SOME REMARKS ON GENERALIZED SOLUTIONS OF DISCONTINUOUS DIFFERENTIAL EQUATIONS

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Abstract: In this paper we consider several concepts of generalized solutions for ordinary differential equations with a discontinuous right hand side. By means of some nontrivial examples, we show that Sentis solutions, Forward-Euler solutions and Carathéodory solutions are independent notions.

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1. Introduction

It is well known that a system of ordinary differential equations

$$\dot{x} = f(x) \ , \qquad x \in \mathbf{R}^n \tag{1}$$

may have no solution (in the usual sense) for some initial state, if the right hand side $f(x) : \mathbf{R}^n \to \mathbf{R}^n$ is not continuous. On the other hand, differential equations with a discontinuous right hand side frequently arise in applications. For this reason, it is interesting to seek generalized notions of solution, for which suitable existence theorems can be proven even in case that $f(x)$ is not continuous. In the literature, we can find many approaches to this problem.

The most popular concept of generalized solution for discontinuous differential equations is due to Filippov [6]. It applies under the assumption that $f(x)$ is Lebesgue measurable and locally bounded: the basic idea consists of replacing (1) by a differential inclusion

$$\dot{x} \in F(x) \tag{2}$$

where

$$F(x) = F_F(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\text{co}} \, \{f(\mathcal{B}(x,\delta) \backslash N)\} \tag{3}$$

(μ denotes the Lebesgue measure of $\mathbf{R}^n$, $\overline{\text{co}}$ denotes the closure of the convex hull, and $\mathcal{B}(x,r)$ is the ball of radius r centered at x). In [7], O. Hájek studies and compares Filippov solutions with other notions, due to Krasowskii and Hermes, and with the classical ones (Newton and Carathéodory solutions). More recently, in the context of control theory, other types of solutions (Euler solutions) have been successfully employed (see Ancona et al [1], Malisoff et al [8]).

In practical problems, choosing the "right" definition may be a matter of convenience, and may depend on the particular application. However, we emphasize that from a mathematical point of view, the relationships among all these notions of solution have not yet been completely understood, although some remarkable work has been done in Bressan [3], Clarke et al [5], Ceragioli [4] (see also Bacciotti [2] for a recent survey, and the reference therein).

In this note we focus on Sentis solutions, introduced by Sentis [9], and Forward-Euler solutions, a variant of the notion of Euler solution, introduced by Bressan [3]. Our contribution is a couple of nontrivial examples, showing the following:

Statement 1 *There exists Sentis solutions which are not Forward-Euler solutions.*

Statement 2 *There exists Carathéodory solutions which are not Forward-Euler solutions.*

The basic definitions are recalled in Sections 2 and 3. The examples are discussed in Sections 4 and 5.

2. Forward Euler Solutions

Throughout this paper, the vector field $f : \mathbf{R}^n \to \mathbf{R}^n$ is assumed to be Lebesgue measurable and locally bounded. Let $I = [a, b] \subseteq \mathbf{R}$ be any closed interval. A function $\varphi(t) : I \to \mathbf{R}^n$ is a *classical (or Newton) solution* of (1) if it admits an extension on some open interval (a', b') (with $[a, b] \subset (a', b')$) which is everywhere differentiable and satisfies $\dot{\varphi}(t) = f(\varphi(t))$ for each $t \in I$. Peano's Theorem states that if f is continuous, then for each $a \in \mathbf{R}$ and for each $\bar{x} \in \mathbf{R}^n$ there exist $b \in \mathbf{R}$ with $a < b$, and there exists at least one classical solution of the initial value problem

$$\begin{cases} \dot{x} = f(x) \\ x(a) = \bar{x} \end{cases} \tag{4}$$

defined on the interval $[a, b]$. The usual proof of Peano's Theorem relies on the construction of a uniformly convergent sequence of certain polygonal approximations. Classical solutions of (1) which can be obtained in this way are called *Euler solutions*. It is well known that the set of Euler solutions of (1) is a proper subset of the set of all the classical solutions, in general. In order to obtain a larger set of solutions, one can exploits an analogous approximation procedure, but with some essential modifications.

Let $m \in \mathbf{N}$, and let $I = [a, b]$ be a given interval. Consider a partition of I

$$a = t_{m,0} < t_{m,1} < \ldots < t_{m,k_m-1} < t_{m,k_m} = b$$

and let $l_m = \max\{t_{m,i+1} - t_{m,i}\ ,\ i = 0, \ldots, k_m - 1\}$. Let us choose a point $x_{m,0} \in \mathbf{R}^n$, and certain vectors $q_{m,i} \in \mathbf{R}^n$ $(i = 0, \ldots, k_m - 1)$. Then construct a continuous, piecewise affine function $\psi_m(t)$ on the interval $[a, b]$ in such a way that $\psi_m(t_{m,0}) = x_{m,0}$, and

$$\psi_m(t_{m,i+1}) = \psi_m(t_{m,i}) + [f(\psi_m(t_{m,i})) + q_{m,i}]\,(t_{m,i+1} - t_{m,i})$$

$(i = 0, \ldots, k_m - 1)$. Any such function $\psi_m(t)$ will be called a *continuous polygonal approximation*. The vectors $q_{m,i}$'s are called *outer perturbations*. When the outer perturbations vanish, we recover the polygonal approximations used in the proof of Peano's Theorem.

Definition 1 *A function $\varphi(t) : [a, b] \to \mathbf{R}^n$ is said to be a* Forward Euler solution *(in short, FE-solution) of (1) if for each $\sigma > 0$ there exists an integer m and a continuous polygonal approximation such that*

$$|\varphi(t) - \psi_m(t)| < \sigma \qquad \forall t \in [a, b]$$

$0 < l_m < \sigma$, $0 \leq |q_{m,i}| < \sigma$, *for each* $i = 0, \ldots, k_m - 1$.

FE-solutions are introduced by Bressan [3]. It is not difficult to prove that in general, the set of the classical solutions of (1) is a proper subset of the set of all its FE-solutions.

3. Sentis Solutions

It is a common opinion that Filippov solutions are "too many" for some applications. To overcome this drawback, in [9] R. Sentis replaces (1) by a differential inclusion (2) where now

$$F(x) = F_S(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\{f(\mathcal{B}(x,\delta)\backslash N)\}} \ . \qquad (5)$$

The set valued map $F_S(x)$ turns out to be upper semicontinuous, locally bounded and compact (but in general not convex) valued. Of course, $F_S(x) \subseteq F_F(x)$; in fact, it is proved by Bacciotti [2] that $F_F(x) = \mathrm{co} F_S(x)$. A new class of solutions for (1) is therefore defined by means of the following approximation procedure.

Let $\bar{x}$ be a point of $\mathbf{R}^n$, $I = [a,b]$ be a closed interval, and $m \in \mathbf{N}$. As in the previous section, let us take a partition of I

$$a = t_{m,0} < t_{m,1} < \ldots < t_{m,k_m-1} < t_{m,k_m} = b$$

where k_m is some positive integer, and let again $l_m = \max\{t_{m,i+1} - t_{m,i}\ ,\ i = 0, \ldots, k_m - 1\}$. Then, for each $i = 0, \ldots, k_m - 1$ choose $\varepsilon_{m,i} \in \mathbf{R}^n$ and construct a right-continuous, piecewise affine function $\psi_m(t)$ on the interval $[a,b]$ in such a way that $\psi_m(t_{m,0}) = \bar{x}$, and

$$\psi_m(t_{m,i+1}) = \psi_m(t_{m,i}) + v_{m,i}(t_{m,i+1} - t_{m,i}) + \varepsilon_{m,i}$$

where $v_{m,i}$ is any element in $F_S(\psi_m(t_{m,i}))$. Any such function $\psi_m(t)$ will be called a *discontinuous polygonal approximation.*

Definition 2 *We say that a function* $\varphi(t) : [a,b] \to \mathbf{R}^n$ *is a* Sentis solution *of (1) if for each* $\sigma > 0$ *there exists an integer* m *and a discontinuous polygonal approximation such that*

$$|\varphi(t) - \psi_m(t)| < \sigma \quad \forall t \in [a,b]$$

$0 < l_m < \sigma$ *and* $0 \leq \sum_{i=0}^{k_m-1} |\varepsilon_m^i| < \sigma$.

Definition 2 can be rephrased by saying that φ is the uniform limit of a sequence of discontinuous polygonal approximations $\{\psi_m\}$, such that l_m is decreasing, $\lim_m l_m = 0$ and

$$\lim_m \sum_{i=0}^{k_m-1} |\varepsilon_m^i| = 0 \ .$$

Note that by construction, $\varphi(a) = \bar{x}$. In Sentis [9], it is proven that every Sentis solution of (1) is a Filippov solution, but the converse is false, in general.

4. Comparison between Sentis solutions and FE-solutions

It is not difficult to give examples of FE-solutions which are not Sentis solutions. The main contribution of this section is the following example. It shows that there may exist Sentis solutions which are not FE-solutions, as well.

Example 1 We construct a two-dimensional vector field $f(x, y)$, as illustrated by Figure 1. The construction starts by taking on the positive x-axis the points $(2^{-n}, 0)$, with $n \in \mathbf{Z}$ (but only the case $n > 0$ is really of interest). Each interval $[2^{-n}, 2^{-n+1}]$ is thought of as divided in three parts by the points $2^{-n}(4/3)$, $2^{-n}(5/3)$. Let us draw the lines joining the points

$$\begin{cases} (2^{-n}(5/3), 2^{-n}(5/3)) \text{ and } (2^{-n+1}(4/3), -2^{-n+1}(4/3)) & \text{if } n \text{ is even} \\ (2^{-n}(5/3), -2^{-n}(5/3)) \text{ and } (2^{-n+1}(4/3), 2^{-n+1}(4/3)) & \text{if } n \text{ is odd.} \end{cases}$$

Note that the absolute value of the slopes of these lines is constant and, to be precise, equal to 13/3. Finally consider for $x \geq 0$ the lines $y = \pm x$. The intersections of all these lines define the vertices of two sequences of triangles (one sequence above the line $y = x$, the other below the line $y = -x$) both "converging" toward the origin. Let us define

$$f(x, y) = \begin{cases} \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{inside the triangles above the line } y = x \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{inside the triangles below the line } y = -x \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{elsewhere.} \end{cases}$$

We claim that the origin is an equilibrium point in the sense of Sentis. To prove it, let us fix $\sigma > 0$, and let $n_0 \in \mathbf{N}$ be such that $n_0 > -\log_2(3\sigma/8)$. We construct a discontinuous piecewise affine approximation in the following way. Starting from the origin, we first move to right, until the point $(2^{-n_0}(4/3), 0)$ is reached. Without loss of generality, we may assume that n_0 is even. Then we jump to the point $(2^{-n_0}(4/3), 2^{-n_0}(4/3))$. Next we move downward, and stop at the point $(2^{-n_0}(4/3), -2^{-n_0-1}(4/3))$. Then we make another jump to the point $(2^{-n_0-1}(4/3), -2^{-n_0-1}(4/3))$. From this point, let us move upward and stop at $(2^{-n_0-1}(4/3), 2^{-n_0-2}(4/3))$. We continue in this way. We obtain a piecewise affine curve on every interval $[0, T]$ whose image is contained in the square $[-\sigma, \sigma] \times [-\sigma, \sigma]$. It is not difficult to estimate that the maximal length of time subintervals is less than $2^{-n_0}(8/3)$. In a similar way, we see that the sum of the distances covered by the jumps is less than the same quantity $2^{-n_0}(8/3)$. The conclusion is an easy consequence of these remarks.

If we would try to prove that the origin is an FE-solution, we should repeat the previous construction, by substituting any discontinuous piecewise affine approximation by a continuous one. But this is impossible. Indeed, in order to reach a triangle of the upper half plane from a triangle of the lower half plane (and vice versa) we should apply an outer perturbation q with $|q| = \cos(\text{arctg}\,(13/3))$.

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5. Comparison between Carathéodory solutions and FE-solutions

A function $\varphi(t) : I \to \mathbf{R}^n$ is a *Carathéodory solution* of (1) if it is differentiable a.e. on the interval I, and it satisfies $\dot\varphi(t) = f(\varphi(t))$ a.e. on I. There are simple examples of FE-solutions which are not Carathéodory solutions. Here, we deal with the opposite inclusion.

Example 2 We define a planar vector field in the following manner (see Figure 2).

- for $y < 0$ we set $f(x,y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$;
- for $y = 0$ and $x \leq 0$, we set $f(x,0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$;
- for $y > 0$, $x \leq 0$, we set $f(x,y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;
- for $y \geq 0$, $n \in \mathbf{N}$, we set $f(\frac{1}{2^n}, y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$;
- for $y \geq 0$, $x > 0$, $x \neq \frac{1}{2^n}$ $(n \in \mathbf{N})$, we set $f(x,y) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Next we define a curve $\varphi(t)$ for $t \in [0,3]$. consider the two sequences of points $s_0 = 0, s_1 = 1, \ldots, s_{n+1} = 2 - \frac{1}{2^n}, \ldots$ and $\sigma_n = \frac{s_{n+1}+s_n}{2}$. Note that $\lim s_n = \lim \sigma_n = 2$. Then we set

$$\varphi(t) = \begin{cases} \begin{pmatrix} -t + 2 - \dfrac{1}{2^n} \\ t - 2 + \dfrac{1}{2^{n-1}} \end{pmatrix} & t \in [s_n, \sigma_n], \quad n = 0,1,\ldots \\ \begin{pmatrix} \dfrac{1}{2^{n+1}} \\ -t + 2 - \dfrac{1}{2^n} \end{pmatrix} & t \in [\sigma_n, s_{n+1}], \quad n = 0,1,\ldots \\ \begin{pmatrix} -t+2 \\ 0 \end{pmatrix} & t \in [2,3] \quad . \end{cases} \tag{6}$$

The vector field $f(x,y)$ is bounded and measurable. The function $\varphi(t)$ is Lipschitz continuous and such that $\dot\varphi(t) = f(\varphi(t))$ a.e. $t \in [0,3]$, so that it is a Carathéodory solution. The restriction of $\varphi(t)$ on the interval $[0,2]$ is clearly an FE-solution, as well as the restriction of $\varphi(t)$ on $[2,3]$, but the entire trajectory $\varphi(t)$ on $[0,3]$ is not. Indeed, assume by contradiction that there is a continuous polygonal approximation $\psi(t)$ such that

$$|\varphi(t) - \psi(t)| < \sigma \ , \quad t \in [0,3]$$

for sufficiently small $\sigma > 0$. Then, there must exist a subinterval $[\tau', \tau''] \subset [0,3]$ such that $\psi(t)$ is affine on $[\tau', \tau'']$ and, in addition,

$$\psi(\tau') \in \{(x,y) : x > 0, y > -\sigma\} \ , \quad \psi(\tau'') \in \{(x,y) : x \leq 0, \ y = 0\} \ .$$

Taking into account the values of the vector field available for $x > 0$, we see that this is impossible, if only small outer perturbations are allowed.

By the way, this example shows that in general, FE-solutions cannot be “cut and pasted” together.

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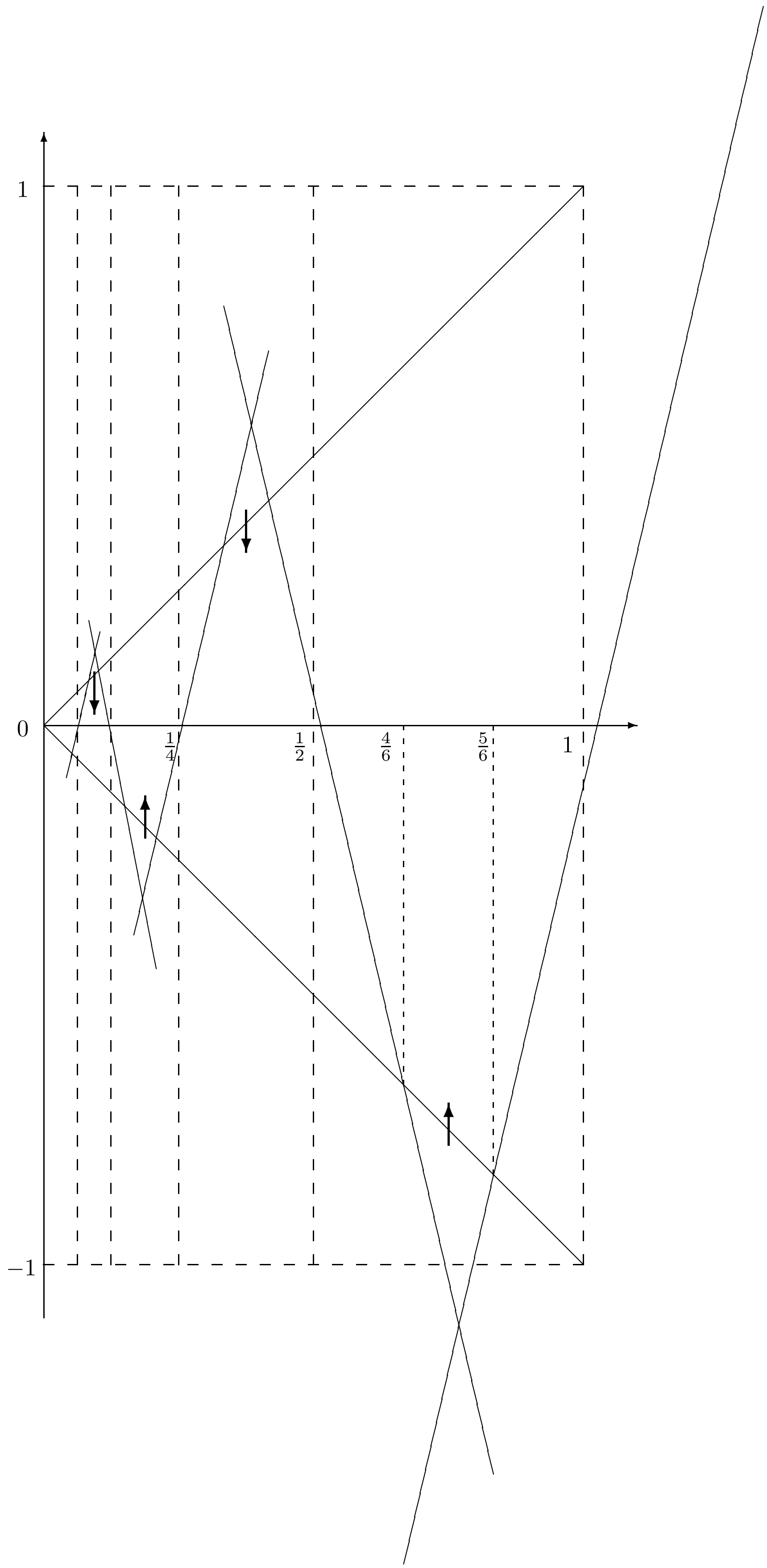

Figure 1: Example 1

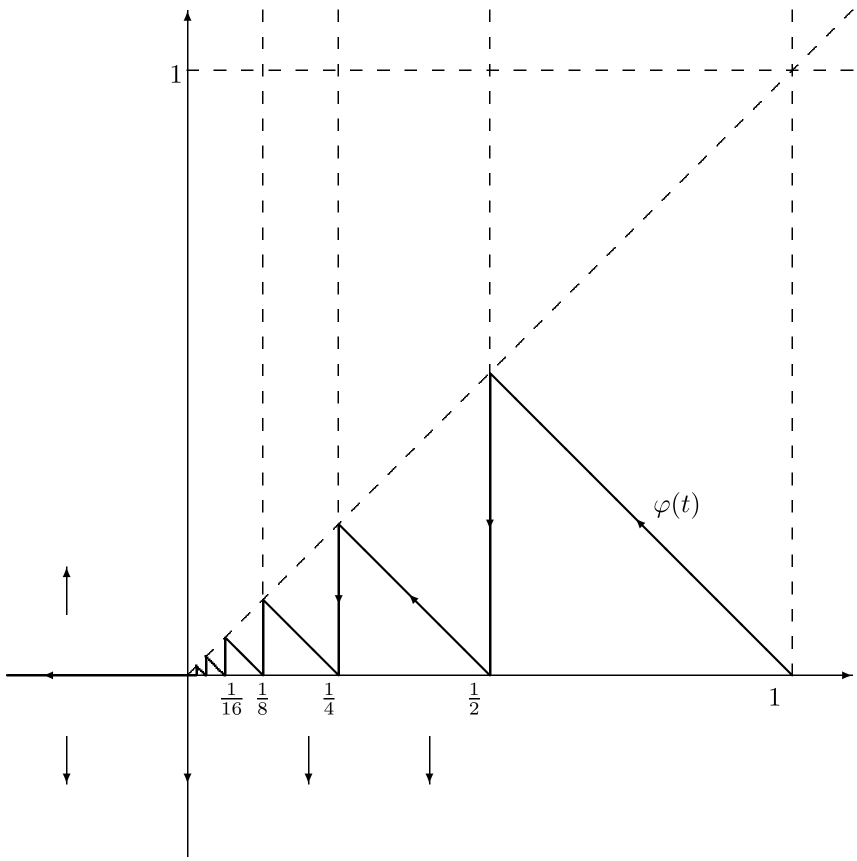

Figure 2: Example 2

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