global* asymptotical stability is concerned, we refer to the global version of Second Liapunov Theorem in [9]. □

Remark 5.2.

(1) It is interesting to compare Theorem 5.1 with analogous results available in literature. In [11] the author assumes that the unforced system associate to (16) admits $V(x) = |x|^2$ as a Liapunov function. The form of the proposed feedback law is reminiscent of the form of classical Jurdjevic and Quinn's feedback law.

In [5], the function $V(f(x)+g(x)u)$ is assumed to be a quadratic polynomial in control; the form of the proposed feedback law seems not related to the classical Jurdjevic–Quinn one ([7]).

Finally, in [4] the authors consider a Liapunov function of class C^2, but the proposed feedback law is proposed in an implicit way.

Here we assume, as in [4], the existence of a C^2 Liapunov function for the unforced system associated to (16), but our feedback law is explicit and appears to be the natural extension of Jurdjevic and Quinn's one to the discrete-time case.

(2) We point out that, as in [4], the feedback *magnitude* can be made as small as desired by tuning the multiplicative constant α.

(3) If $V(x)$ is quadratic (as in [11] and [5]), then the assumption (iv) is automatically fulfilled.

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