A discussion about stabilizing periodic and near-periodic switching signals

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Abstract: A rather natural way to address the stabilizability problem for switched systems is to make use of periodic switching laws. The success is guaranteed, provided that the discrete dynamical system associated to the periodic law is stable.
In this paper we discuss a more general type of stabilizing switching laws, called near-periodic, which can be implemented when the associated discrete dynamical system has a non-trivial stable submanifold, and a suitable controllability condition is fulfilled.

Keywords: Switched systems, stabilizability, time-dependent switching rules.

1. INTRODUCTION

The term *switched system* is often used in the engineering literature to denote a collection of dynamical components which are activated, one at a time, according to a certain rule to be specified separately. In this way, for each initial state a trajectory is generated.

The so-called *stabilization problem* for switched systems consists of determining the switching rule (possibly dependent on the initial state) in such a way that all the corresponding trajectories converge to the origin. The most common stabilizing switching rules studied in the literature are state-dependent (Liberzon (2003), Peleties and deCarlo (1991), Petterson and Lennarston (2001)). However, state-dependent rules lead to dynamical equations with discontinuous right-hand-side: this causes some mathematical problems about existence and continuability of solutions (Ceragioli (2006)). Moreover, even if the solutions exist, it is not straightforward to recognize whether they can be actually reproduced by a time-dependent switching signal. On the other hand, the existence of stabilizing time-dependent switching rules with special structure (periodic with very short or very large period) has been directly investigated in Sun and Ge (2005), Morse (1996), Bacciotti and Mazzi (2010).

In the present paper, we discuss a method which enables us to construct time-dependent stabilizing switching rules in a systematic way. The starting point of the method is the remark that any periodic switching rule can be associated to a discrete time dynamical system. Even if the periodic switching signal does not generate trajectories converging to the origin for each initial state, it often has a non-trivial stable manifold. This manifold can be clearly exploited in order to produce trajectories converging to the origin, provided that a suitable controllability condition is satisfied. The switching laws obtained by this method are basically time-dependent, but since the switching times arise when the trajectories intersect a fixed geometric object, they can be also reviewed as state dependent.

The basic definitions and notions needed in this paper are exposed in Section 2. In Section 3 we discuss the crucial notion of stabilizing periodic switching signal. The main section is Section 4, where we introduce the new notion of stabilizing near-periodic switching signal together with its main ingredients: the stable submanifold of the associated discrete dynamical system and the controllability condition. We also indicate how to ensure these conditions.

For the sake of simplicity (and coherence with the examples), we limit ourselves to systems whose components are linear. However, similar ideas can be applied also in the case of nonlinear components.

2. PRELIMINARY DEFINITIONS AND REMARKS

Let $N \geq 2$ be a fixed integer, and let $\mathcal{N} = \{1, \dots, N\}$ be equipped with the discrete topology. Let $\mathcal{U}_{\mathcal{N}}$ be the set of all the *switching signals*, that is all the right continuous, piecewise constant maps $\sigma : [0, +\infty) \to \mathcal{N}$. The discontinuity points of a switching signal σ form a finite or infinite (possibly empty) subset of the open half line $(0, +\infty)$ and are called the *switching times* of σ. We denote by I_σ the set whose elements are $t_0 = 0$ and all the switching times of σ, indexed in such a way that $0 = t_0 < t_1 < t_2 < \dots$. If the set of the switching times is infinite, then clearly $\lim_{i \to +\infty} t_i = +\infty$. If it is finite and $\max t_i = t_{i_*}$, then we agree to set $t_{i_*+1} = +\infty$.

Throughout this paper, we use $\mathcal{F}$ to denote a family of $d \times d$ real matrices $\{A_n\}_{n \in \mathcal{N}}$, where $d \geq 1$ is a fixed integer. A *linear switched system* in $\mathbf{R}^d$ with index set $\mathcal{N}$ is defined by $\mathcal{F}$, together with a map $\Sigma : \mathbf{R}^d \to \mathcal{U}_{\mathcal{N}}$ which assigns a switching signal $\sigma(t) = \Sigma_{x_0}(t)$ to each point $x_0 \in \mathbf{R}^d$, regarded as initial state. A linear switched system will be denoted by $(\mathcal{F}, \Sigma)$. A switched system for which Σ is constant i.e., the same switching signal $\sigma(t)$ is applied for each initial state x_0, will be simply denoted by $(\mathcal{F}, \sigma)$.

For each matrix A_n the system of linear differential equations in $\mathbf{R}^d$

$$\dot{x} = A_n x \tag{1}$$

is called the n-th component of $\mathcal{F}$. As well known, for each initial state $x_0 \in \mathbf{R}^d$ there is a unique differentiable curve defined by (1): it is called the *trajectory* of (1) issued from x_0 and it is represented by $x = e^{tA_n} x_0$ $(t \in \mathbf{R})$. Analogously, for each switched system $(\mathcal{F}, \Sigma)$ and each x_0 there exists a unique continuous curve $\chi(t, x_0) : [0, +\infty) \to \mathbf{R}^d$ satisfying the condition $\chi(0, x_0) = x_0$, and such that

$$\chi(t, x_0) = e^{(t-t_i)A_{\sigma(t_i)}} \chi(t_i, x_0) \ , \quad \forall t \in [t_i, t_{i+1}) \ , \ \ \forall t_i \in I_\sigma$$

where $\sigma(t) = \Sigma_{x_0}(t)$. We say that $\chi(t, x_0)$ is the *switched trajectory* of $(\mathcal{F}, \Sigma)$, issued from the initial state x_0.

It is interesting to characterize those systems $(\mathcal{F}, \Sigma)$ such that for each initial state x_0 one has

$$\lim_{t \to +\infty} \chi(t, x_0) = 0 \ . \tag{2}$$

As well known, this problem is not trivial, since even if all the matrices A_n are Hurwitz (i.e., have all their eigenvalues in the open left complex plane) it may happen that for certain switching signals and certain initial states the corresponding trajectory does not converge to the origin (Liberzon (2003); Sun and Ge (2005)). The following definitions are taken from Sun and Ge (2005).

Definition 1. The family of matrices $\mathcal{F}$ is said to be *point-wise stabilizable* if there exists a map Σ such that for each $x_0 \in \mathbf{R}^d$ the corresponding switched trajectory of $(\mathcal{F}, \Sigma)$ satisfies (2).

Definition 2. The family of matrices $\mathcal{F}$ is said to be *consistently stabilizable* if there exists a switched signal $\sigma(t)$ such that for each $x_0 \in \mathbf{R}^d$ the corresponding switched trajectory of $(\mathcal{F}, \sigma)$ satisfies (2).

In other words, point-wise stabilizability means that all the initial states can be eventually driven toward the origin, but different switching rules might be required for different initial states. On the contrary, consistent stabilizability means that the same switching rule works for all the initial states.

Clearly, if $\mathcal{F}$ is consistently stabilizable then it is also point-wise stabilizable, but the converse is false in general: this is shown by an example in (Sun and Ge, 2005) (p. 58), and also by other examples presented later in this paper.

To construct (consistently) stabilizing switching signals is trivial, if there exists at least one $n \in \mathcal{N}$ such that A_n is Hurwitz. In this case one can take $\sigma(t) \equiv n$. However, there may be situations where switching among two or more components is compulsory. We are especially interested in switching rules whose structure is partially fixed: for instance, it seems natural to assume that the activation of the various components obeys to a pre-assigned sequence, while other details, such as durations (i.e., the times elapsed between two consecutive switching) are available for design. In such a situation, a bad choice of the durations can lead to instability, even if some or all the matrices are Hurwitz. In the next section, we focus on those cases where the choice of the switching signal is further restrained to have a periodic structure.

3. PERIODIC SWITCHING

A *periodic switched system* is a pair $(\mathcal{F}, \sigma)$ such that σ is periodic for $\mathcal{F}$. More precisely, a switching signal $\sigma(t)$ is said to be *periodic* (of period T) for $\mathcal{F}$ if there exists a string of real numbers $t_0, \ldots, t_H$ (where H is an integer, $H \geq 1$) and a string of indices $n_1, \ldots, n_H \in \mathcal{N}$ such that:

- $0 = t_0 < t_1 < \ldots < t_H = T$;
- $\sigma(t) = n_h$ for $t \in [t_{h-1}, t_h)$, for each $h = 1, \ldots, H$;
- $\sigma(t) = \sigma(t - mT)$ for $t \in [mT, (m+1)T)$, $m = 1, 2, \ldots$.

The points $t_n + mT$, with $n \in \mathcal{N}, m = 0, 1, \ldots$ coincide with the switching times, provided that $n_1 \neq n_2, n_2 \neq n_3 \ldots n_H \neq n_0$. The durations will be denoted by $\tau_h = t_h - t_{h-1}$ $(h = 1, \ldots, H)$. Note that σ is constant when $H = 1$.

Definition 3. The family of matrices $\mathcal{F}$ is said to be *periodically stabilizable* if there exists a switching signal $\sigma(t)$ such that σ is periodic for $\mathcal{F}$ and for each $x_0 \in \mathbf{R}^d$ the corresponding switched trajectory of $(\mathcal{F}, \sigma)$ satisfies (2).

Remark 1. Of course, if the family of matrices $\mathcal{F}$ is periodically stabilizable, then it is consistently stabilizable. In Sun and Ge (2005) it is proven that these two properties are actually equivalent.

The existence of high frequency periodic stabilizing switching signals has been proven in Tokarzewski (1987) (see also Sun and Ge, 2005), under the assumption that there exists a convex combination of $A_1, \ldots, A_N$ which is Hurwitz[1]. The existence of periodic stabilizing switching signals with large durations can be deduced from Lemma 2 of Morse, 1996 provided that all the matrices A_n are Hurwitz. The case of a pair of planar oscillators has been studied in Bacciotti (2008).

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The (linear) discrete dynamical system associated to a periodic switched system $(\mathcal{F}, \sigma)$ is

$$x_{k+1} = \Phi_\sigma x_k \qquad k = 0, 1, 2, \ldots \tag{3}$$

where $\Phi_\sigma = e^{\tau_H A_{n_H}} \ldots e^{\tau_1 A_{n_1}}$. When the periodic signal σ is clear from the context, we simply write Φ instead of Φ_σ. Note also that $\Phi x = \chi(T, x)$.

If the origin is an asymptotically stable fixed point for the associated discrete dynamical system (3), then $\mathcal{F}$ is periodically stabilized by $\sigma(t)$ (Tokarzewski (1987), Sun and Ge (2005)). As well known, the origin is asymptotically stable for (3) if and only if all the eigenvalues of Φ lie in the interior of the unit disc of the complex plane. If the number of eigenvalues of Φ meeting this condition is $\nu < d$ (counting multiplicity), then the initial states asymptotically driven to the origin along the solutions of (3) form a ν-dimensional invariant subspace of $\mathbf{R}^d$, called the *stable subspace*.

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[1] This assumption was originally introduced in Peleties and deCarlo (1991) to prove the existence of state dependent stabilizing switching rules.

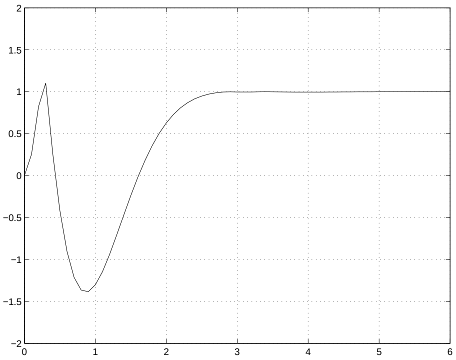

Fig. 1. Graph of the function (6)

Remark 2. Of course, the investigation of the stability properties of (3) is easier when all the matrices $A_1, \dots, A_N$ commute (Liberzon (2003)). In this paper we do not make this assumption.

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To illustrate the notions introduced so far and the approach based on the associated discrete dynamical system (3), we revisit the example of a pair of linear systems, both with sink configuration. Although the same example can be found elsewhere, we point out many new aspects and details.

Example 1. Consider in $\mathbf{R}^2$ the family $\mathcal{F}$ formed by the matrices

$$A_1 = \begin{pmatrix} -1 & -5 \\ \frac{1}{5} & -1 \end{pmatrix} , \qquad A_2 = \begin{pmatrix} -1 & -\frac{1}{5} \\ 5 & -1 \end{pmatrix} .$$

A_1 and A_2 are both Hurwitz. Let $\tau > 0$ be given. For sake of simplicity, we limit ourselves to periodic switching laws of period 2τ such that $\sigma(t) = 1$ for $t \in [0, \tau)$, and $\sigma(t) = 2$ for $t \in [\tau, 2\tau)$. We have:

$$e^{\tau A_1} = \begin{pmatrix} \cos\tau & -5\sin\tau \\ \frac{1}{5}\sin\tau & \cos\tau \end{pmatrix} e^{-\tau},$$

$$e^{\tau A_2} = \begin{pmatrix} \cos\tau & -\frac{1}{5}\sin\tau \\ 5\sin\tau & \cos\tau \end{pmatrix} e^{-\tau}$$

and

$$e^{\tau A_2} e^{\tau A_1} = \begin{pmatrix} 1 - \frac{26}{25}\sin^2\tau & -\frac{26}{2}\sin\tau\cos\tau \\ \\ \frac{26}{2}\sin\tau\cos\tau & 1 - 26\sin^2\tau \end{pmatrix} e^{-2\tau} \qquad (4)$$

The characteristic equation of (4) is:

$$\lambda^2 - e^{-2\tau}\left(2 - \frac{676}{25}\sin^2\tau\right)\lambda + e^{-4\tau} = 0. \qquad (5)$$

By Schur-Cohn Lemma (LaSalle (1986)), at least one eigenvalue lies in the interior of the unit circle. The other one lies in the unit circle if and only if

$$f(\tau) = 1 - e^{-2\tau}\left|2 - \frac{676}{25}\sin^2\tau\right| + e^{-4\tau} > 0 . \qquad (6)$$

The graph of $f(\tau)$ is plotted in Figure 1. In particular, we see that there exist $\theta_1 \in (0, \pi/2)$ and $\theta_2 \in (\pi/2, \pi)$ such that:

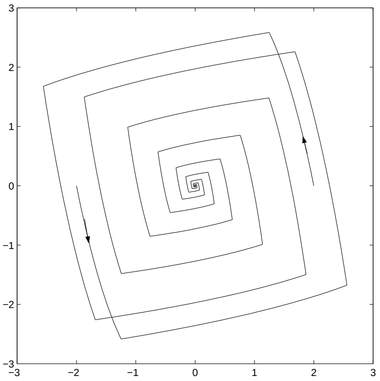

Fig. 2. Example 1 with $t \approx \pi/8$: the origin is asymptotically stable

- $\forall \tau \in (0, \theta_1)$, $\mathcal{F}$ is periodically stabilized by σ (see Figure 2);
- $\forall \tau > \theta_2$, $\mathcal{F}$ is periodically stabilized by σ;
- $\forall \tau \in (\theta_1, \theta_2)$, the switched system $(\mathcal{F}, \sigma)$ is unstable.

We point out that in the third case, the periodic switching signal σ can be exploited in order to construct a point-wise stabilizer, according to the following strategy: first use a different switching signal (depending on the initial state) in order to reach the stable subspace of the associated discrete dynamical system, then use σ, trying to correct possible round-off errors in order to maintain the evolution on the stable subspace as far as possible.

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The previous example shows in particular that the stability properties of a periodic switched system may depend on the value of the period T. The fact that the system of Example 1 admits stabilizing periodic switching signals for both small and large values of τ agrees with the results of Tokarzewski (1987), Sun and Ge (2005), Morse (1996), Bacciotti and Mazzi (2010) already quoted in Remark 1. Note that it is not possible to completely destabilize the family of the Example 1 by means of a periodic switching signal.

The reversed time version of Example 1 is interesting, as well: it suggests a method for point-wise stabilization which exploits the stable subspace (if any) of the discrete time system associated to a suitable periodic switching signal. This idea is formalized in the next section.

4. NEAR-PERIODIC SWITCHING

As already mentioned, there exist families of matrices not consistently stabilizable (and hence, not periodically stabilizable) that can be stabilized by applying a different switching signal for each initial state.

We are especially interested in switching signals of the type specified in the following definition.

Definition 4. A map $\Sigma : \mathbf{R}^d \to \mathcal{U}_N$ is said to be *near-periodic* (of period T) if there exist a periodic switching signal $\sigma(t)$ (of period T) and a map $T_0(x) : \mathbf{R}^d \to [0, +\infty)$ such that for each $x_0 \in \mathbf{R}^d$,

$$\Sigma_{x_0}(t) = \sigma(t - T_0(x_0)) \qquad \forall t \geq T_0(x_0) .$$

The switched system $(\mathcal{F}, \Sigma)$ is said to be near-periodic if the map Σ is near-periodic.

In other words, a switched system is near-periodic if it behaves as a periodic one after an initial transient interval (depending on the initial state), where the switching rule may obey to a different law.

Definition 5. We say that $\mathcal{F}$ is *near-periodically stabilizable* if there exists a near-periodic map Σ such that for each $x_0 \in \mathbf{R}^d$, the corresponding switched trajectory of $(\mathcal{F}, \Sigma)$ satisfies (2).

As already noticed, the system of Example 1 is near-periodically stabilizable in the interval (θ_1, θ_2) where periodic stabilizability fails. The construction of stabilizing near-periodic switching signals can be easily achieved provided that $\mathcal{F}$ possesses the following two properties:

(P1) there exists a periodic switching signal such that the associated discrete dynamical system (3) has a nontrivial stable subspace W_s;
(P2) for each $x_0 \in \mathbf{R}^d$ $(x_0 \neq 0)$, there exists a switching signal (depending on x_0) such that the corresponding switched trajectory of $\mathcal{F}$ issuing from x_0 intersects W_s in finite time.

It is therefore interesting to find conditions ensuring the validity of (P1) and (P2).

4.1 Existence of stable subspaces

The following Proposition provides a sufficient condition for (P1).

Proposition 1. Let $\mathcal{F} = \{A_n\}_{n \in \mathcal{N}}$ be a family of $d \times d$ real matrices. Assume that for some $\alpha_1, \ldots, \alpha_N$ ($\alpha_n > 0$, $\sum_{n=1}^N \alpha_n = 1$) the matrix $\bar{A} = \sum_n \alpha_n A_n$ has at least one eigenvalue with negative real part. Then Property (P1) is met.

Proof Let $\tau_n = \alpha_n T$, for each $n \in \mathcal{N}$ and some $T > 0$. Let Φ be the matrix of the associated discrete system. For sufficiently small T, there exists a matrix $C(T)$ such that

$$\Phi = e^{C(T)} .$$

Such a matrix $C(T)$ can be represented by the Baker-Campbell-Hausdorff expansion (Varadarajan (1974))

$$C(T) = (\sum_n \alpha_n A_n)T + G(T)T^2$$

where $G(T)$ is bounded. Recall that the eigenvalues depend continuously on the elements of a matrix. By taking a small enough T and using the assumption that $\sum_n \alpha_n A_n$ has at least one eigenvalue with negative real part, we arrive to the conclusion that $C(T)$ has at least one eigenvalue with negative real part, as well. The statement easily follows.

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Remark 3. As already recalled, the much stronger assumption that for some choice of $\alpha_1, \ldots, \alpha_N$ the matrix $\bar{A}$ is Hurwitz, has been used in Peleties and deCarlo (1991) in order to construct state dependent switching rules (not convertible in a time dependent one, in general), and in Tokarzewski (1987) (see also Sun and Ge (2005)) in order to construct high frequency periodic stabilizers.

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The assumption of Proposition 1 is fulfilled in particular if for some index $\bar{n}$, the matrix $A_{\bar{n}}$ has at least one eigenvalue with negative real part. Indeed, we can take a convex combination with $\alpha_n << 1$ for each $n \neq \bar{n}$, so that $\sum_n \alpha_n A_n$ can be viewed as a small perturbation of $A_{\bar{n}}$. On the other hand, Property (P1) may be valid even if all the matrices A_n and their convex combinations have all their eigenvalues in the open right half plane: this happens for instance in the reversed time version of Example 1.

It can be proven that if the matrices $A_1, \ldots, A_N$ are symmetric, then the existence of an index $\bar{n}$ such that the matrix $A_{\bar{n}}$ has at least one eigenvalue with negative real part is a necessary condition for point-wise stabilization (Sun and Ge, 2005). Combining these observations, we obtain the following result.

Corollary 1. Let $\mathcal{F} = \{A_n\}_{n \in \mathcal{N}}$ be a family of $d \times d$ symmetric real matrices. If $\mathcal{F}$ is point-wise stabilizable, then Property (P1) is met.

The assumption of Corollary 1 can be slightly relaxed, by asking that for each $A \in \mathcal{F}$, $A^{\mathbf{t}} \in \mathcal{F}$. In fact, we conjecture that at least for pairs of matrices $\{A_1, A_2\}$ of $\mathbf{R}^2$, point-wise stabilizability implies Property (P1) without need of any additional assumption.

Proposition 2. Assume that all the matrices $A_1, \ldots, A_N$ are Hurwitz. Then, for each periodic switching signal, Property (P1) is met.

Proposition 2 implies that, as we already argued from Example 1, a family of Hurwitz matrices cannot be completely destabilized by applying a periodic switching signal. We also see that the choice $\tau_1 = \tau_2$ is not restrictive, in Example 1.

The conclusion of Proposition 2 can be proved to be valid under the following slightly weaker assumption: for each $n = 1, \ldots, N$, $\operatorname{tr} A_n < 0$ (where tr denotes the trace of the matrix A_n).

Proof Recall that for each $t \in \mathbf{R}$ and each square matrix A, $\det e^{tA} = e^{\operatorname{tr}(tA)} = e^{t(\operatorname{tr} A)}$. If $\operatorname{tr} A_n < 0$, then

$$0 < \det e^{tA_n} < 1$$

for each t and each $n \in \mathcal{N}$. Hence,

$$\det \Phi = \det e^{\tau_N A_N} \ldots e^{\tau_1 A_1} = \det e^{\tau_N A_N} \cdot \ldots \cdot \det e^{\tau_1 A_1} < 1 .$$

On the other hand, $\det \Phi = \lambda_1 \cdot \ldots \cdot \lambda_d$, where $\lambda_1, \ldots, \lambda_d$ are the (non necessarily distinct) eigenvalues of Φ. It follows that at least one eigenvalue of Φ lies in the interior of the unit disc of the complex plane.

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4.2 Controllability

Of course, Property (P1) is not sufficient to guarantee that $\mathcal{F}$ is near-periodically stabilizable.

The simplest way to check Property (P2) is to look at the family $\mathcal{F}^\circ$ formed by the (nonlinear) vector fields

$$f_n(p) = A_n p - \left(\frac{p^{\mathbf{t}} A_n p}{p^{\mathbf{t}} p}\right) p \ , \qquad n = 1, \ldots, N \qquad (7)$$

where $p \in \mathbf{R}^d$, and ${}^{\mathbf{t}}$ denotes transposition. For each n, the expression which defines the vector field (7) is homogeneous of degree one. It is straightforward to verify that $f_n(p)$ is tangent to any sphere $\mathcal{B}_r = \{p^{\mathbf{t}} p = r\}$ $(r > 0)$. In fact, $f_n(\bar{p})$ is the projection of $A_n \bar{p}$ on the tangent space to the sphere $\mathcal{B}_{\bar{r}}$ with $\bar{r} = \bar{p}^{\mathbf{t}} \bar{p}$, at the point $\bar{p}$.

By virtue of homogeneity, we can limit ourselves to $r = 1$. Assume that for some fixed periodic switching signal Property (P1) holds. Let $W_s^\circ = W_s \cap \mathcal{B}_1$ and let $\mathcal{R}^\circ(p_0) \subset \mathcal{B}_1$ be the set of points reachable in finite time from $p_0 \in \mathcal{B}_1$, along switched trajectories of $\mathcal{F}^\circ$. Clearly, Property (P2) will be ensured if for each $p_0 \in \mathcal{B}_1$ one has $\mathcal{R}^\circ(p_0) \cap W_s^\circ \neq \emptyset$. This is true in particular if $\mathcal{F}^\circ$ is completely controllable on $\mathcal{B}_1$.

Thus, the problem is reduced to test controllability of a family of vector fields on a $(d-1)$-dimensional manifold. To this purpose, one needs to compute Lie brackets and to look for some additional recurrence condition: a very general result of this type can be found for instance in Jurdjevic (1997), p.78.

Example 2. Consider the family $\mathcal{F}$ with $d = 2$, whose members are the matrices

$$A_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ , \qquad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ .$$

The component defined by A_1 has a saddle point, the component defined by A_2 has an improper node. This family is point-wise stabilizable (apart from a different choice of coordinates and coefficients, this example has been considered in Bacciotti (2004).

Stabilizing near-periodic maps Σ can be constructed according to the method described above. Indeed, a periodic switching signal such that the associated discrete dynamical system has a non-trivial stable subspace exists, by virtue of Proposition 1. The components of the family $\mathcal{F}^\circ$ are readily computed:

$$f_1(x, y) = \begin{pmatrix} \dfrac{y(x^2 - y^2)}{x^2 + y^2} \\ -\dfrac{x(x^2 - y^2)}{x^2 + y^2} \end{pmatrix} \ ,$$

$$f_2(x, y) = \begin{pmatrix} \dfrac{y(2x^2 + y^2)}{x^2 + y^2} \\ \dfrac{xy^2}{x^2 + y^2} \end{pmatrix} \ .$$

The vector field f_1 has four equilibria on $\mathcal{B}_1$, corresponding to $(\pm\sqrt{2}, \pm\sqrt{2})$. The vector field f_2 has two equilibria, corresponding to $(0, \pm 1)$. Figure 3 shows the phase portrait of f_1 and f_2 on $\mathcal{B}_1$. By direct inspection, it is clear that $\mathcal{F}^\circ$ is completely controllable.

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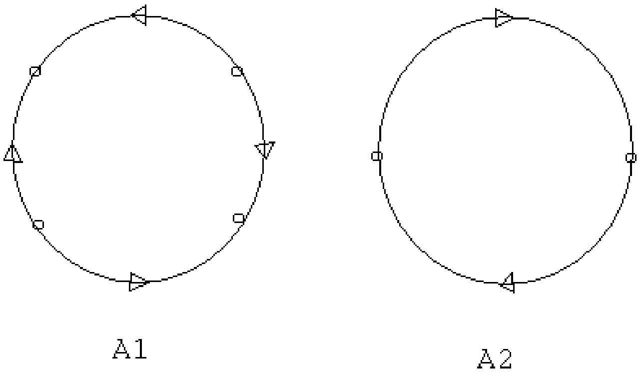

Fig. 3. Phase portrait of the projection on the unit sphere of the linear systems of Example 2.

5. FINAL REMARKS

In this paper we propose a method which can be used to design in a systematic way switching signals whose corresponding trajectories converge to the origin. The method applies to systems whose components are linear, but the underlying idea can be extended, in principle, to systems with nonlinear components. Although a state-dependent interpretation is plausible, our method basically generates time-dependent switching signals, so that it fits well into the open-loop control philosophy.

In a forthcoming paper, we show that our method is applicable in order to prove that any point-wise stabilizable and controllable family of linear vector fields is actually near-periodically stabilizable.

Since the method exploits the stable subspace of a discrete dynamical system associated to a suitable periodic switching signal, it is natural to expect that some troubles can arise in numerical simulations. Because of round-off errors, the distance from the origin of the simulated trajectories is actually seen to approach zero within an appropriate number of iterations. But, as the number of iterations increases, it could begin to increase. To avoid such drawback, the process needs to be monitored and re-started when needed, updating the initial state. This remark seems to suggest that the method proposed in this paper is a valid tool for ascertaining the existence of stabilizing switching signals, while some care is required in practical applications. On the other hand, we emphasize that lack of robustness is an intrinsic drawback of any open-loop control approach.

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