Stabilization by Damping Control

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Abstract

In engineering practice, damping control is a usual method for stabilization. In this paper we consider nonlinear affine systems. We show that if there exists a feedback law which provides sufficiently fast convergence of trajectories (rational stabilization) then the system can also be stabilized by damping control. Our construction relies on a Liapunov function, which is basically defined as the value function of an optimal regulation problem.

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1 Introduction

In this paper we are interested in finite dimensional, time-invariant, affine systems of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{1}$$

where $x \in \mathbb{R}^n$, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$. The vector fields $f, g_1, \ldots, g_m$ are required to be of class C^1, and $f(0) = 0$. We say that the system is *stabilizable* if there exists a feedback law $u = k(x)$ which, when replaced in (1), renders the origin a globally, asymptotically stable equilibrium. In particular, we say that the system is *stabilizable by damping control* if there exists a feedback stabilizer of the form

$$u = k(x) = -\gamma(\nabla V(x) G(x))^{\mathbf{t}} \ . \tag{2}$$

where $G(x)$ is the matrix whose columns are $g_1(x), \ldots, g_m(x)$, γ is a positive constant and $V(x)$ is some positive definite, radially unbounded function which is continuous on the whole of $\mathbb{R}^n$, and of class C^1 possibly except at the origin. Most of requirements about $V(x)$ are motivated by its possible use as a Liapunov function for the closed loop system.

We address the following question: *find conditions which ensure that a stabilizable system also admits a stabilizer in damping form.*

It is well known that any stabilizable, linear system can actually be stabilized by damping control, as well. This remarkable fact is related to the existence of solutions of a quadratic regulation problem and the associated algebraic Riccati equation ([1], [5]). On the other hand, there exist systems which cannot be stabilized by damping control.

Example Let us consider the simple one dimensional equation

$$\dot{x} = x^3 + ux^2 \tag{3}$$

which can be stabilized for instance by the linear feedback $u = -2x$. Assume that there exists a stabilizer of the form $u = -\gamma V'(x)x^2$, $V(x)$ being a function with the required properties. For any x in a right neighborhood of the origin one must have $1 - \gamma x V'(x) < 0$, that is $V'(x) > (1/\gamma x)$. By the mean value theorem the last inequality implies $V(x) > (1/\gamma)$ and since $V(0) = 0$, it cannot be continuous at $x = 0$.

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As noticed in [12], the interest of control laws of the form (2) is related to certain robustness properties of the closed loop system. According to [12], an answer to the proposed question can be given by the following argument. If there exists a continuous stabilizing feedback $u = k(x)$ for (1), then Kurzweil's converse theorem provides a C^∞ Liapunov function $V(x)$ for the closed loop system. Of course, $V(x)$ is a control Liapunov function for (1). Then a stabilizing feedback of the form

$$u = k(x) = -p(x)(\nabla V(x)G(x))^{\mathbf{t}} \ . \tag{4}$$

can be constructed, for instance, by means of a suitable modification of Sontag's universal formula. The robustness properties of the closed loop system depend on the fact that (4) is at the same time the solution of an inverse optimal regulation problem. Note that the function $p(x)$ in (4) is not constant in general (so that (4) is not a damping control in our sense). In order to improve the robustness properties, in [13] the authors address the problem of redesigning $V(x)$ in such a way to replace (4) by a feedback of the same form, but with $p(x) = \gamma$ =constant. To this purpose, they give a sufficient condition which involves the homogeneous approximations of $f, g_1, \ldots, g_m$ and V with respect to a given dilation. We emphasize that in practice, in order to check this condition, one needs to know an explicit expression for $V(x)$, which in general is not easy.

A related, but different approach is pursued in Section 2 of the present paper. We assume that the system admits a special type of stabilizer $u = k(x)$, which provides in particular "rational" convergence of trajectories. This allows us to construct a function V which is proved to be of class C^1. Then we define the feedback according to (2), and we show that it asymptotically stabilizes the system by using V as a Liapunov function. Note that in this way, the preliminary knowledge of a control Liapunov function is avoided.

The idea is partially borrowed from [11], where it is exploited in order to identify the region of attraction of the closed loop system: to this purpose, the author reduces the Hamilton-Jacobi equation to Zubov's equation.

Of course, our assumption about rational stabilizability is restrictive, but it is needed in order to validate all the steps of the proof, and in fact it is implicit in [11]. We remark that it is fulfilled for certain classes of homogeneous systems. Section 3 contains two technical lemmas.

We point out that a somewhat opposite idea is pursued in [6], where the authors show that a function V which defines a damping control can be modified in order to obtain a control Liapunov function.

2 The generality of damping control

Let $u = k(x)$ be a global stabilizer of class C^1 for system (1). In what follows, we denote by $\varphi_{k(\cdot)}(t, x)$ the unique solution of the closed loop system

$$\dot{x} = f(x) + G(x)k(x) \tag{5}$$

such that $\varphi_{k(\cdot)}(0, x) = x$. Moreover, we denote by $\| \cdot \|$ both the norm of vectors and the induced norm of matrices.

Since $k(\cdot)$ is a global stabilizer, the flow $\varphi_{k(\cdot)}(t, x)$ is defined for each $t \geq 0$ and each $x \in \mathbb{R}^n$. It is well known that in certain circumstances, the decay of the trajectories of an asymptotically stable system can be estimated by an algebraic function of t. Let us introduce a definition. It can be considered an extension to input systems of the definition of rational stability given in [2], pp. 150-152.

Definition 1 *Given the system (1), a* global rational stabilizer *is a feedback $u = k(x)$ of class C^1, with $k(0) = 0$, which enjoyes the following properties.*

(i) there exist positive numbers M_1, c_1, p such that

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^n \qquad \|\varphi_{k(\cdot)}(t, x)\| \leq \frac{M_1 \|x\|}{(1 + c_1 \|x\|^p t)^{1/p}}$$

(ii) there exist positive numbers M_2, c_2 and $q \in \mathbb{R}$ such that

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^n \qquad \|\frac{\partial}{\partial x}\big(\varphi_{k(\cdot)}(t, x)\big)\| \leq M_2 (1 + c_2 \|x\|^p t)^q$$

(iii) there exist positive numbers M_3, c_3, μ and r, with $\mu > \frac{p}{2}$, $r > \max\{\frac{1}{2}, q + 1\}$, such that

$$\forall t \geq 0,\ \forall x \in \mathbb{R}^n \qquad \|k(\varphi_{k(\cdot)}(t, x))\| \leq \frac{M_3 \|x\|^\mu}{(1 + c_3 \|x\|^p t)^r}\ .$$

Remark 1 Conditions (i) and (ii) are guaranteed when the closed loop system (5) exhibits exponential decay (hence, in the case of linear systems) or, in more generality, when its right hand side is of class C^2 and homogeneous of degree $p + 1$ (see [7], [8], [14], [2]). In force of (i), condition (iii) is fulfilled if the feedback $u = k(x)$ is a homogeneous function of degree $\mu > p(q + 1)$. The example (3) with the feedback $u = -2x$ does not fulfill condition (iii).

Theorem 1 *Consider an affine system of the form (1), satisfying the following assumptions.*

($\mathbf{A_1}$) *There exists a globally, rationally stabilizing feedback $u = k(x)$.*

($\mathbf{A_2}$) *There exist positive constants A, C, R, α, b such that*

(I) $\|f(x)\| \le A\|x\|^\alpha + C$ *for* $\|x\| > R$,

(II) $\|G(x)\| \le b$.

Then, there exists a map $V(x)$ such that the system can be also stabilized by a damping control of the form (2), with $\gamma = 1/2$.

Proof. Let us define

$$L(x) = \frac{\|x\|^{2l} + \|k(x)\|^2}{2}$$

where $l > \max\{\frac{\alpha}{2}, \frac{p}{2}, \frac{(pq+p+1)}{2}\}$. First of all, we observe that

$$V(x) = \int_0^{+\infty} L(\varphi_{k(\cdot)}(t;x))\,dt \tag{6}$$

is well defined for each $x \in \mathbb{R}^n$. Indeed, it is clear that $V(0) = 0$. For $x \ne 0$, the integral

$$\int_0^{+\infty} \|\varphi_{k(\cdot)}(t;x)\|^{2l}\,dt \tag{7}$$

converges by virtue of the assumption $2l > p$, while the integral

$$\int_0^{+\infty} \|k(\varphi_{k(\cdot)}(t;x_0))\|^2\,dt \tag{8}$$

converges by virtue of the assumption $r > 1/2$. It is also clear that $V(x) > 0$ for $x \ne 0$.

In order to prove that $V(x)$ is of class C^1 for $x \ne 0$, we consider the integral

$$\int_0^{+\infty} \frac{\partial}{\partial x}\big(L(\varphi_{k(\cdot)}(t,x))\big)\,dt\ . \tag{9}$$

We have

$$\begin{aligned}\frac{\partial}{\partial x}\big(L(\varphi_{k(\cdot)}(t,x))\big) \quad &= \quad l\|\varphi_{k(\cdot)}(t;x)\|^{2l-2}\left[\varphi_{k(\cdot)}(t,x)\right]^{\mathbf{t}} \frac{\partial}{\partial x}\big(\varphi_{k(\cdot)}(t,x)\big) + \\ &\quad\quad \left[k(\varphi_{k(\cdot)}(t,x))\right]^{\mathbf{t}} (Dk)(\varphi_{k(\cdot)}(t,x))\frac{\partial}{\partial x}\big(\varphi_{k(\cdot)}(t,x)\big)\ .\end{aligned}$$

Thus

$$\|\frac{\partial}{\partial x}\big(L(\varphi_{k(\cdot)}(t,x))\big)\| \quad \leq \quad \Bigg[l\|\varphi_{k(\cdot)}(t;x)\|^{2l-1} + \\ \|(Dk)(\varphi_{k(\cdot)}(t,x))\| \cdot \|k(\varphi_{k(\cdot)}(t,x))\| \Bigg] \cdot \|\frac{\partial}{\partial x}\big(\varphi_{k(\cdot)}(t,x)\big)\| \ .$$

Let $\mathcal{C}$ be any compact subset of $\mathbb{R}^n$, not containing the origin. Since the closed loop system (5) is globally asymptotically stable, the solutions are uniformly bounded. Hence, there exists $R > 0$ such that $\|\varphi_{k(\cdot)}(t;x)\| \leq R$ for each $x \in \mathcal{C}$ and each $t \geq 0$. Since k is of class C^1, (Dk) is bounded along the solutions issuing from $\mathcal{C}$. Using (i), (ii), (iii), we therefore conclude that for $x \in \mathcal{C}$,

$$\|\frac{\partial}{\partial x}\big(L(\varphi_{k(\cdot)}(t,x))\big)\| \leq (1 + c_2\|x\|^p t)^q \left(\frac{N_1}{(1 + c_1\|x\|^p t)^{(2l-1)/p}} + \frac{N_2}{(1 + c_3\|x\|^p t)^r} \right) \tag{10}$$

where N_1, N_2 are some new constants. By assumption, $r - q > 1$ and l has been choosen in such a way that $\frac{2l-1}{p} - q > 1$. Hence, (9) converges uniformly for $x \in \mathcal{C}$. The conclusion follows according to well known results of classical calculus.

We have still to prove that $V(x)$ is continuous at the origin. To this purpose, we estimate (7) and (8) using the inequalities (i) and (iii). Then the integrals can be explicitly computed. Taking into account that $2l > p$ and $2\mu > p$, we see that $V(x)$ is indeed continuous at the origin.

As far as radially unboundedness is concerned, we remark that by virtue of $(\mathbf{A_2})$, Lemma 2 is applicable, if we specify for each x the input $u_x(t) = k(\varphi_{k(\cdot)}(t,x))$.

We are now ready to get the conclusion. By a direct computation we obtain

$$\frac{d}{dt}V(\varphi_{k(\cdot)}(t,x))|_{t=0} = \nabla V(x)[f(x) + G(x)k(x)] = -L(x) \tag{11}$$

for each $x \in \mathbb{R}^n$, $x \neq 0$. Let

$$h(x) = \|x\|^{2l} + (k(x) + (\nabla V(x)G(x))^{\mathbf{t}})^2 \ .$$

From (11) we see that

$$\nabla V(x)f(x) - \frac{1}{2}\|\nabla V(x)G(x)\|^2 = -\frac{h(x)}{2} \ . \tag{12}$$

Consider the affine system (1) with the feedback law $u = -\frac{1}{2}(\nabla V(x)G(x))^{\mathbf{t}}$. The left hand side of (12) coincides with the derivative of V with respect to the closed loop system. Hence, (1) with the feedback law $u = -\frac{1}{2}(\nabla V(x)G(x))^{\mathbf{t}}$ admits V as a global, strict Liapunov function. This implies that the origin is globally asymptotically stable for the closed loop system and the statement is proved.

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Remark 2 In the proof of Theorem 1, the function V is basically defined as the value function of an inverse optimal regulation problem. Equation (12) is nothing else but the associated Hamilton-Jacobi equation.

Remark 3 Assumption ($\mathbf{A_2}$) is restrictive, but it is needed only in order to show that the Liapunov function is radially unbounded. If one is just interested in local results, it can be dropped out.

Example Let us consider the scalar system

$$\dot{x} = u|x|^{7/4} \ . \tag{13}$$

The feedback $u = -\mathrm{sgn}\, x|x|^{5/4}$ stabilizes. The requirements of Definition 1 are fulfilled with $p = 2$, $q = -3/2$, $\mu = 5/4$, $r = 5/8$. Hence, it is a rational stabilizer. The assumptions of Theorem 1 are locally satisfied. According to (6), with the choice $l = 5/4$ we compute $V(x) = 2|x|^{1/2}$. Thus, we see that the given feedback is in damping form itself. As a matter of fact, for simple systems of the form $\dot{x} = u|x|^{\beta}$ it is easy to check that any feedback $u = -\mathrm{sgn}\, x|x|^{\alpha}$ is a rational stabilizer if and only if it is in damping form.

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3 Technical lemmas

This section contains two lemmas. Lemma 1 is needed in the proof of Lemma 2, while Lemma 2 is recalled in the proof of Theorem 1. We do not report the proof of Lemma 1, which is long and technical: it can be found in [3]. In what follows, by *admissible input* we mean any measurable, locally bounded function $u(t) : [0, +\infty) \to \mathbb{R}^m$. Moreover, we denote by $\varphi(t; x, u(\cdot))$ the (unique) solution of (1) corresponding to the initial state x and the admissible input $u(\cdot)$.

Lemma 1 *Let $x \in \mathbb{R}^n$ and $\beta > 0$ be fixed. Assume that for some admissible input $u(t)$*

$$\int_0^{+\infty} \frac{||\varphi(t; x, u(\cdot))||^{\beta} + ||u(t)||^2}{2} \, dt < \infty \ .$$

Then,

$$\lim_{t \to +\infty} \varphi(t; x, u(\cdot)) = 0 \ .$$

Lemma 2 *Assume that there exist positive numbers A, C, b, R, α such that*

(I) $\|f(x)\| \leq A\|x\|^{\alpha} + C$ for $\|x\| > R$,

(II) $\|G(x)\| \leq b$.

Let $\beta \geq \alpha$. Assume further that for each $x \in \mathbb{R}^n$, there exists an admissible input $u_x(t)$ such that

$$V(x) = \int_0^{+\infty} \frac{||\varphi(t; x, u_x(\cdot))||^\beta + ||u_x(t)||^2}{2}\, dt < \infty \ .$$

Then, the function $V(x)$ is radially unbounded.

Proof.

Let us assume that the conclusion is false. Then, there exists $M > 0$ such that for each $K > 0$ we can find a point x_K with $\|x_K\| > K$ for which

$$V(x_K) \leq M \ . \tag{14}$$

Without loss of generality, we can assume $R > 1$ and $M > R^\beta$. Let us take $K > 2C + \sqrt[\beta]{M} + 2AM + 2b\sqrt{M}$. We remark that

$$\|x_K\|^\beta > K^\beta > M \ . \tag{15}$$

Let us set for simplicity $\varphi_K(t) = \varphi(t; x_K, u_{x_K}(\cdot))$. Using Lemma 1, we have that $\varphi_K(t) \to 0$ for $t \to +\infty$. Let $T_K > 0$ be such that $\|\varphi_K(T_K)\|^\beta = M$ while

$$\|\varphi_K(t)\|^\beta \geq M \qquad \forall t \in [0, T_K] \ . \tag{16}$$

This is the same thing as

$$\|\varphi_K(t)\| \geq \sqrt[\beta]{M} > R > 1 \qquad \forall t \in [0, T_K] \ . \tag{17}$$

On the other hand we have

$$M \geq V(x_K) = \frac{1}{2}\int_0^{+\infty} \left(||\varphi_K(t)||^\beta + \|u_{x_K}(t)\|^2\right) dt \geq \frac{1}{2}\int_0^{+\infty} ||\varphi_K(t)||^\beta \, dt \geq \frac{1}{2} M T_K$$

from which we deduce that $T_K \leq 2$. By virtue of (I) and (II), we also have

$$\begin{aligned} K - \sqrt[\beta]{M} &\leq \|x_K\| - \|\varphi_K(T_K)\| \\ &\leq \|\varphi_K(T_K) - x_K\| \\ &\leq \int_0^{T_K} (\|f(\varphi_K(t))\| + \|G(\varphi_K(t))\| \cdot \|u_{x_K}(t)\|)\, dt \\ &\leq \int_0^{T_K} (A \cdot \|\varphi_K(t)\|^\alpha + C)\, dt + D\int_0^{T_K} \|u_{x_K}(t)\|\, dt \ . \end{aligned}$$

Using Hölder inequality, and recalling that $R > 1$, we obtain therefore

$$K - \sqrt[\beta]{M} \quad \leq \quad A\int_0^{T_K} \|\varphi_K(t)\|^\beta\, dt + CT_K + b\sqrt{T_K}\left(\int_0^{T_K} \|u_{x_K}(t)\|^2\, dt\right)^{\frac{1}{2}} \ .$$

By invoking again (14) and the fact that $T_K \leq 2$, we conclude that

$$K - \sqrt[\beta]{M} \leq 2AM + T_K C + 2b\sqrt{M} \ . \tag{18}$$

Finally, we get

$$T_K \geq \frac{K - \sqrt[\beta]{M} - 2AM - 2b\sqrt{M}}{C} > 2 \ ,$$

a contradiction.

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In particular, Lemma 2 applies to the value function of a solvable optimal regulation problem associated to system (1). In this sense, it may have an independent interest.

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