Nonpathological Lyapunov functions and discontinuous Carathéodory systems

Andrea Bacciotti and Francesca Ceragioli [a]

[a] *Dipartimento di Matematica del Politecnico di Torino, C.so Duca degli Abruzzi, 24 - 10129 Torino - Italy*

Abstract

Differential equations with discontinuous righthand side and solutions intended in Carathéodory sense are considered. For these equations sufficient conditions which guarantee both Lyapunov stability and asymptotic stability in terms of nonsmooth Lyapunov functions are given. An invariance principle is also proven.

Key words: Lyapunov functions, stability, stabilizability, discontinuous control, invariance principle, nonpathological functions, Carathéodory solutions

1 Introduction

The interest in the study of Lyapunov-like theorems for discontinuous systems is essentially motivated by the connection between stability and stabilizability problems. Since the papers by Sussmann (1979), Artstein (1983) and Brockett (1983) it is clear that, in order to deal with general stabilization problems, discontinuous feedback laws are needed. With the introduction of discontinuous feedback laws, the theoretical problem of giving an appropriate definition of solution for a system with discontinuous righthand side, comes out. Different approaches have been followed in the literature (Filippov, 1988; Clarke *et. al.*, 1997; Ancona and Bressan, 1999). Filippov solutions in particular have been used in order to deal with discontinuous stabilizability problems. For these solutions Lyapunov methods have been widely developed (see, e.g., Aubin and Cellina,1994; Clarke *et al.*, 1998; Bacciotti and Rosier, 2001; Teel and Praly, 2000). Some recent papers show that the stabilization problem can be very well approached by means of Carathéodory solutions (Ancona and Bressan, 1999, 2002, 2004; Bressan, 1998; Rifford, 2002, 2003; Kim and Ha, 1999, 2004). If the notion of Carathéodory solution is accepted as a good notion of solution for discontinuous systems, in particular for systems coming from discontinuous stabilization problems, it is then interesting to develope an appropriate Lyapunov theory for them. Indeed, as far as the authors know, such a generalization of Lyapunov methods has not been treated in the literature before. We consider nonsmooth Lyapunov functions: nonsmoothness of Lyapunov functions is in fact unavoidable when studying stability properties of discontinous systems. In particular, Lipschitz continuous Lyapunov functions which are also nonpathological (Valadier, 1989) are considered and a notion of derivative introduced in Shevitz and Paden (1994), improved in Bacciotti and Ceragioli (1999) and then used in Bacciotti and Ceragioli (1999, 2003), is used throughout the paper.

The paper consists of two main sections. In Section 2 the fundamental tools needed in order to obtain the main results are collected. In particular, the definition and basic properties of nonpathological functions are recalled. In Section 3 a Lyapunov like theorem and an invariance principle are stated and proved. In Section 4 these results are illustrated by means of some examples and counterexamples. Finally, some conclusions are driven in Section 5.

⋆ This paper was presented at NOLCOS 2004. Corresponding author F. Ceragioli. Tel. ++390115647518. Fax ++390115647599.

Email address: `andrea.bacciotti@polito.it`, `francesca.ceragioli@polito.it` (Andrea Bacciotti and Francesca Ceragioli).

2 Tools

2.1 Carathéodory solutions

Consider the autonomous differential equation:

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ locally bounded.

Definition 1 *Let I be an interval of $\mathbf{R}$. A function $\varphi : I \to \mathbf{R}^n$ is said to be a* Carathéodory solution *of (1) on I if $\varphi(t)$ is absolutely continuous and $\frac{d}{dt}\varphi(t) = f(\varphi(t))$ for almost every $t \in I$.*

In the following only Carathéodory solutions are considered and they are simply addressed as solutions. Moreover S_{x_0} denotes the set of solutions of (1) with initial condition $x(0) = x_0$.

The vector field $f(x)$ in (1) is in general not continuous, then existence of solutions of (1) is not guaranteed by classical theorems. General sufficient conditions on the vector field $f(x)$ in order to have existence of solutions of (1) have been studied in Pucci (1971),Bressan (1988) and Ancona and Bressan (1999). In the following we will make the following basic assumption:

(H) for any initial condition $x_0 \in \mathbf{R}^n$ at least one solution of (1) exists and all solutions are defined on the interval $[0, +\infty)$.

Definitions of stability in the case of discontinuous systems with solutions intended in Carathéodory sense do not differ from the usual ones. From now on, the origin is assumed to be an equilibrium position for the system, i.e. $f(0) = 0$.

Definition 2 *System (1) is said to be*

- Lyapunov stable *at the origin if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any x_0 with $\|x_0\| < \delta$ and for any $\varphi(t) \in S_{x_0}$, $\|\varphi(t)\| < \epsilon$ for all $t \geq 0$;*
- locally asymptotically stable *at the origin if it is Lyapunov stable and moreover there exists $\eta > 0$ such that for any x_0 with $\|x_0\| < \eta$ and for any $\varphi(t) \in S_{x_0}$, $\lim_{t\to+\infty} \varphi(t) = 0$.*

2.2 Nonpathological functions and nonpathological derivative

From now on, functions considered are nonpathological. We recall the definition of nonpathological function given in Valadier (1989). In the following, $\partial_C V(x)$ denotes the Clarke gradient of the function $V(x)$ at the point x (Clarke, 1983).

Definition 3 *A function $V : \mathbf{R}^n \to \mathbf{R}$ is said to be* nonpathological *if it is locally Lipschitz continuous and for every absolute continuous function $\varphi : I \to \mathbf{R}^n$ and for a.e. $t \in I$, the set $\partial_C V(\varphi(t))$ is a subset of an affine subspace orthogonal to $\dot{\varphi}(t)$.*

Nonpathological functions form a wide class which includes Clarke regular functions, semiconcave and semiconvex functions (see Clarke, 1983, for the definition of Clarke regular function and Cannarsa and Sinestrari, 2004, for the definitions of semiconvex and semiconcave functions). Nonpathological functions can be easily handled thanks to the following proposition (Valadier, 1989).

Proposition 1 *If $V : \mathbf{R}^n \to \mathbf{R}$ is nonpathological, and $\varphi : \mathbf{R} \to \mathbf{R}^n$ is absolutely continuous, then the set $\{p \cdot \dot{\varphi}(t), p \in \partial_C V(\varphi(t))\}$ is reduced to the singleton $\{\frac{d}{dt}V(\varphi(t))\}$ for almost every t.*

The notion of nonpathological derivative of a map with respect to a differential equation is now introduced. This notion is analogous to the notion of set-valued derivative of a map with respect to a differential inclusion introduced in Shevitz and Paden (1994) and improved in Bacciotti and Ceragioli (1999) (note that in Bacciotti and Ceragioli (1999), Clarke regular functions instead of nonpathological functions were used; the analogous set-valued derivative for nonpathological functions has been studied in Ceragioli (2000)).

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a nonpathological function and let (1) be given. Let

$$A_V = \{x \in \mathbf{R}^n : \ \exists c \in \mathbf{R} \ \text{ such that}$$
$$p \cdot f(x) = c \ \forall p \in \partial_C V(x)\}.$$

<u>Remark</u> **1** *It follows from (H) that*

(i) A_V is dense in $\mathbf{R}^n$;
(ii) if $V(x)$ is Clarke regular or semiconcave or semiconvex, then $\mathbf{R}^n \backslash A_V$ has null Lebesgue measure.

The definition of nonpathological derivative is now given.

Definition 4 *If $x \in A_V$, the* nonpathological derivative of the map $V(x)$ with respect to (1) at x *is the number*

$$\dot{\overline{V}}_f(x) = p \cdot f(x),$$

where p is any vector in $\partial_C V(x)$.

The use of the nonpathological derivative is explained by the following corollary of Proposition 1 (see also Example 1 in Section 3).

Corollary 1 *Let the function $V : \mathbf{R}^n \to \mathbf{R}$ be nonpathological, and let $\varphi(t)$ be any solution of system (1). Then $\varphi(t) \in A_V$ and $\frac{d}{dt}V(\varphi(t)) = \dot{\overline{V}}_f(\varphi(t))$ for almost every t.*

3 Results

In this section two propositions which generalize Lyapunov method to systems with discontinuous righthand side and solutions intended in Carathéodory sense are proved.

Proposition 2 *Let $V : \mathbf{R}^n \to \mathbf{R}$ be positive definite, locally Lipschitz continuous and nonpathological. Let A_V be defined as in Definition 4. Assume that*

$$\forall x \in A_V, \quad \dot{\overline{V}}_f(x) \leq 0. \tag{2}$$

Then

(i) system (1) is Lyapunov stable at the origin;
(ii) if moreover there exists a function $W : \mathbf{R}^n \to \mathbf{R}$ continuous and positive definite such that $\dot{\overline{V}}_f(x) \leq -W(x)$ for all $x \in A_V$ then system (1) is locally asymptotically stable.

PROOF. (i) In order to prove Lyapunov stability of the origin it is sufficient to prove that for any solution $\varphi(t)$ of (1) the composite function $V(\varphi(t))$ is nonincreasing. According to Corollary 1, $\varphi(t) \in A_V$ for almost every t, and $\frac{d}{dt}V(\varphi(t)) = \dot{\overline{V}}_f(\varphi(t))$ is nonpositive thanks to (2). The remaining part of the proof is standard.

(ii) From (i) it follows that for all $\epsilon > 0$ there exists δ such that for any x_0 with $\|x_0\| < \delta$ and for any $\varphi(t) \in S_{x_0}$ it holds $\|\varphi(t)\| < \epsilon$ for all $t \geq 0$. Next it is shown that $\lim_{t\to+\infty} \varphi(t) = 0$ for any x_0 with $\|x_0\| < \delta$ and for any $\varphi(t) \in S_{x_0}$.

First it is shown that $\lim_{t\to+\infty} V(\varphi(t)) = 0$. It has already been proved in (i) that $\frac{d}{dt}V(\varphi(t))$ is nonpositive. $V(\varphi(t))$ is then nonincreasing and bounded from below and then there exists $\lim_{t\to+\infty} V(\varphi(t)) = l$ with $l \geq 0$. Assume by contradiction that $l > 0$. Thanks to continuity of $V(x)$ and to the fact that $V(0) = 0$, there exists $0 < \mu < \epsilon$ such that if $\|x\| < \mu$ then $V(x) < l$. Let $C = \{x \in \mathbf{R}^n : \mu \leq \|x\| \leq \epsilon\}$ and let $N = \max\{-W(x), x \in C\}$. Since W is positive definite, $N < 0$. It is clear that $\mu \leq \|x_0\| < \delta \leq \epsilon$. For any $\varphi(t) \in S_{x_0}$, $\varphi(t) \in C$ for all $t \geq 0$ and for almost every t, $\frac{d}{dt}V(\varphi(t)) = \dot{\overline{V}}_f(\varphi(t)) \leq -W(\varphi(t)) \leq N$. From this inequality it follows that $V(\varphi(t)) \leq V(x_0) + Nt$ and then $\lim_{t\to+\infty} V(\varphi(t)) = -\infty$, which contradicts the fact that $V(x)$ is positive definite.

Next it is shown that $\lim_{t\to+\infty} V(\varphi(t)) = 0$ implies $\lim_{t\to+\infty} \varphi(t) = 0$. The statement is proved by contradiction. Assume there exists a sequence $\{t_n\}$ with $t_n \to +\infty$ such that there exists $\sigma > 0$ such that $\|\varphi(t_n)\| \geq \sigma$ for all n. Thanks to (i), $\varphi(t)$ is bounded, then there exists a subsequence $\{t_{n_k}\}$ such that $\varphi(t_{n_k}) \to y$ for some y with $\|y\| \geq \sigma$. It follows $0 = \lim_{t_{n_k}\to+\infty} V(\varphi(t_{n_k})) = V(y) > 0$, that is a contradiction. □

Next a LaSalle-like invariance principle is proved (see Shevitz and Paden, 1994, Ryan, 1998, Bacciotti and Ceragioli, 1999 for other versions of the invariance principle). In order to get it, some regularity for the vector field is needed.

Definition 5 *A vector field $f(x)$ is said to have the* solutions closure property *if for any sequence $\{\varphi_n(t)\}$ of solutions of (1) such that $\varphi_n(t) \to \varphi(t)$ uniformly on compact subsets of $\mathbf{R}$, one has that also $\varphi(t)$ is a solution of (1).*

Of course any continuous vector field $f(x)$ has the solutions closure property. An important class of discontinuous vector fields with the solutions closure property is the class of patchy vector fields (see Ancona and Bressan (1999)).

In the statement of Proposition 3 the notion of weakly invariant set is needed.

Definition 6 *A set M is said to be* weakly invariant *for (1) if for any $x_0 \in M$ there exists $\varphi(t) \in S_{x_0}$ such that $\varphi(t) \in M$ for all $t \geq 0$.*

Proposition 3 *Assume that the vector field $f(x)$ has the solutions closure property. Let $V : \mathbf{R}^n \to \mathbf{R}$ be positive definite and nonpathological. Let A_V be defined as in Definition 4 and assume (2). Assume that for some $l > 0$ the connected component L_l of the level set $\{x \in \mathbf{R}^n : V(x) \leq l\}$ such that $0 \in L_l$ is bounded. Let $Z_f^V = \{x \in A_V : \dot{\overline{V}}_f(x) = 0\}$ and let M be the largest weakly invariant subset of $\overline{Z_f^V} \cap L_l$. Then for any $x_0 \in L_l$ and any $\varphi(t) \in S_{x_0}$*

$$\lim_{t\to+\infty} \mathrm{dist}\,(\varphi(t), M) = 0. \tag{3}$$

Lemma 1 *If $f(x)$ has the solutions closure property, then for any x_0 and any $\varphi(t) \in S_{x_0}$ the fact that the positive limit set $\Omega(\varphi)$ is nonempty implies that it is weakly invariant.*

The proof of this lemma is very similar to the proof of the analogous lemma for Filippov solutions (see Filippov (1988), page 130).

PROOF. Let $x_0 \in L_l$ and let $\varphi(t) \in S_{x_0}$. First of all, note that $\varphi(t)$ is bounded. In fact if $\varphi(t)$ is not bounded on one hand there exists $\bar{t}$ such that $\varphi(\bar{t}) \notin L_l$ and on the other hand, by the proof of Proposition 2 (i), $V(\varphi(\bar{t})) \leq V(x_0) \leq l$, which is a contradiction.

Denote by $\Omega(\varphi)$ the positive limit set of $\varphi(t)$. Since $\varphi(t)$ is bounded then $\Omega(\varphi) \neq \emptyset$ and $\lim_{t\to+\infty} \mathrm{dist}\,(\varphi(t), \Omega(\varphi)) = 0$. Next it is proved that $\Omega(\varphi) \subset \overline{Z_f^V} \cap L_l$. From the

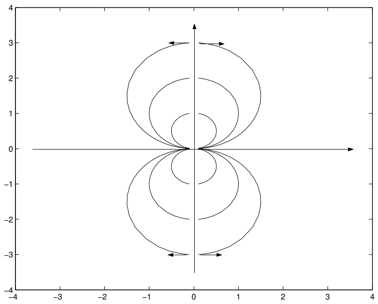

Fig. 1. Trajectories of system (4) with u as in (5).

fact that $\Omega(\varphi)$ is weakly invariant it will follow (3). Let $z \in \Omega(\varphi)$. Consider the composite function $V(\varphi(t))$, which is nonincreasing and bounded from below. From this fact it follows that there exists $\lim_{t\to+\infty} V(\varphi(t)) = c$ and then $V(z) = c$ for every $z \in \Omega(\varphi)$. In fact, if $z \in \Omega(\varphi)$, there exists a sequence $\{t_n\}$, $t_n \to +\infty$ such that $\lim_{n\to+\infty} \varphi(t_n) = z$ and, thanks to the continuity of $V(x)$, $V(z) = \lim_{n\to+\infty} V(\varphi(t_n)) = \lim_{t\to+\infty} V(\varphi(t)) = c$. Due to Lemma 1, $\Omega(\varphi)$ is weakly invariant, then there exists $\tilde{\varphi}(t) \in S_z$ such that $\tilde{\varphi}(t) \in \Omega(\varphi)$ for all $t \geq 0$. It follows that $V(\tilde{\varphi}(t)) = c$ for all $t \geq 0$ and $\frac{d}{dt}V(\tilde{\varphi}(t)) = 0$ for all $t \geq 0$. Since $\tilde{\varphi}(t) \in A_V$ and $\dot{\overline{V}}_f(\tilde{\varphi}(t)) = \frac{d}{dt}V(\tilde{\varphi}(t)) = 0$ for almost all $t \geq 0$, $\tilde{\varphi}(t) \in Z_f^V$ for almost all $t \geq 0$. Let now $\{t_i\}$ be a sequence such that $t_i \to 0$ and $\tilde{\varphi}(t_i) \in Z_f^V$ for all i. Since $\tilde{\varphi}(t)$ is continuous it finally follows that $z = \lim_{i\to+\infty} \tilde{\varphi}(t_i) \in \overline{Z_f^V}$. □

Corollary 2 *Assume that $f(x)$ has the solutions closure property. Let $V : \mathbf{R}^n \to \mathbf{R}$ be positive definite and nonpathological, and let A_V be defined as in Definition 4. If (2) holds and $\dot{\overline{V}}_f(x) = 0$ if and only if $x = 0$, then system (1) is locally asymptotically stable.*

4 Examples

Proposition 2 and 3 are now illustrated by means of some examples. Example 1, besides being an application of Proposition 2, aims to explain the use of the nonpathological derivative. Example 2 is a counter-example showing that Proposition 2 is not true if assumption (H) does not hold. In Example 3 Proposition 3 is applied to prove the asymptotic stability of a planar bilinear "switched system". This example aims also to remark that discontinuous feedback laws also appear when control systems with finite control set are studied. Finally Example 4 is a counter-example showing that the assumption about the solutions closure property of the vector field in Proposition 3 is actually needed.

Example 1 Consider the two-dimensional, single input, driftless system

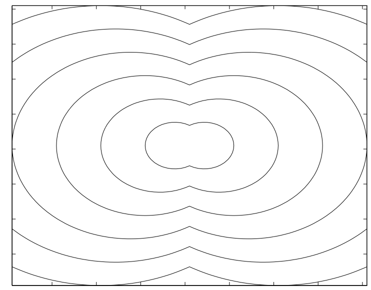

Fig. 2. Level curves of $V(x, y) = (4x^2 + 3y^2)^{1/2} - |x|$.

$$\begin{cases} \dot{x} = (x^2 - y^2)u \\ \dot{y} = 2xyu \end{cases} \tag{4}$$

(the so-called Artstein's circles example, see Artstein,1983, Sontag 1999). This system can not be stabilized by means of a continuous feedback. A discontinuous stabilizing feedback is (see Fig. 1)

$$u(x, y) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0. \end{cases} \tag{5}$$

Denote by $f(x, y)$ the righthand side of the implemented system. The function $V(x, y) = \sqrt{4x^2 + 3y^2} - |x|$ satisfies the assumptions of Proposition 2. As a sum of a function of class C^1 and a concave function, $V(x, y)$ is semiconcave and then nonpathological, but it is not differentiable when $x = 0$ (the level curves are piecewise arcs of circumferences: see Fig. 2).

$A_V = \mathbf{R}^2 \setminus \{(x, y) : x = 0\}$ and if $(x, y) \in A_V$ then $\dot{\overline{V}}_f(x, y) = \nabla V(x, y) \cdot f(x, y) = -W(x, y)$, where $W(x, y) = \frac{4|x|^3 + 2|x|y^2 - (x^2 - y^2)\sqrt{4x^2+3y^2}}{\sqrt{4x^2+3y^2}}$ is continuous and positive definite. Assumptions of Proposition 2 are satisfied, then the implemented system is asymptotically stable.

Note that in this example the set where $V(x, y)$ is not differentiable is a curve, the y-axis, which is not included in A_V, because the vector field $f(x, y)$ is transversal to it. In this case, in order to apply Proposition 2, the condition involving the nonpathological derivative needs not to be checked at all points of $\mathbf{R}^2$.

Consider now the slightly modified vector fields

$$f^+(x, y) = \begin{cases} f(x, y) & \text{if } x \neq 0 \\ (0, y) & \text{if } x = 0 \end{cases} \tag{6}$$

$$f^-(x, y) = \begin{cases} f(x, y) & \text{if } x \neq 0 \\ (0, -y) & \text{if } x = 0. \end{cases} \tag{7}$$

Note that for $x = 0$ both these vector fields are parallel to the y-axis, i.e. to the curve where $V(x, y)$ is not dif-

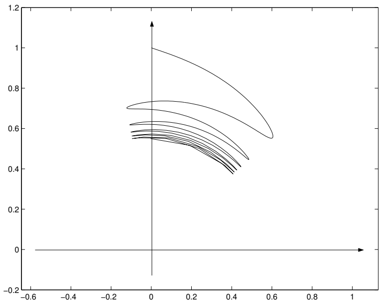

Fig. 3. A trajectory of the system in Example 2.

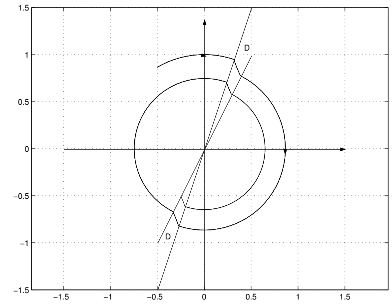

Fig. 4. A trajectory of the system in Example 3.

ferentiable, and $A_V = \mathbf{R}^2$. The condition involving the nonpathological derivative needs now to be checked at all points of $\mathbf{R}^2$. The numbers $\dot{\overline{V}}_{f^+}(x,y)$ and $\dot{\overline{V}}_{f^-}(x,y)$ can be easily computed. It turns out that the system defined by $f^-(x,y)$ satisfies the assumptions of Proposition 2 and hence it is asymptotically stable, while the system defined by $f^+(x,y)$ does not satisfy the assumptions of Proposition 2. The system defined by $f^+(x,y)$ is clearly unstable.

Example 2 Consider the two dimensional system whose righthand side is:

$$\begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } \rho = 0 \\ \begin{pmatrix} -\frac{x}{\rho} - \frac{y}{(\rho-\frac{1}{n})^2}\sin\frac{2}{\rho-\frac{1}{n}} \\ -\frac{y}{\rho} + \frac{x}{(\rho-\frac{1}{n})^2}\sin\frac{2}{\rho-\frac{1}{n}} \end{pmatrix} & \text{if } \rho \in (\frac{1}{n}, \frac{1}{n-1}] \\ \begin{pmatrix} -x \\ -y \end{pmatrix} & \text{if } \rho > 1 \end{cases}$$

where $\rho = \sqrt{x^2+y^2}, n \in \mathbf{N}, n \geq 2$. Consider the function $V(x,y) = \frac{x^2+y^2}{2}$. It holds that $A_V = \mathbf{R}^2$ and

$$\dot{\overline{V}}_f(x,y) = \begin{cases} 0 & \text{if } \rho = 0 \\ -\sqrt{x^2+y^2} & \text{if } 0 < \rho \leq 1 \\ -(x^2+y^2) & \text{if } \rho > 1, \end{cases}$$

then $\dot{\overline{V}}_f(x,y) \leq -\min\{\sqrt{x^2+y^2}, \frac{x^2+y^2}{2}\}$. Nevertheless the system is not asymptotically stable. Proposition 2 (ii) can not be applied due to the fact that solutions of the system are not defined on the interval $[0,+\infty)$. Indeed trajectories of the system in polar coordinates are (see Fig. 3):

$$\begin{cases} \rho(t) = \rho_0 - t \\ \theta(t) = \sin^2\frac{1}{\rho_0 - t - \frac{1}{n}} + \theta_0 - \sin^2\frac{1}{\rho_0 - \frac{1}{n}} \end{cases}$$

where $\rho_0 \in (\frac{1}{n}, \frac{1}{n-1}]$.

Example 3 Consider the two-dimensional control system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = uA\begin{pmatrix} x \\ y \end{pmatrix} + (1-u)B\begin{pmatrix} x \\ y \end{pmatrix}, \tag{8}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{9}$$

and the feedback law

$$u(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in \mathbf{R}^2 \backslash D \end{cases} \tag{10}$$

where $D = \{(x,y) : x > 0 \text{ and } 2x < y < 3x \text{ or } x < 0 \text{ and } 3x < y < 2x\}$.

The function $V(x,y) = \frac{x^2+y^2}{2}$ is C^∞, nevertheless we can not use it in order to apply classical Lyapunov theorems to the implemented system due to the fact the feedback law is discontinuous. Denote by $f(x,y)$ the righthand side of the implemented system. $A_V = \mathbf{R}^2$, $\dot{\overline{V}}_f(x,y) = \nabla V(x,y)\cdot f(x,y) \leq 0$ for all $(x,y) \in \mathbf{R}^2$ and $\overline{Z}^V_f = \mathbf{R}^2 \backslash D$. The largest weakly invariant subset of $\overline{Z}^V_f$ is the origin, then the implemented system is asymptotically stable (see Fig. 4).

Example 4 Consider the equation $\dot{x} = f(x)$ where $x \in \mathbf{R}$, $f(0) = 0$, $f(x) = -f(-x)$ and $f(x)$ is defined in the following way:

$$f(x) = \begin{cases} -x + \frac{1}{n} & \text{if } x \in (\frac{1}{n}, \frac{1}{n-1}], \\ -x + 1 & \text{if } x > 1. \end{cases} \tag{11}$$

where $n \in \mathbf{N}, n \geq 2$.

Remark that:

a- $f(x) \neq 0$ for all $x \neq 0$,
b- for any initial condition $x_0 \in \mathbf{R}$ there exists a Carathéodory solution, which moreover is unique and defined on the interval $[0,+\infty)$,
c- the origin is stable, but not asymptotically stable.

Consider $V(x) = \frac{x^2}{2}$. It holds

$$\dot{\overline{V}}_f(x) = \begin{cases} x(-x + \frac{1}{n}) & \text{if } x \in (\frac{1}{n}, \frac{1}{n-1}], \\ x(-x+1) & \text{if } x > 1, \end{cases}$$

then $\overline{Z_f^V} = \{0\}$.

In this case Proposition 3 can not be applied due to the fact that the vector field $f(x)$ does not have the solutions closure property (note that for any solution $\varphi(t)$ with non zero initial condition the positive limit set limit set $\Omega(\varphi)$ is not invariant).

Finally remark that there does not exist a positive and contiuous function $W(x)$ such that $\dot{\overline{V}}_f(x) \leq -W(x)$.

5 Conclusion

Differential equations with discontinuous righthand side and solutions intended in Carathéodory sense have been considered. The interest of such equations is motivated by the fact that they naturally arise in stabilizability problems in connection with important classes of discontinuous feedback laws. For these equations sufficient conditions which guarantee both Lyapunov stability and asymptotic stability in terms of nonsmooth Lyapunov functions are proved. Moreover an appropriate version of LaSalle invariance principle is given.

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