SYSTEMS WITH CONTINUOUS TIME AND DISCRETE TIME COMPONENTS

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We introduce a class of difference/differential equations which is sufficiently large to include systems of interest for applications, but at the same time sufficiently easy to handle. In this framework, we give in particular a detailed and rather complete study of the asymptotic behavior of pairs of oscillators. We finally introduce appropriate notions of stability and extend Liapunov first and second theorem.

Keywords: Hybrid systems; Stability; Liapunov functions.

1. Introduction

Motivated by a wide range of industrial and technological applications, there has been a rapid growth of interest in the recent engineering literature about hybrid systems. The term "hybrid" often is applied informally to denote systems which combine subsystems of different nature, and whose time evolution is characterized by discontinuities in the state (impulses) and/or in the velocity (switches). A large variety of systems with very complex behavior fall in this class. For this reason, it is hard to figure an axiomatic definition of hybrid system. Recently, there have been some interesting attempts, see Refs. 1–6. However, we remark that all these definitions turn out to be extremely formal and abstract (this is the obvious price to be payed if we want to include more and more general classes of systems), and hence difficult to handle.

In this paper we consider a class of systems, large enough to include, as particular cases, finite dimensional continuous time systems, discrete time systems, open loop switched systems, feedback systems with quantized control, sample data systems, certain types of delayed differential equations and certain types of hybrid systems with timed automata. They will be called *systems with continuous time and discrete time components.*

Systems with continuous time and discrete time components admit a simple mathematical representation and hence, although their generality is limited (for instance, they cannot account for impulse effects), they have the advantage of being rather concrete. On the other hand, we notice that certain aspects of systems with continuous time and discrete time components are not covered by the definitions given in Ref. 3 (where switches can be interpreted as state discontinuities but not as changes of the dynamical rules), and in Refs. 1,2,4,5 (where the state space of the discrete time component is finite).

As in Refs. 3,4,6, we address the stability problem. Far from being surprising, our results are natural generalizations of Liapunov first and second theorems. Nevertheless, their proofs are not completely obvious and cannot be deduced from the existing literature. In particular, we note that our results, compared with Ref. 6, are more precise and require less conservative assumptions, in spite of a less general setting.

We now shortly explain the organization of the paper. Section 2 contains the definition of system with continuous time and discrete time components, comments on its generality and further comparisons with analogous definitions available in the literature. In Section 3 we discuss in detail an example: we will see that in spite of a very simple structure, the asymptotic behavior of a system with continuous time and discrete time components may be very complex. The notions of stability and asymptotic stability are stated in Section 4. Section 5 is devoted to the extension of Liapunov first theorem: we give a proof of it and some other remarks. Two slight different extensions of Liapunov second theorem are finally presented in Section 6.

2. Description of the model

We are interested in objects defined by the following set of data:

- an integer $n \geq 1$;
- a locally compact metric space Q
- a continuous map $f(x, q) : \mathbf{R}^n \times Q \to \mathbf{R}^n$;
- a continuous map $g(x, q) : \mathbf{R}^n \times Q \to Q$;
- a sequence $\{d_k\}$, such that $d_k > 0$ for each $k = 0, 1, 2, \ldots$ and $\lim_{k\to+\infty} \sum_{i=0}^{k} d_i = +\infty$.

Throughout this paper, such objects are called *systems with continuous time and discrete time components* (in short, CTDTC-systems), and conventionally represented by writing

$$\begin{cases} \dot{x} = f(x, q_k) \\ q_{k+1} = g(x, q_k) \ . \end{cases} \tag{1}$$

For any given $\bar{t} \in \mathbf{R}$, we set $\tau_0 = \bar{t}$, $\tau_1 = \tau_0 + d_0$, $\tau_2 = \tau_1 + d_1, \ldots, \tau_{k+1} = \tau_k + d_k, \ldots$. By a *solution of (1) corresponding to the initial condition* $(\bar{t}, \bar{x}, \bar{q}) \in \mathbf{R} \times \mathbf{R}^n \times Q$, we mean any pair $(\varphi(t), \{u_k\})$ such that:

- $\varphi(t) : [\bar{t}, +\infty) \to \mathbf{R}^n$ is a curve with $\varphi(\bar{t}) = \bar{x}$, which is assumed to be continuous at every $t \geq \bar{t}$;
- $\{u_k\}$ is a sequence in Q, with $u_0 = \bar{q}$;
- for each $k = 0, 1, 2, \ldots$ and each $t \in (\tau_k, \tau_{k+1})$, $\varphi(t)$ is differentiable and

$$\dot{\varphi}(t) = f(\varphi(t), u_k) \ ;$$

- for each $k = 0, 1, 2, \ldots$

$$u_{k+1} = g(\varphi(\tau_{k+1}), u_k) \ .$$

The idea underlying this notion of solution can be intuitively described in this way. Starting from the point $\bar{x}$, the continuous time component evolves according to the differential equation

$$\dot{x} = f(x, \bar{q}) = f(x, u_0)$$

on the interval $[\tau_0, \tau_1]$, while the discrete time component remains unchanged. At the instant τ_1, the discrete time component is updated, according to

$$u_1 = g(\varphi(\tau_1), u_0) \ .$$

Then, on the subsequent interval $[\tau_1, \tau_2]$ the continuous time component evolves according to the new equation

$$\dot{x} = f(x, u_1)$$

and so on. To clear up the notation, it is convenient to introduce the map

$$h(t) = k \quad \text{for} \quad t \in [\tau_k, \tau_{k+1}) \ , \qquad (k = 0, 1, 2, \ldots)$$

so that a solution can be written as a curve

$$t \mapsto (\varphi(t), u_{h(t)}) : [\bar{t}, +\infty) \to \mathbf{R}^n \times Q \ .$$

Note that such a curve is not continuous, in general. Note also that in order to guarantee existence of solutions which are actually defined for each $t \geq \bar{t}$, the continuity of f is not sufficient in general, due to the possible finite escape time phenomenon. To prevent it, we assume that the vector field $f(\cdot, q)$ is complete for each $q \in Q$ (sufficient conditions for completeness are well known and can be found on the more popular handbooks about ordinary differential equations). In what follows, uniqueness of solutions plays no role at all.

Remark 2.1. The following remarks illustrate the generality of CTDTC-systems.

(a) If Q reduces to a singleton, then (1) reduces to a finite dimensional, time-invariant system of ordinary differential equations.
(b) If $n = 0$ and $Q = \mathbf{R}^m$, then (1) reduces to a finite dimensional, discrete time dynamical system.
(c) If $d_k = 1$ for each $k = 0, 1, 2, \ldots$, $Q = \mathbf{R}^n$, $g(x, q) = x$ and $\bar{t} = 0$, then (1) reduces to

$$\dot{x}(t) = f(x(t), x([t])) \tag{2}$$

where $[t]$ denotes the greatest integer less than or equal to t. Note that (2) is a retarded differential equation of the type considered in Ref. 7.
(d) If $g(x, q) = g(x) : \mathbf{R}^n \to Q$, then (1) describes a system with a feedback connection, where the actuator is a digital device which is able to change its value only at the prescribed instants $\tau_0, \tau_1, \ldots$. This situation is similar to what happens in the so-called *quantized control problems*, where one initially starts with a feedback $g(x)$ which can vary continuously and free of constraints, but then the levels of quantization must be found in such a way to preserve the achievement of the control goal.
(e) If $d > 0$ is a fixed time size and $d_k = d$ for each $k = 0, 1, 2, \ldots$, a CTDTC-system can be thought of as sample-data systems.[8]
(f) If Q is a finite set (endowed with the discrete metric) and $g(x, q) = g(q) : Q \to Q$, then the sequence $\{u_k\}$ is independent of the evolu-

tion of x and can be computed in advance. Then (1) reduces to a switched system of the type considered in Refs. 9,10.

Remark 2.2. One feature of our definition of CTDTC-system is that changes in the continuous time dynamics can occur only at the prescribed instants $\tau_0, \tau_1, \ldots$. This feature is shared by similar notions available in the literature (see for instance Refs. 1–4). We point out that if Q is finite, our definition of CTDTC-system can be viewed as a special case of the definition of hybrid system studied in Refs. 2,4, the differences being that here the discrete state transitions are uniquely determined by a function, rather than by a relation (or by a set valued map as in Ref. 3), and the *reset map* is the identity (which implies in particular that systems with impulsive effects are not comprised in (1)). On the other hand, the generalization to sets Q which are not necessarily finite sets, enables us to include a wider range of applications, as indicated by the examples above.

Remark 2.3. Functions f and g do not depend explicitly on time. Nevertheless, because of the constraint on the updating times, the translation of a solution $(\varphi(t+T), u_{h(t+T)})$ in general is no more a solution. In other words, the semigroup property does not hold. Accordingly, we should not expect that a CTDTC-system behaves as a time-invariant one.

3. Oscillatory systems: an example

In spite of its simplicity, a CTDTC-system may exhibit very complex and unexpected behaviors. In this section we discuss with some details the system

$$\begin{cases} \dot{x} = -q_k y \\ \dot{y} = \dfrac{x}{q_k} \\ q_k = \dfrac{1}{q_{k-1}} \end{cases} \tag{3}$$

where $n = 2$, $Q = \{\omega, \frac{1}{\omega}\}$, and $\omega > 1$ is a given real number. Equivalently, we can look at (3) as a switched system formed by the pair of harmonic oscillators

$$\begin{cases} \dot{x} = -\omega y \\ \dot{y} = \dfrac{x}{\omega} \end{cases} \tag{4}$$

and

$$\begin{cases} \dot{x} = -\dfrac{y}{\omega} \\ \dot{y} = \omega x \ . \end{cases} \tag{5}$$

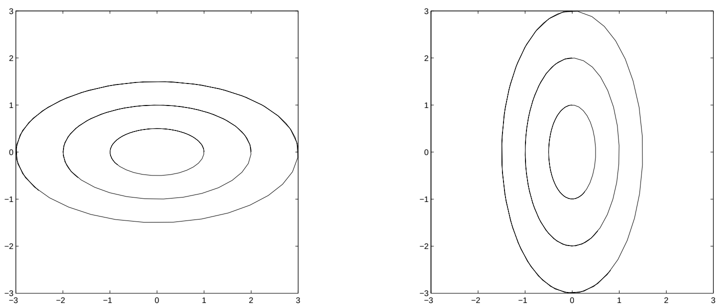

Fig. 1. *Some trajectories of system (4) on the left and some trajectories of system (5) on the right ($\omega = 2$)*

Some trajectories of these systems are plotted in Figure 1. Note that both (4) and (5) are stable at the origin. For simplicity, we limit ourselves to the case where $\bar{t} = 0$, $d_k = T$ (i.e., $\tau_k = kT$), T being a fixed real number, $T \in (0, 2\pi]$. Moreover, without loss of generality we agree that $\bar{q} = \omega$. It is clear and well known[9] that if we choose $\tau_k = k\pi/2$, then the behavior of the solutions depends on the initial condition $(\bar{x}, \bar{y}, \bar{q})$. For instance, if $(\bar{x}, \bar{y}, \bar{q}) = (1, 0, \omega)$ then it is natural to guess that $(x(t), y(t))$ converges to the origin for $t \to +\infty$, while if $(\bar{x}, \bar{y}, \bar{q}) = (0, 1, \omega)$ then $(x(t), y(t))$ becomes larger and larger as $t \to +\infty$. In fact, examples similar to the present one are often invoked in order to show that a switched system may exhibit features which are not recognizable in the singular subsystems.

Here, our purpose is to analyze how the behavior of the system actually depends on the choices of ω and T. We show in particular that for "many" values of ω the system is actually stable[a], and that for the remaining values of ω the behavior of the solution corresponding to a fixed initial condition is extremely sensitive to the choice of T: in particular, we will see that the

[a]Stability of systems with continuous time and discrete time components will be formally defined later; for the moment, the term "stable" is used in the obvious heuristic meaning.

occurrence of trajectories convergent to the origin is extremely rare, and practically impossible to simulate in machine experiments.

We start by computing, for $t = T$, the fundamental matrix of system (4)

$$\Phi_{(4)}(T) = \begin{pmatrix} \cos T & -\omega \sin T \\ \dfrac{\sin T}{\omega} & \cos T \end{pmatrix}$$

and the fundamental matrix of system (5)

$$\Phi_{(5)}(T) = \begin{pmatrix} \cos T & -\dfrac{\sin T}{\omega} \\ \omega \sin T & \cos T \end{pmatrix} .$$

The idea is to look at (3) as a discrete time system of $\mathbf{R}^2$, whose state is updated at the instants $0, 2T, 4T, 6T, \ldots$. More precisely, we study the system

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \Phi(T) \begin{pmatrix} x_{k-1} \\ y_{k-1} \end{pmatrix} , \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \tag{6}$$

where

$$\Phi(T) = \Phi_{(5)}(T)\Phi_{(4)}(T) = \begin{pmatrix} \cos^2 T - \dfrac{\sin^2 T}{\omega^2} & -\dfrac{1+\omega^2}{\omega} \sin T \cos T \\ \dfrac{1+\omega^2}{\omega} \sin T \cos T & \cos^2 T - \omega^2 \sin^2 T \end{pmatrix} .$$

It is clear, and not difficult to prove, that the stability properties of (3) can be deduced from those of (6) (see Ref. 11, Ch. 8). To compute the eigenvalues of $\Phi(T)$, we must solve the equation

$$p_{\Phi(T)}(\lambda) = \lambda^2 - \left[2\cos^2 T - \frac{1+\omega^2}{\omega^2} \sin^2 T \right] \lambda + \frac{(1+\omega^2)^2}{\omega^2} \sin^2 T \cos^2 T$$
$$+ \left(\cos^2 T - \frac{\sin^2 T}{\omega^2} \right) \left(\cos^2 T - \omega^2 \sin^2 T \right)$$

whose discriminant is

$$\Delta(T) = \frac{(1+\omega^2)^2}{\omega^2} \sin^2 T \left[\frac{(1+\omega^2)^2}{\omega^2} \sin^2 T - 4 \right] .$$

Note that $\Delta(T) = 0$ only in the following two cases:

(C_1) $\sin^2 T = 0$, that is $T = \pi$ or $T = 2\pi$.

(C_2) $0 < \sin^2 T = \dfrac{4\omega^2}{(1+\omega^2)^2} < 1$, which gives rise to exactly 4 distinct solutions in $(0, 2\pi)$.

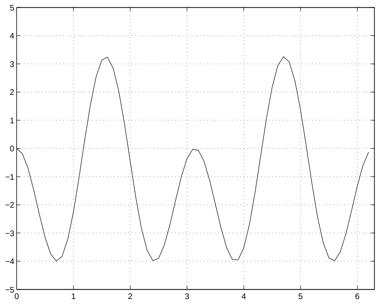

Fig. 2. *Graph of* $\Delta(T)$

The graph of $\Delta(T)$ is plotted in Figure 2 for $\omega = 1.5$. Let us examine first the case $\Delta < 0$. It is not difficult to check that in this case $\Phi(T)$ has a pair of conjugate (distinct) eigenvalues, lying exactly on the boundary of the unit circle of the complex plane. Hence, system (3) is stable.

Now, we pass to consider the case $\Delta(T) > 0$. Here, we have real eigenvalues. It is not difficult to see that one of them is always less than -1, while the other is inside the interval $(-1, 1)$. Hence in this case the system is unstable. More precisely, system (6) has a saddle point at the origin: the stable manifold coincides with the x-axis and the unstable manifold with the y-axis. If we assign an initial condition on the x-axis, we therefore expect that the trajectory converge toward the origin. Surprisingly, this prediction seems to be contradicted by numerical experiments: see Figure 3, where $T = \pi/2$. What actually happens is that, π being an irrational number, round off errors are inevitable in machine computations; as a consequence, it is impossible to keep a simulated trajectory inside the stable manifold when k becomes larger and larger. Note that $\Delta(\pi/2) > 0$ for every $\omega > 1$, and that the measure of $\{T \in (0, 2\pi) : \Delta(T) > 0\}$ goes to zero as $\omega \to 1^+$.

Finally, when $\Delta(T) = 0$ the eigenvalues of $\Phi(T)$ coincide: they are both equal to 1 in case (C_1), and equal to -1 in case (C_2). Moreover, the eigenvalue is simple in case (C_1), so that the system is stable, but not in case (C_2), so that the system is not stable.

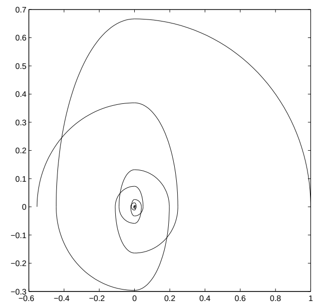

Fig. 3. *Trajectory with* $\omega = 1.5$, $T = \pi/2$ *starting from* $(1, 0)$

Of course, we can look at the problem from an other point of view; for instance we can fix T and take ω as a parameter. Let us considered for instance the choice $T = \pi/4$. Our investigation reveals that with this choice, the system is stable only if $\omega < \sqrt{3 + \sqrt{8}} = 2.4142\ldots$ (see Figures 4 and 5).

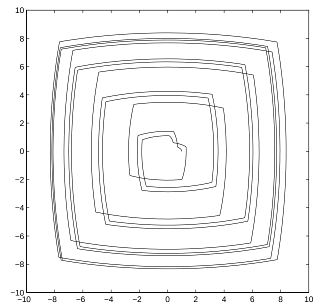

Fig. 4. *Trajectory with* $\omega = 2.4$, $T = \pi/4$ *starting from* $(1, 0)$

Remark 3.1. We know that for certain values of T there are trajectories of system (4) which diverge. Because of periodicity, the same happens of course if we replace T by $T + 2k\pi$ (for each k). This shows that for simple stability we do not have an analogue of Lemma 2 in Ref. 12.

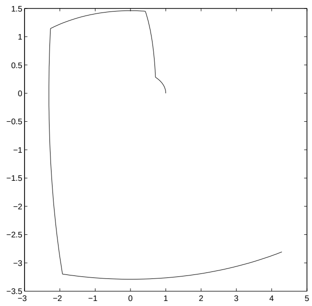

Fig. 5. *Trajectory with $\omega = 2.5$, $T = \pi/4$ starting from* $(1, 0)$

4. Stability notions

Motivated by the example of the previous section, we give some definitions of stability which seem to be appropriate for CTDTC-systems.

Let M be a compact subset of Q. Let us denote $\mathbf{d}_M(q) = \min_{p\in M} \mathbf{d}(p, q)$, where $\mathbf{d}$ is the distance function of Q. The euclidian norm of $x \in \mathbf{R}^n$ is denoted by $|x|_{\mathbf{R}^n}$. Moreover, we set $E = \mathbf{R}^n \times Q$. For $(x, q) \in E$, let us denote $N(x, q) = \max\{|x|_{\mathbf{R}^n}, \mathbf{d}_M(q)\}$. We also denote $B_{\mathbf{R}^n}(r) = \{x : |x|_{\mathbf{R}^n} < r\}$, $B_Q(r) = \{q : \mathbf{d}_M(q) < r\}$ and $B_E(r) = \{(x, q) : N(x, q) < r\}$, where $r > 0$. The subscripts $\mathbf{R}^n$, Q and E will be dropped out, when there is no risk of ambiguity. From now on, we assume

(A_1) $f(0, p) = 0$ for each $p \in M$
(A_2) $g(0, M) \subseteq M$

that is, the origin of $\mathbf{R}^n$ is an equilibrium point for the continuous time component of the system for each $p \in M$, and M is a positively invariant set for the discrete time component, when $x = 0$.

Definition 4.1. A CTDTC-system is *uniformly stable* with respect to $\{0\} \times M$ if for each $\varepsilon > 0$ there exists $\eta > 0$ such that

$$N(\bar{x}, \bar{q}) < \eta \Longrightarrow N(\varphi(t), u_{h(t)}) < \varepsilon$$

for each $\bar{t} \in \mathbf{R}$, each $t \geq \bar{t}$ and each solution $(\varphi(t), u_{h(t)})$ corresponding to the initial condition $(\bar{t}, \bar{x}, \bar{q})$.

A CTDTC-system is *uniformly-uniformly stable* with respect to $\{0\} \times M$ if it is uniformly stable for each choice of the sequence $\{d_k\}$.

Definition 4.2. A CTDTC-system is *(locally) uniformly-uniformly asymptotically stable* with respect to $\{0\} \times M$ if it is uniformly-uniformly stable and, in addition, there exists $\delta_0 > 0$ such that for each $\bar{t} \in \mathbf{R}$

$$N(\bar{x}, \bar{q}) < \delta_0 \Longrightarrow \lim_{t \to +\infty} N(\varphi(t), u_{h(t)}) = 0$$

for each solution $(\varphi(t), u_{h(t)})$ corresponding to the initial condition $(\bar{t}, \bar{x}, \bar{q})$.

We emphasize that the definitions above depend on the choice of the origin, as a special steady state of $\mathbf{R}^n$, and of the set $M \subseteq Q$: in what follows, we omit to mention them explicitly for the sake of simplicity, since no ambiguity is possible. We remark also that Definitions 4.1 and 4.2 could be referred to a more general set $M_0 \times M$, where M_0 is a compact subsets of $\mathbf{R}^n$ not reduced to the origin. Again, the origin has been chosen to simplify the exposition. Instead, as far as the discrete dynamics are concerned, an analogous simplification is not convenient. The reason is that in some applications Q might be a finite set with no distinguished elements. If in addition Q is endowed with the discrete topology, then by (A_2), M plays no role at all in checking stability: in particular, when M is a singleton, the problem becomes trivial. Note that if we are interested in stability of the continuous time component alone, we can take $M = Q$ and look at the discrete time component as a stabilizing device.

5. A sufficient condition for stability

The following result is the natural extension of Liapunov First Theorem to nonlinear CTDTC-systems. With respect to the well known classical case, the interplay between the continuous time dynamics and the discrete time one requires some more care in the proof.

Theorem 5.1. *Let the CTDTC-system (1) be given. Assume that there exists $r > 0$ and a continuous map $V : B_E(r) \to \mathbf{R}$ such that:*

(i) $V(x,q)$ is positive definite at $\{0\} \times M$, that is $V(x,q) \geq 0$ and $V(x,q) = 0$ for each $(x,q) \in B_E(r)$ implies $(x,q) \in \{0\} \times M$;
(ii) for each $q \in B_Q(r)$, the map $x \mapsto V(x,q)$ is of class C^1 on $B_{\mathbf{R}^n}(r)$;
(iii) $\nabla_x V(x,q) f(x,q) \leq 0$, for each $(x,q) \in B_E(r)$;
(iv) $V(x, g(x,q)) \leq V(x,q)$, for each $(x,q) \in B_E(r)$.

Then, the system is uniformly-uniformly stable.

Proof. Let $0 < R < r$ so that V is defined and continuous on $\overline{B_E(R)}$. Let $m_R = \inf_{N(x,q)=R} V(x,q)$. It is clear that the set $\{(x,q) : N(x,q) = R\}$ is compact. Hence, m_R is actually a minimum and, by (i), $m_R > 0$. The set

$$\Omega = \{(x,q) : V(x,q) < m_R\}$$

is open, and $\{0\} \times M \subset \Omega$. Let Ω_0 be the connected component (i.e., the largest connected subset) of Ω which contains $\{0\} \times M$. Of course, $\Omega_0 \subset B_E(R)$. Let us consider the continuous map $\tilde{g}(x,q) = (x, g(x,q)) : E \to E$.

We claim that $\tilde{g}(\Omega_0) \subset \Omega_0$. Indeed, from $(x,q) \in \Omega_0$ and (iv) it follows

$$V(x, g(x,q)) \leq V(x,q) < m_R$$

which in turn implies $\tilde{g}(x,q) \in \Omega$. Moreover, if $p \in M$, then by virtue of (A_2) we have $\tilde{g}(0,p) = (0, g(0,p)) \in \{0\} \times M$; on the other hand, $\tilde{g}(0,p) \in \tilde{g}(\Omega_0)$, since $\{0\} \times M \subset \Omega_0$. This means that

$$\tilde{g}(\Omega_0) \cap \{0\} \times M \neq \emptyset \ .$$

The claim is proven, since the continuous image $\tilde{g}(\Omega_0)$ of the connected set Ω_0 is connected.

Pick now $\varepsilon > 0$ such that $B_E(\varepsilon) \subset \Omega_0 \subset B_E(R)$. Let

$$m_\varepsilon = \inf_{\varepsilon \leq N(x,q) \leq R} V(x,q) \ .$$

Again, we have that m_ε is a minimum and $m_\varepsilon > 0$. Let $\delta > 0$ such that

$$N(x,q) < \delta \Longrightarrow V(x,q) < m_\varepsilon \ .$$

Of course, $\delta < \varepsilon$. Let $\bar{t} \in \mathbf{R}$, $(\bar{x}, \bar{q}) \in B_E(\delta)$, and let $(\varphi(t), u_{h(t)})$ be any solution of (1) such that $(\varphi(\tau_0), u_0) = (\bar{x}, \bar{q})$. We want to prove that $(\varphi(t), u_{h(t)}) \in B_E(\varepsilon)$ for each $t \geq \tau_0$. To this purpose, using the mathematical induction principle, we show that the statement

$$(\varphi(t), u_{h(t)}) \in B_E(\varepsilon) \quad \text{for each} \quad t \in [\tau_k, \tau_{k+1})$$

is true for each $k = 0, 1, 2, \ldots$. We proceed according to the following pattern.

First step ($k = 0$). Using the fact that $V(\bar{x}, \bar{q}) < m_\varepsilon$, we prove that

$$(\varphi(t), u_0) \in B_E(\varepsilon) \quad \text{for each} \quad t \in [\tau_0, \tau_1] \tag{7}$$

and, in addition,

$$(\varphi(\tau_1), u_1) \in B_E(\varepsilon) \quad \text{and } V(\varphi(\tau_1), u_1) < m_\varepsilon \ . \tag{8}$$

Inductive step ($k > 0$). Assuming that $(\varphi(\tau_k), u_k) \in B_E(\varepsilon)$ and $V(\varphi(\tau_k), u_k) < m_\varepsilon$, we prove that

$$(\varphi(t), u_k) \in B_E(\varepsilon) \tag{9}$$

for each $t \in [\tau_k, \tau_{k+1}]$,

$$(\varphi(\tau_{k+1}), u_{k+1}) \in B_E(\varepsilon) \quad \text{and} \quad V(\varphi(\tau_{k+1}), u_{k+1}) < m_\varepsilon \ . \tag{10}$$

Proof of the first step. Let $T \in (\tau_0, \tau_1]$ be such that $(\varphi(t), u_0) \in B_E(\varepsilon)$ for each $t \in [\tau_0, T]$. Because of (iii), we obviously have

$$V(\varphi(t), u_0) \le V(\bar{x}, \bar{u}) < m_\varepsilon \tag{11}$$

for each $t \in [\tau_0, T]$. Since $\varphi(t)$ is continuous, using (11) and arguing by contradiction, it is immediate to check the validity of (7) (note in particular that since $d_M(u_0) < \varepsilon$, $|\varphi(T_0)| = \varepsilon$ for some T_0 implies $N(\varphi(T_0), u_0) = \varepsilon$). In fact, we have $V(\varphi(\tau_1), u_0) < m_\varepsilon$.

From $(\varphi(\tau_1), u_0) \in B_E(\varepsilon) \subset \Omega_0$ it follows $(\varphi(\tau_1), u_1) \in \Omega_0$. Next, by (iv)

$$V(\varphi(\tau_1), u_1) \le V(\varphi(\tau_1), u_0) < m_\varepsilon \ .$$

This in turn implies that $(\varphi(\tau_1), u_1)$ cannot belong to $\Omega_0 \setminus B_E(\varepsilon)$. The validity of (8) is so proven.

Proof of the inductive step. Taking into account the inductive assumption, the proof that $(\varphi(t), u_k)$ remains in $B_E(\varepsilon)$ for $t \in [\tau_k, \tau_{k+1}]$ can be carried out as in the case $k = 0$. In particular, we can conclude that

$$(\varphi(\tau_{k+1}), u_k) \in B_E(\varepsilon) \tag{12}$$

and

$$V(\varphi(\tau_{k+1}), u_k) < m_\varepsilon \; . \tag{13}$$

From (12) it follows that $(\varphi(\tau_{k+1}), u_{k+1}) \in \Omega_0$, and from (13) and (iv) it follows $V(\varphi(\tau_{k+1}), u_{k+1}) < m_\varepsilon$. Hence, $(\varphi(\tau_{k+1}), u_{k+1})$ cannot belong to $\Omega_0 \setminus B_E(\varepsilon)$.

The statement is proven, taking into account that no special role is played in the proof by the sequence $\{d_k\}$. □

Remark 5.1. Conditions (iii), (iv) imply that for each solution $(\varphi(t), u_{h(t)})$, the map $\gamma(t) = V(\varphi(t), u_{h(t)})$ is nonincreasing, at least as far as $(\varphi(t), u_{h(t)})$ remains in $B_E(r)$. This fact will be used later, in the proofs of Theorems 6.1 and 6.2. However, it does not allow us to obtain a simpler proof of Theorem 5.1 (on the contrary of what happens for the classical Liapunov first theorem) since in general $(\varphi(t), u_{h(t)})$ is not continuous.

Remark 5.2. Basically, stability is a local notion. It is transformed in a global one when combined with Lagrange stability.[13] A corresponding global version of Theorem 5.1 can be stated under the additional assumption

(v) for each $q \in Q$, the function $x \mapsto V(x, q)$ is defined for each $x \in \mathbf{R}^n$ and radially unbounded.

It is worthwhile to mention that assumption (v) is actually made in Ref. 6 to obtain a merely local result: it also enables the authors to bypass the more involved aspects of the proof (due to the co-existence of discrete and continuous dynamics) and all become simpler.

As in the classical Liapunov theory, the assumption about the differentiability of $x \mapsto V(x, u)$ can be weakened; in fact, it is sufficient to ask that it is lower semicontinuous; accordingly, an appropriate notion of generalized derivative must be used in (ii) (see Ref. 14).

In a topological context, even the monotonicity condition (ii) can be relaxed.[6] We finally point out that, by allowing time-varying Liapunov functions, a converse of Theorem 5.1 has been obtained in Ref. 6.

Remark 5.3. We can interpret $V(x, q)$ as a family of Liapunov functions indexed by $q \in Q$, and condition (iv) as a kind of compatibility condition for multiple Liapunov functions.[9,15–17] To this respect, we point out that the condition imposed in Refs. 9,15–17 ore formally more general but they require an explicit knowledge of the solutions. Our condition is more conservative, but easier to apply in practice.

Remark 5.4. For the case where Q is finite, conditions (iii) and (iv) of Theorem 5.1 are the same as the conditions 1) and 2) of Theorem IV.1 of Ref. 4. To this respect, we remark that in Ref. 4 the authors are mainly interested in attractivity of certain compact sets (generalization of LaSalle invariance principle); in a sense, our Theorem 5.1 complements Theorem IV.1 of Ref. 4, showing that the selected compact set is not only attractive, but also stable.

Example 5.1. We can apply Theorem 5.1 in order to prove stability for the following system:

$$\begin{cases} \dot{x} = -(1 + q_k^2)y \\ \dot{y} = x \\ q_k = \dfrac{q_{k-1}}{2} \end{cases} \tag{14}$$

where $n = 2$, $Q = \mathbf{R}$ and $M = \{0\}$. The continuous time component of this system can be viewed as a family of harmonic oscillators (however, we notice that the eigenvalues here are different from those of example treated in Section 3). Define $V(x, y, q) = x^2 + (1 + q^2)y^2 + q^2$. It is not difficult to check directly then the assumptions of Theorem 5.1 are satisfied (a trajectory is shown in Figure 6).

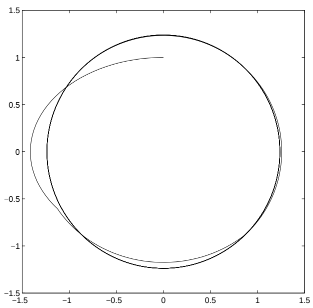

Fig. 6. *Trajectory of system (14) with* $d_k \equiv \pi/2$ *starting from* $(0, 1, 1)$

6. Sufficient conditions for asymptotic stability

In order to obtain sufficient conditions for asymptotic stability, we need strengthened forms of conditions (ii), (iii) and (iv). Recall that a function

$\alpha : [0, r_0) \to [0, +\infty)$ (where r_0 is some positive real number possibly dependent on α) is of class $\mathcal{K}$ if it is continuous, strictly increasing and such that $\alpha(0) = 0$. We are able to prove two theorems under alternative assumptions.

Theorem 6.1. *Let a CTDTC-system (1) be given, and assume that (i), (ii), (iv) hold. Assume further that:*

(ii') for each $q \in B_Q(r)$, the map $x \mapsto V(x, q)$ is of class C^1 on $B_{\mathbf{R}^n}(r)$ moreover, the function $W(x, q) = \nabla_x V(x, q) f(x, q) =$ is continuous on $B_E(r)$;

(iii') $W(x, q)$ is negative definite i.e., $W(x, q) \le 0$ for each $(x, q) \in B_E(r)$ and $W(x, q) = 0$ if and only if $(x, q) \in \{0\} \times M$.

Then, the CTDTC-system is uniformly-uniformly asymptotically stable.

Proof. According to Theorem 5.1, the system is uniformly-uniformly stable; thus, for any positive fixed number $r_0 < r$ we can find $\delta_0 > 0$ such that for each triple $(\bar{t}, \bar{x}, \bar{q})$ and each solution $(\varphi(t), u_{h(t)})$ such that $\varphi(\bar{t}) = \bar{x}$, $u_0 = \bar{q}$, one has

$$N(\bar{x}, \bar{q}) < \delta_0 \Longrightarrow N(\varphi(t), u_{h(t)}) < r_0$$

for each $t \ge \bar{t}$. Let let $\gamma(t) = V(\varphi(t), u_{h(t)})$. Because of (iii), (iv), $\gamma(t)$ is nonincreasing on $[\bar{t}, +\infty)$, so that

$$\lim_{t \to +\infty} \gamma(t) = L \ge 0 \ .$$

and $\gamma(t) \ge L$ for each $t \ge \bar{t}$. Assume by contradiction that $L > 0$. Using (i), we can find $\mu \in (0, \delta_0)$ such that $V(x, q) < L$ for $N(x, q) < \mu$. Of course, we must have

$$\mu \le N(\varphi(t), u_{h(t)}) \le r_0$$

for each $t \ge \bar{t}$. Let now $c = \sup_{\mu \le N(x,q) \le r_0} W(x, q)$. By (v), c is actually a maximum and $c < 0$. Now,

$$\gamma(t) = \gamma(\tau_0) + \sum_{k=0}^{h(t)-1} \int_{\tau_k}^{\tau_{k+1}} W(\varphi(s), u_k)\, ds + \int_{\tau_{h(t)}}^{t} W(\varphi(s), u_{h(t)})\, ds < \gamma(\bar{t}) + ct \quad .$$

This would imply $\lim_{t\to+\infty}\gamma(t) = -\infty$, which is impossible since $\gamma(t) \geq L > 0$. □

Theorem 6.2. *Let a CTDTC-system (1) be given, and assume that (i), (ii), (iv) hold. Assume further that:*

(iv') there exists a map $\rho \in \mathcal{K}$ such that for each $(x, q) \in B_E(r)$

$$V(x, g(x, q)) \leq V(x, q) - \rho(N(x, q))\ .$$

Then, the CTDTC-system is uniformly-uniformly asymptotically stable.

Proof. Let $\gamma(t)$ and $L \geq 0$ be as in the first part of the proof of Theorem 6.1. Conditions (iii), (iv) imply that

$$\gamma(\tau_i) = V(\varphi(\tau_i), u_i) \geq V(\varphi(\tau_{i+1}), u_i) \geq V(\varphi(\tau_{i+1}), u_{i+1}) = \gamma(\tau_{i+1}) \quad (15)$$

for each $i = 0, 1, 2, \ldots$. This yields $\lim_{i\to+\infty} V(\varphi(\tau_{i+1}), u_i) = \lim_{i\to+\infty} V(\varphi(\tau_{i+1}), u_{i+1}) = L$. Now, using (iv') we obtain

$$\begin{aligned} 0 = L - L &= \lim_{i\to+\infty} [V(\varphi(\tau_{i+1}), u_{i+1}) - V(\varphi(\tau_{i+1}), u_i)] \\ &\leq - \lim_{i\to+\infty} \rho(N(\varphi(\tau_{i+1}), u_i)) \leq 0\ . \end{aligned}$$

It follows $\lim_{i\to+\infty} \rho(N(\varphi(\tau_{i+1}), u_i)) = 0$ and hence $\lim_{i\to+\infty} N(\varphi(\tau_{i+1}), u_i) = 0$. Using again (15) and the continuity of V, we conclude that also $\lim_{i\to+\infty} V(\varphi(\tau_{i+1}), u_{i+1}) = \lim_{i\to+\infty} \gamma(\tau_{i+1}) = L = 0$. Recalling that $\gamma(t)$ is nonincreasing, we finally infer $\lim_{t\to+\infty} \gamma(t) = 0$. It is now easy to conclude the proof. □

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