

OPTIMAL REGULATION AND DISCONTINUOUS STABILIZATION

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Abstract: The relationship between a minimization problem on the infinite horizon and local stabilizability is studied for affine control systems. A sufficient condition for a system to be locally stabilizable in the sense of Filippov solutions is given in the case that the value function associated to the minimization problem is locally Lipschitz continuous.

Keywords: optimization, nonlinear control systems, discontinuous controls, stabilizability.

1. INTRODUCTION

Affine control systems of the form

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^m u_i g_i(x) \quad (1)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, the functions $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$, $g_i : \mathbf{R}^n \rightarrow \mathbf{R}^n$, $i = 1, \dots, m$, are of class C^1 and G is the matrix whose columns are g_1, \dots, g_m are considered.

Admissible inputs are piecewise continuous and right continuous functions $u : \mathbf{R} \rightarrow \mathbf{R}^m$. The set of admissible inputs is denoted by \mathcal{U} and the solution of equation (1) such that $\varphi(0; x_0, u(\cdot)) = x_0$ is denoted by $\varphi(t; x_0, u(\cdot))$. For every admissible input and every initial condition there exists a piecewise classical solution which is unique.

The relationship between the two problems of local stabilization and minimization of a cost functional on the infinite horizon is studied. More precisely, the two problems can be stated in the following manner.

(A) System (1) is said to be *(locally) stabilizable* if there exists a map $u = k(x) : \mathbf{R}^n \rightarrow \mathbf{R}^m$, called *feedback law*, such that the origin is a (locally)

asymptotically stable equilibrium position for the closed-loop system

$$\dot{x} = f(x) + G(x)k(x), \quad (2)$$

i.e.

(i) for all $\epsilon > 0$ there exists $\delta > 0$ such that for each solution $\varphi(t)$ of (2), $|\varphi(0)| < \delta$ implies $|\varphi(t)| < \epsilon$ for all $t \geq 0$ (Lyapunov stability),

(ii) there exists $\eta > 0$ such that for each solution $\varphi(t)$ of (2) such that $|\varphi(0)| < \eta$ one has $\lim_{t \rightarrow +\infty} \varphi(t) = 0$ (attractivity).

Note that, till now, the stabilization problem has not been well specified. In fact the regularity asked to the feedback has not been stated yet. It is well known that the class of continuous feedbacks is not sufficiently large in order to solve general stabilization problems (see (Artstein, 1983; Brockett, 1983)). For this reason in the following discontinuous feedbacks are considered. As a consequence, it is important to specify in which generalized sense solutions of the closed-loop system are intended. System (1) is said to be either *Carathéodory* or *Filippov (locally) stabilizable* according to the fact that solutions of the closed-loop system (2) are intended respectively either in Carathéodory's or in Filippov's sense

(see (Ancona and Bressan, 1999; Filippov, 1988)) for different definitions of solutions; moreover note that, due to a result in (Rifford, 2000), stabilization in the sense of the so called sampling solutions used in (Clarke *et al.*, 1997) essentially reduces to Carathéodory stabilization in the case of affine systems. In the following local stabilizability will be simply addressed stabilizability.

The minimization problem is now introduced.

(B) The cost functional associated to (1) is

$$J(x_0, u(\cdot)) = \frac{1}{2} \int_0^{+\infty} h(\varphi(t; x_0, u(\cdot))) + \frac{|u(t)|^2}{\gamma} dt$$

where $h : \mathbf{R}^n \rightarrow \mathbf{R}$ is continuous and positive definite and $\gamma \in \mathbf{R}^+$. The problem of minimizing the functional J for every initial condition x_0 is considered. The value function $V : \mathbf{R}^n \rightarrow \mathbf{R}$ associated to the minimization problem is

$$V(x_0) = \inf_{u \in \mathcal{U}} J(x_0, u).$$

Problem (B) is said to be solvable if for every x_0 the infimum in the definition of V is actually a minimum. If problem (B) is solvable, an optimal control corresponding to the initial condition x_0 is denoted by $u_{x_0}^*(\cdot)$.

The relationship between problems (A) and (B) is completely understood in the case f is linear, G is a constant matrix and the function h is quadratic: the two problems are actually equivalent (and they are also equivalent to the solvability of Riccati's equation). Analogous relationships for nonlinear systems have been studied in the case the value function V is C^1 and feedback laws (which can be taken in the damping form) are continuous (Riccati's equation is replaced by Hamilton-Jacobi's equation in this case, see (Bernstein, 1993; van der Schaft, 1992; Bacciotti and Ceragioli, 2001)). Note that the assumption on the regularity of V is very strong. In fact, in general, the value function is proved to be just lower semi-continuous. It would be then very interesting to see to what extent the mentioned framework can be carried out with weaker assumptions on the regularity of V . Here the case V is locally Lipschitz continuous is considered. This case is particularly interesting since there are some results of Lipschitz regularity of the value function for optimization problems on the infinite horizon (see (Da Lio, 2000)). By means of a sort of Barbalat's lemma for control systems, it is easy to prove that solvability of problem (B) implies asymptotic controllability (see next section for precise definition of this property). This in turns implies stabilizability in Carathéodory sense (see (Ancona and Bressan, 1999)). Here stabilization

in Filippov's sense is mainly studied. This can not be simply deduced by asymptotic controllability. In Section 3 a condition which, added to solvability of problem (B), guarantees Filippov stabilizability by means of a discontinuous damping feedback is given.

2. THE MAIN RESULTS

In the present section it is investigated to what extent solvability of the minimization problem implies stabilizability. Carathéodory stabilizability is easily obtained. In fact the following Theorem 2 states that solvability of problem (B) implies asymptotic controllability. It is then sufficient to recall the result in (Ancona and Bressan, 1999) to get Carathéodory stabilizability. On the other hand Filippov stabilizability can not be simply deduced by asymptotic controllability: some extra assumptions in order to get it are needed.

The following preliminary result states that the optimal control can be taken in a feedback form. $\partial_C V(x)$ denotes Clarke gradient of V at x (see (Clarke, 1983)).

Proposition 1. Assume that (B) is solvable. Then, the optimal control can be expressed in feedback form:

$$u_{x_0}^*(0) = -\gamma(p_0 G(x_0))^t = \quad (3)$$

$$-\gamma(p_0 g_1(x_0), \dots, p_0 g_m(x_0))^t$$

where p_0 is a suitable element in $\partial_C V(x_0)$.

Note that $u_{x_0}^*(0)$ is well defined, since admissible inputs are assumed to be right continuous.

PROOF. From Lemma 6 in next section it follows that there exists $p_0 \in \partial_C V(x_0)$ such that

$$-p_0 \cdot (f(x_0) + G(x_0)u_{x_0}^*(0)) - \frac{h(x_0)}{2} - \frac{|u_{x_0}^*(0)|^2}{2\gamma} = 0$$

By writing Lemma 7 for $p = p_0$ it follows that for every $u_0 \in \mathbf{R}$

$$-p_0 \cdot (f(x_0) + G(x_0)u_0) - \frac{h(x_0)}{2} - \frac{|u_0|^2}{2\gamma} \leq 0$$

and then

$$-p_0 \cdot (f(x_0) + G(x_0)u_{x_0}^*(0)) - \frac{h(x_0)}{2} - \frac{|u_{x_0}^*(0)|^2}{2\gamma} =$$

$$\max_{u \in \mathbf{R}^m} \left\{ -p_0 \cdot (f(x_0) + G(x_0)u) - \frac{h(x_0)}{2} - \frac{|u|^2}{2\gamma} \right\}$$

i.e.

$$-p_0 \cdot G(x_0)u_{x_0}^*(0) - \frac{|u_{x_0}^*(0)|^2}{2\gamma} =$$

$$\max_{u \in \mathbf{R}^m} \left\{ -p_0 \cdot G(x_0)u - \frac{|u|^2}{2\gamma} \right\} =$$

$$\max_{u \in \mathbf{R}^m} \left\{ -\frac{1}{2\gamma}(\gamma p_0 G(x_0) + u)^2 + \frac{\gamma}{2}(p_0 G(x_0))^2 \right\}.$$

The maximum in the righthand side of the previous inequality is finally obtained for $u_{x_0}^*(0) = -\gamma(p_0 G(x_0))^{\dagger}$. \square

The definition of asymptotic controllability is now given.

System (1) is said to be (locally) asymptotically controllable if

(i) there exists $\eta > 0$ such that for each x_0 with $|x_0| < \eta$ there exists an input $u_{x_0} \in \mathcal{U}$ such that

$$\lim_{t \rightarrow +\infty} \varphi(t; x_0, u_{x_0}(\cdot)) = 0,$$

(ii) for each $\epsilon > 0$ there exists $\delta > 0$ ($\delta \leq \eta$) such that if $|x_0| < \delta$, there exists a control $u_{x_0}(\cdot)$ as in (i) such that $|\varphi(t; x_0, u_{x_0}(\cdot))| < \epsilon$ for each $t \geq 0$.

Moreover, there must exist $\delta_0 > 0$ and $\eta_0 > 0$ such that if $|x_0| < \delta_0$, then $u_{x_0}(t)$ can be chosen in such a way that $|u_{x_0}(t)| < \eta_0$ for $t \geq 0$.

Theorem 2. If (B) is solvable and V is locally Lipschitz continuous then (1) is (locally) asymptotically controllable.

PROOF. For any initial condition x_0 , system (1) with $u = u_{x_0}^*(t)$ is considered. This system is proved to satisfy Lyapunov stability and attractivity.

The value function V is proved to be a Lyapunov function for the system, i.e. it is proved that it decreases along solutions of the system. Note that system (1) with $u = u_{x_0}^*(t)$ should be seen as a system with memory. Note also that, as in the memory free case, the decrease of the positive definite function V along solutions of the system guarantees Lyapunov stability of the equilibrium position.

In the following part of the proof $\varphi(\cdot; x_0, u_{x_0}^*(\cdot))$ is simply denoted by $\varphi(\cdot)$. Consider the function $V \circ \varphi(\cdot)$. V is locally Lipschitz continuous and φ is absolutely continuous, then $V \circ \varphi(\cdot)$ is absolutely continuous and for almost all t there exists $\frac{d^+}{dt} V \circ \varphi(t)$. In the following it is proved that $\frac{d^+}{dt} V \circ \varphi(t) \leq 0$ for almost all t .

As a consequence of the dynamic programming principle $u_{x_0}^*(t)$ is optimal on any interval $[\bar{t}, +\infty)$, so that

$$\begin{aligned} \frac{V(\varphi(\bar{t} + T)) - V(\varphi(\bar{t}))}{T} = \\ -\frac{1}{2T} \int_{\bar{t}}^{\bar{t}+T} \left(h(\varphi(s)) + \frac{|u_{x_0}^*(s)|^2}{\gamma} \right) ds. \end{aligned}$$

The limits as $T \rightarrow 0^+$ of both the sides of the previous equality exist for almost all \bar{t} and then $\frac{d^+}{dt} V \circ \varphi(\bar{t}) = -\frac{h(\varphi(\bar{t}))}{2} - \frac{|u_{x_0}^*(\bar{t})|^2}{2\gamma} \leq 0$.

Attractivity of the origin is a consequence of Lemma 4. Finally boundedness of the control is a consequence of the stability proved in (i), the feedback form (3) obtained for the optimal control and the fact that the set-valued map $\partial_C V$ is upper semi-continuous with compact and convex values. \square

Note that from the proof of the theorem it follows that the control law which gives (local) asymptotic controllability is the optimal control.

As already mentioned, from the result in (Ancona and Bressan, 1999) it follows that if (B) is solvable, then (1) is Carathéodory stabilizable. An interesting question is whether Carathéodory stabilization can be achieved by means of a feedback in the damping form.

The problem of Filippov stabilizability is now considered.

It has already been remarked that the optimal feedback law can be expressed in a damping form by means of a selection of $\partial_C V$. Such a selection is the function from \mathbf{R}^n to \mathbf{R}^n which associates to any x_0 a vector p_0 in $\partial_C V(x_0)$ according to Lemma 6. Nothing is known on the regularity of this function, so its effect if implemented in the control system is also unknown. It is natural to consider the same feedback law that is considered in the case V is C^1 , i.e. $k(x) = -\gamma(\nabla V(x)G(x))^{\dagger}$. The corresponding implemented system is

$$\dot{x} = f(x) - \gamma G(x)(\nabla V(x)G(x))^{\dagger}. \quad (4)$$

Thanks to Rademacher's theorem, the righthand side of this system is almost everywhere defined. Moreover it is locally bounded and measurable (see (Bacciotti and Ceragioli, 1999)), then it makes sense to consider its Filippov solutions, i.e. solutions of the differential inclusion

$$\dot{x} \in f(x) - \gamma G(x)(\partial_C V(x)G(x))^{\dagger} \quad (5)$$

(see (Filippov, 1988) for the definition of Filippov solutions and (Paden and Sastry, 1997) for the computation of (5)).

Note that from Proposition 1 it follows that the differential inclusion (5) is weakly asymptotically stable, in the sense that there exists one of its solutions (the optimal solution of problem (B)) which converges to the origin in a Lyapunov stable way. As already mentioned, in order to have strong asymptotic stability of the differential inclusion (5), i.e. Filippov stabilizability of the control system, some extra conditions are needed.

In particular, the value function V is assumed to be Clarke regular (see (Clarke, 1983)).

Theorem 3. Assume that (B) is solvable, V is locally Lipschitz continuous, Clarke regular and satisfies the following condition: for each $x \in \mathbf{R}^n$, for each fixed $i = 1, \dots, m$, $\underline{D}_C V(x, g_i(x))$ and $\overline{D}_C V(x, g_i(x))$ have the same sign, and, in addition

$$\begin{aligned} & \text{if } \underline{D}_C V(x, g_i(x)) \geq 0 \text{ then} \\ & \overline{D}_C V(x, g_i(x)) \leq 2\underline{D}_C V(x, g_i(x)) \end{aligned} \quad (6)$$

$$\begin{aligned} & \text{if } \overline{D}_C V(x, g_i(x)) \leq 0 \text{ then} \\ & \underline{D}_C V(x, g_i(x)) \geq 2\overline{D}_C V(x, g_i(x)). \end{aligned} \quad (7)$$

Then (1) is Filippov stabilizable.

PROOF. Due to a result in (Bacciotti and Ceragioli, 1999), in order to prove asymptotic stability of (5), it is sufficient to prove that, for each fixed x , if $r \in \mathbf{R}$ is such that there exists $p \in \partial_C V(x)$ such that $r = q \cdot (f(x) - \gamma G(x)(pG(x))^{\dagger})$ for all $q \in \partial_C V(x)$, then $r < 0$. Let $r \in \mathbf{R}$ have the previous property. According to Lemma 6 q can be chosen equal to p_0 so that $r = p_0 \cdot (f(x) - \gamma G(x)(p_0 G(x))^{\dagger})$. From Lemma 6 and (3) it follows that

$$p_0 \cdot f(x) - \gamma(p_0 G(x))^2 = -\frac{h(x)}{2} - \frac{|u_x(0)|^2}{2\gamma}$$

and then

$$r = -\frac{h(x)}{2} - \frac{\gamma}{2} \sum_{i=1}^m (p_0 \cdot g_i(x)) [2p \cdot g_i(x) - p_0 \cdot g_i(x)].$$

Fix \bar{x} and \bar{t} and suppose that they are such that condition (7) is satisfied (analogous considerations may be repeated for (6)). Let p be fixed arbitrarily in $\partial_C V(\bar{x})$ and let $a = p_0 \cdot g_{\bar{i}}(\bar{x})$, $b = p \cdot g_{\bar{i}}(\bar{x})$. It is now proved that $2ab - a^2 = a(2b - a) \geq 0$. From condition (7) it follows that both a and b are negative and $\min\{p \cdot g_{\bar{i}}(\bar{x}), p \in \partial_C V(\bar{x})\} \geq 2 \max\{p \cdot g_{\bar{i}}(\bar{x}), p \in \partial_C V(\bar{x})\}$, then, in particular $2b - a \leq 0$. Since analogous considerations can be repeated for each $i = 1, \dots, m$ it finally follows that $r < 0$. \square

Note that condition (6,7) is evidently not the less restrictive condition in order to get Filippov stabilizability. Less restrictive conditions may also involve the function h .

3. SOME LEMMAS

The first lemma (that is only stated by sake of brevity), can be seen as a sort of Barbalat's lemma for control systems. In particular, it states that

solvability of the minimization problem implies attractivity of the origin for the system in which the control is the optimal one.

Lemma 4. Let x_0 be fixed. Assume that for some admissible input $u(t)$, $J(x_0, u(\cdot)) < \infty$. Then,

$$\lim_{t \rightarrow +\infty} \varphi(t; x_0, u(\cdot)) = 0.$$

It is now proved a lemma which works as a chain rule. Let $\dot{\varphi}^+(t)$ denote the right derivative of the function $\varphi : \mathbf{R} \rightarrow \mathbf{R}^n$ at the point t and by $\partial_C V(x)$ Clarke gradient of the function $V : \mathbf{R}^n \rightarrow \mathbf{R}$ at the point x (see (Clarke, 1983)).

Lemma 5. Let $I \subseteq \mathbf{R}$, $\varphi : I \rightarrow \mathbf{R}^n$ be an absolutely continuous function and $V : \mathbf{R}^n \rightarrow \mathbf{R}$ be a Lipschitz continuous function. If $\bar{t} \in I$ is such that there exists both $\frac{d^+}{dt} V(\varphi(\bar{t}))$ and $\dot{\varphi}^+(\bar{t})$, then there exists $p_0 \in \partial_C V(\varphi(\bar{t}))$ such that $\frac{d^+}{dt} V(\varphi(\bar{t})) = p_0 \cdot \dot{\varphi}^+(\bar{t})$.

PROOF. Consider the differential quotient:

$$\frac{V(\varphi(\bar{t}+h)) - V(\varphi(\bar{t}))}{h} = \frac{1}{h} \{V(\varphi(\bar{t}) + h\dot{\varphi}^+(\bar{t}) + o(h)) - V(\varphi(\bar{t}) + h\dot{\varphi}^+(\bar{t})) + V(\varphi(\bar{t}) + h\dot{\varphi}^+(\bar{t})) - V(\varphi(\bar{t}))\}.$$

Since V is locally Lipschitz continuous, there exists $L > 0$ such that

$$\left| \frac{V(\varphi(\bar{t}+h\dot{\varphi}^+(\bar{t})+o(h)) - V(\varphi(\bar{t})+h\dot{\varphi}^+(\bar{t}))}{h} \right| \leq L o(1),$$

then

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \{V(\varphi(\bar{t}) + h\dot{\varphi}^+(\bar{t}) + o(h)) - V(\varphi(\bar{t}) + h\dot{\varphi}^+(\bar{t}))\} = 0.$$

By assumption, there exists

$$\frac{d^+}{dt} V(\varphi(\bar{t})) = \lim_{h \rightarrow 0^+} \frac{V(\varphi(\bar{t}+h)) - V(\varphi(\bar{t}))}{h},$$

then there exists also

$$\lim_{h \rightarrow 0^+} \frac{V(\varphi(\bar{t})+h\dot{\varphi}^+(\bar{t})) - V(\varphi(\bar{t}))}{h}.$$

Let $\dot{V}(\varphi(\bar{t})) = \{p \cdot \dot{\varphi}^+(\bar{t}), p \in \partial_C V(\varphi(\bar{t}))\}$ and recall that $\underline{D}_C V(\varphi(\bar{t}), \dot{\varphi}^+(\bar{t})) = \min \mathcal{V}(\varphi(\bar{t}))$ and $\overline{D}_C V(\varphi(\bar{t}), \dot{\varphi}^+(\bar{t})) = \max \mathcal{V}(\varphi(\bar{t}))$. It follows that

$$\begin{aligned} \frac{d^+}{dt} V(\varphi(\bar{t})) &= \lim_{h \rightarrow 0^+} \frac{V(\varphi(\bar{t}) + h\dot{\varphi}^+(\bar{t})) - V(\varphi(\bar{t}))}{h} \\ &= \begin{cases} \liminf_{h \rightarrow 0^+} \frac{V(\varphi(\bar{t})+h\dot{\varphi}^+(\bar{t})) - V(\varphi(\bar{t}))}{h} \\ \limsup_{h \rightarrow 0^+} \frac{V(\varphi(\bar{t})+h\dot{\varphi}^+(\bar{t})) - V(\varphi(\bar{t}))}{h} \end{cases} \\ &= \begin{cases} \underline{D}^+ V(\varphi(\bar{t}), \dot{\varphi}^+(\bar{t})) \geq \underline{D}_C V(\varphi(\bar{t}), \dot{\varphi}^+(\bar{t})) \\ \overline{D}^+ V(\varphi(\bar{t}), \dot{\varphi}^+(\bar{t})) \leq \overline{D}_C V(\varphi(\bar{t}), \dot{\varphi}^+(\bar{t})) \end{cases}. \end{aligned}$$

Due to the fact that the set-valued map $\partial_C V$ has compact and convex values, $\dot{V}(\varphi(\bar{t})) \subset \mathbf{R}$ is a bounded and closed interval. From this fact it follows that $\frac{d^+}{dt} V(\varphi(\bar{t})) \in \dot{V}(\varphi(\bar{t}))$ and then the thesis. \square

The following two lemmas are essentially based on the dynamic programming principle.

Lemma 6. Assume that (B) is solvable and V is locally Lipschitz continuous. For each $x_0 \in \mathbf{R}^n$ there exists $p_0 \in \partial_C V(x_0)$ such that

$$p_0 \cdot (f(x_0) + G(x_0)u_{x_0}^*(0)) = -\frac{h(x_0)}{2} - \frac{|u_{x_0}^*(0)|^2}{2\gamma}$$

PROOF. Due to the definition of the value function and to dynamic programming principle

$$\begin{aligned} \frac{V(\varphi(T; x_0, u_{x_0}^*(\cdot))) - V(x_0)}{T} = \\ -\frac{1}{2} \frac{1}{T} \int_0^T \left(h(\varphi(t; x_0, u_{x_0}^*(\cdot))) + \frac{|u_{x_0}^*(t)|^2}{\gamma} \right) dt. \end{aligned}$$

By the continuity of h and φ and right-continuity of $u_{x_0}^*(\cdot)$, there exists

$$\begin{aligned} \lim_{T \rightarrow 0^+} -\frac{1}{2} \frac{1}{T} \int_0^T \left(h(\varphi(t; x_0, u_{x_0}^*(\cdot))) + \frac{|u_{x_0}^*(t)|^2}{\gamma} \right) dt \\ = -\frac{h(x_0)}{2} - \frac{|u_{x_0}^*(0)|^2}{2\gamma}. \end{aligned}$$

Then there exists also $\frac{d^+}{dt} V(\varphi(0; x_0, u_{x_0}^*(\cdot))) = \lim_{T \rightarrow 0^+} \frac{V(\varphi(T; x_0, u_{x_0}^*(\cdot))) - V(x_0)}{T}$.

By Lemma 5 there exists $p_0 \in \partial_C V(x_0)$ such that

$$\frac{d^+}{dt} V(\varphi(0; x_0, u_{x_0}^*(\cdot))) = p_0 \cdot (f(x_0) + G(x_0)u_{x_0}^*(0)),$$

then the thesis follows. \square

Lemma 7. Assume that (B) is solvable and V is locally Lipschitz continuous. For each $x_0 \in \mathbf{R}^n$, for each $u_0 \in \mathbf{R}^m$ and for each $p \in \partial_C V(x_0)$ it holds

$$p \cdot (f(x_0) + G(x_0)u_0) \geq -\frac{h(x_0)}{2} - \frac{|u_0|^2}{2\gamma}$$

PROOF. Consider the control law

$$u(t) = \begin{cases} u_0 & t \in [0, T] \\ u_y^*(t - T) & t > T \end{cases}$$

and let $\eta = \varphi(T; y, u_0)$.

By the dynamic programming principle for each y

$$V(y) \leq \frac{1}{2} \int_0^T \left(h(\varphi(t; y, u_0)) + \frac{|u_0|^2}{\gamma} \right) dt + V(\eta)$$

and then

$$\begin{aligned} \liminf_{T \rightarrow 0, y \rightarrow x_0} -\frac{1}{2} \frac{1}{T} \int_0^T \left(h(\varphi(t; y, u_0)) + \frac{|u_0|^2}{\gamma} \right) dt \\ \leq \liminf_{T \rightarrow 0, y \rightarrow x_0} \frac{V(\eta) - V(y)}{T}. \end{aligned}$$

Consider now the two handsides of the inequality separately.

Left-hand side.

There exists $\theta \in [0, T]$ such that

$$\begin{aligned} -\frac{1}{2} \frac{1}{T} \int_0^T \left(h(\varphi(t; y, u_0)) + \frac{|u_0|^2}{\gamma} \right) dt = \\ -\frac{h(\varphi(\theta; y, u_0))}{2} - \frac{|u_0|^2}{2\gamma}. \end{aligned}$$

By continuous dependence of solutions of Cauchy problem from initial data it follows that

$$\begin{aligned} \liminf_{T \rightarrow 0, y \rightarrow x_0} \varphi(\theta; y, u_0) = x_0 \text{ and then} \\ \liminf_{T \rightarrow 0, y \rightarrow x_0} -\frac{1}{2} \frac{1}{T} \int_0^T \left(h(\varphi(t; y, u_0)) + \frac{|u_0(t)|^2}{\gamma} \right) dt \\ = -\frac{h(x_0)}{2} - \frac{|u_0|^2}{2\gamma}. \end{aligned}$$

Righthand side. Denote

$$A(y, T) = \frac{1}{T} \{V(y + (f(y) + G(y)u_0)T + o(T)) - V(y + (f(y) + G(y)u_0)T)\},$$

$$B(y, T) = \frac{1}{T} \{V(y + [(f(x_0) + G(x_0)u_0 + Df(x_0)(y - x_0) + u_0 DG(x_0)(y - x_0) + o(\|y - x_0\|))]T) - V(y + [f(x_0) + G(x_0)u_0]T)\},$$

$$C(y, T) = \frac{V(y + [f(x_0) + G(x_0)u_0]T) - V(y)}{T}.$$

It follows that

$$\frac{V(\eta) - V(y)}{T} = A(y, T) + B(y, T) + C(y, T).$$

By Lipschitz continuity of V there exists $L > 0$ such that $|A(y, T)| \leq L \frac{o(T)}{T}$, then

$\lim_{T \rightarrow 0, y \rightarrow x_0} A(y, T) = 0$. Moreover

$$|B(y, T)| \leq L \{(|Df(x_0)| + \|u_0\| \|DG(x_0)\|) \|y - x_0\| + o(\|y - x_0\|)\} \text{ and then, for all } T, \lim_{y \rightarrow x_0} B(y, T) = 0. \text{ Finally } \liminf_{T \rightarrow 0, y \rightarrow x_0} C(y, T) = \underline{D}C V(x_0, f(x_0) + G(x_0)u_0) \leq p \cdot (f(x_0) + G(x_0)u_0) \text{ for all } p \in \partial_C V(x_0).$$

By comparing the two sides of the inequality the thesis is obtained. \square

4. CONCLUSION

In this paper the relationship between an optimal regulation problem on the infinite horizon with nonsmooth value function and (possibly discontinuous) stabilization of affine input control systems has been studied. In particular it has been given a condition which guarantees stabilizability in the sense of Filippov solutions.

5. REFERENCES

Ancona, F. and A. Bressan (1999). Patchy vector fields and asymptotic stabilization. *Esaim-Cocv* 4, 445-472.

- Artstein, Z. (1983). Stabilization with relaxed controls. *Nonlinear Analysis* **7**, 1163–1173.
- Bacciotti, A. and F. Ceragioli (1999). Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions. *Esaim-Cocv* **4**, 361–376.
- Bacciotti, A. and F. Ceragioli (2001). Nonlinear stabilization and optimal regulation: the smooth case. *Preprint del Dipartimento di Matematica del Politecnico di Torino*.
- Bernstein, D.S. (1993). Nonquadratic cost and nonlinear feedback control. *International Journal of Robust and Nonlinear Control* **3**, 211–229.
- Brockett, R. (1983). Asymptotic stability and feedback stabilization. In: *Differential Geometric Control Theory* (R. Millman R. Brockett and H. Sussmann, Eds.). Birkhäuser. Boston.
- Clarke, F.H. (1983). *Optimization and Nonsmooth Analysis*. Wiley and Sons. New York.
- Clarke, F.H., Yu.S. Ledyaev, E.D. Sontag and A.I. Subbotin (1997). Asymptotic controllability implies feedback stabilization. *IEEE Trans. Automat. Control* **42**, 1394–1407.
- Da Lio, F. (2000). On the Bellmann equation for infinite horizon problems with unbounded cost. *Applied Mathematics and Optimization* **41**, 171–197.
- Filippov, A.F. (1988). *Differential Equations with Discontinuous Right Hand sides*. Kluwer. Dordrecht.
- Paden, B. and S. Sastry (1997). A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators. *IEEE Transaction on Circuits and Systems* **34**, 73–81.
- Rifford, L. (2000). Semiconcave control-Lyapunov functions and stabilizing feedbacks. preprint Università di Padova.
- van der Schaft, A.J. (1992). L_2 -gain analysis of nonlinear systems and nonlinear state feedback H_∞ control. *IEEE Transaction on Automatic Control* **37**, 770–784.