Stability of dynamical polysystems via families of Liapunov functions

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Abstract

In this paper we deal with the problem of stability and asymptotic stability of critical points of dynamical polysystems. We obtain results concerning polysystems with and without constraints, by means of uniform families of Liapunov functions.

Dynamical polysystems may be interpreted as the topological counterpart of switched systems: our results are compared to those previously obtained in the literature.

Keywords: Dynamical polysystem, stability, asymptotic stability, Liapunov function, switched system, constrained switched system.

1 Introduction

Families of vector fields were extensively used in geometric control theory to represent control systems whose admissible inputs are piecewise constant. The topological counterpart of a family of vector fields is called a dynamical polysystem. Roughly speaking, a *dynamical polysystem* is a collection of dynamical systems (in the classical topological sense, see for instance [1]) which will be referred to as *subsystems*. To each dynamical polysystem, we associate a set of continuous curves (called here *admissible evolutions*), generated by glueing together arcs of trajectories of its subsystems. The project of extending stability theory and the method of Liapunov functions to dynamical polysystems in a topological framework was initiated in [2, 3], where stability of dynamical polysystems was defined with respect to their reachable sets, and studied by the aid of what today would be called a "common Liapunov function".

Dynamical polysystems may also be thought as topological representations of switched differential systems. Stability of switched systems has been widely investigated in the recent literature [4] (see also [5, 6, 7, 8]). In particular, Branicky's theorem ([9]) states that stability of a switched system can be established by using "multiple Liapunov functions", which means that

(a) each subsystem has its own Liapunov function, and

(b) an additional condition is imposed, in order to ensure consistency among the Liapunov functions of the single subsystems

(see [10] for early work about this subject). Note that condition (a) alone is not sufficient: counterexamples are well known and can be found for instance in [4, 6].

In this paper we adopt the topological point of view. Basically, we have in mind two possible scenarios. In the first one, the admissible evolutions may be subject to restrictions; in the second one the trajectories are generated in a completely free way.

In the first case, we propose a reformulation of Branicky's theorem and a new proof based on the mathematical induction principle (in [9] the proof is actually sketched only for pairs of dynamical systems). Our statement represents also a slight generalization of the original one. Indeed, Branicky's theorem applies to finite families of dynamical systems, or to families for which the set of indices is compact. We point out that the proof actually depends on a uniformity condition which must be satisfied by the corresponding family of Liapunov functions. This remark allows us to admit also infinite (even non-compact) families of dynamical systems, provided that every admissible evolution exploits only a finite number of them.

For systems without restrictions, we use a variation technique in order to obtain stability results, under the assumption that a Liapunov function is known for at least one of the dynamical systems of the family. We emphasize that this is not a common Liapunov function in general; however, we still need a compatibility condition which involves the totality of trajectories.

The aforementioned results are formally stated and proved in Section 3. In Section 4 we strengthen the previous conditions: we obtain in this way some results about asymptotic stability. The basic material is introduced in Section 2.

Finally we point out some other bibliographic references where stability of switched and/or hybrid systems is studied in a topological framework ([11, 12, 13, 14, 15]). In particular, in [12] the authors prove an interesting extension of Branicky's theorem, which is independent from ours. In fact, combining the two extensions one can obtain a still more general criterion.

2 Basic definitions and preliminaries

In this section we formally define dynamical polysystems and their families of evolutions. We also introduce the definitions of stability used in this paper, and the notions of common Liapunov function and of uniform family of Liapunov functions.

2.1 Driving signals and polysystems

Let U be a nonempty set of indices. By *driving signal* we mean any function $\sigma : [0,+\infty) \to U$ for which the following holds: there exist a divergent sequence of real numbers $0 = t_0 < t_1 < t_2 < \dots$ and a sequence of (not necessarily distinct) indices $u_0, u_1, u_2, \dots \in U$ such that

$$\sigma(t) = u_i \ \text{ for } \ t \in [t_i, t_{i+1}) \ , \qquad i = 0, 1, 2, \dots \ . \tag{1}$$

The numbers $t_0, t_1, t_2, \dots$ will be called *updating times*[1]. The set of all driving signals will be denoted by $\mathcal{U}$.

Now, let X be a locally compact metric space, with distance d. As usual, $B(x,r)$ denotes the ball of radius r centered at x, and $S(x,r)$ denotes the topological boundary of $B(x,r)$.

A *dynamical polysystem* is a pair $(\mathcal{S}, \Sigma)$ where $\mathcal{S}$ is a family of continuous dynamical systems

$$\phi_u^t(x) : \mathbf{R} \times X \to X \ , \qquad u \in U$$

and Σ is a set valued map, which associates to each $x \in X$ a subset $\Sigma(x) \subseteq \mathcal{U}$. The map Σ specifies for each initial state, the set of admissible driving signals. A curve $\varphi(t) : [0,+\infty) \to X$ is said to be an *admissible evolution* of $(\mathcal{S}, \Sigma)$ if there exists $\sigma \in \Sigma(\varphi(0))$ such that

$$\varphi(t) = \phi_{u_i}^t(\varphi(t_i)) \qquad t \in [0, t_{i+1} - t_i)$$

where the u_i's and t_i's are as in (1). Note that for each $x \in X$ and each $\sigma \in \Sigma(x)$ there is a unique admissible evolution, issuing from x and corresponding to σ. It will be denoted by $\Phi_\sigma^t(x)$.

Remark 1 *A dynamical polysystem on $X = \mathbf{R}^n$ is typically described by a finite dimensional, time invariant control system*

$$\dot{x} = f(x, u) \tag{2}$$

where the admissible inputs are piecewise constant. In (2), $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ and U is a nonempty subset of $\mathbf{R}^m$. It is assumed that the vector field $f(\cdot, u)$ is of class C^1 and complete for each $u \in U$.

Remark 2 *Our approach includes two particular cases which have often appeared in recent literature: a first one, when $\Sigma(x_0)$ contains only one driving signal for each x_0 (possibly the same driving signal for each x_0) [9, 12], and a second one, somehow opposite to the latter, when $\Sigma(x_0) = \mathcal{U}$ for each x_0 [5, 8].*

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[1] Since we do not require $u_i \neq u_{i+1}$ for all indices i, the sequence of the updating times is not unique; the advantage of this apparent imprecision is that it allows us to encompass within the same notation, functions with infinitely many changes of values and functions which are constant on some interval $[T, +\infty)$.

2.2 Stability and asymptotic stability

Assume that the members of $\mathcal{S}$ have a common equilibrium $x_e \in X$, that is

$$\phi_u^t(x_e) = x_e \ , \quad \forall u \in U \ , \ \forall t \in \mathbf{R} \ .$$

Definition 1 *We say that* x_e *is* stable *for* $(\mathcal{S}, \Sigma)$ *if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$x \in B(x_e, \delta) \ , \ \sigma \in \Sigma(x) \Longrightarrow \Phi_\sigma^t(x) \in B(x_e, \varepsilon) \ \forall t \geq 0 \ .$$

Definition 2 *A point* x *is* attracted *by* x_e *if for each* $\sigma \in \Sigma(x)$ *one has*

$$\lim_{t \to +\infty} \Phi_\sigma^t(x) = x_e \ .$$

Finally, we say that x_e *is* asymptotically stable *for* $(\mathcal{S}, \Sigma)$ *if it is stable, and the set of points attracted by* x_e *contains a neighborhood of* x_e*.*

For reader's convenience, we recall the definition of common Liapunov function [4, 8].

Definition 3 *Let* $\mathcal{S}$ *be a given family of dynamical systems on* X*. A continuous function* $V : X \to [0, +\infty)$ *is said to be a* common (weak) Liapunov function *for* $\mathcal{S}$ *if it is positive definite*[2] *at* x_e *and the map*

$$t \mapsto V(\phi_u^t(x)) \tag{3}$$

is non-increasing on $[0, +\infty)$*,* $\forall x \in X, \ \forall u \in U$*.*

It is clear that if $\mathcal{S}$ admits a common Liapunov function then x_e is stable for the dynamical polysystem $(\mathcal{S}, \Sigma)$, for any Σ. The following examples show that in general the converse is false.

Example 1 It is well known that even when U is a singleton and $n = 1$, it may be impossible to find a continuous Liapunov function for a stable system. A classical example is given by

$$\dot{x} = f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^3 \sin^2 \frac{1}{x} & \text{if } x \neq 0 \end{cases} \ .$$

∎

Example 2 By a slight modification of the previous example, we can construct a dynamical polysystem of the form (2) with $n = m = 1$, $U = \{1, 2\}$ such that

- $f(\cdot, u)$ admits a C^1 weak Liapunov function V_u $(u = 1, 2)$
- the system is stable

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[2] that is, $V(x_e) = 0$ and $V(x) > 0$ for $x \neq x_e$

- there exist no common weak <u>continuous</u> Liapunov function.

We can take for instance

$$f(x,u)=\begin{cases}0 & \text{if } x=0\\ (-1)^u x^3\sin\frac{1}{x} & \text{if } x\neq 0\end{cases}\ .$$

∎

Remark 3 *The term "common Liapunov function" is used with a different meaning in [16], where the authors are interested in stabilizing switching strategies. More precisely, in [16] the map (3) is required to be non-increasing only on some region* $\Omega_u\subset X$*; the stability result is obtained provided that the regions* Ω_u *overlap and* $\cup_{u\in U}\Omega_u=X$.

Remark 4 *The monotonicity condition about (3) can be weakened, by assuming the existence of a continuous map* $h:[0,+\infty)\to[0,+\infty)$*, with* $h(0)=0$*, such that* $V(\phi_u^t(x))\leq h(V(x))$ *for each* $x\in X$*,* $u\in U$ *and* $t\geq 0$ *(see [12]).*

2.3 Families of Liapunov functions

We need to introduce some new notation. It is convenient to assume that U is endowed with the discrete topology. Thus, it is clear that $\sigma\in\mathcal{U}$ is discontinuous at some $\bar t$, if and only if $\bar t$ is an updating time t_i and $u_i\neq u_{i-1}$.

For each $\sigma\in\mathcal{U}$ and each $u\in U$, we denote by $L_\sigma(u)$ the (possibly empty) set of times $t\in[0,+\infty)$ such that $\sigma(t)=u$ and σ is discontinuous at t. Moreover, we denote by $I_\sigma(u)$ the set $\sigma^{-1}(u)$: it is the (finite or countable, bounded or unbounded) union of intervals where σ takes exactly the value u.

We say that a set valued map of admissible driving signals Σ is *complete* if for each $x\in X$, each $\sigma\in\Sigma(x)$, and each $\tau>0$ we have $\tilde\sigma\in\Sigma(x)$, where

$$\tilde\sigma(t)=\begin{cases}\sigma(t) & \text{if } t<\tau\\ \sigma(\tau) & \text{if } t\geq\tau\end{cases}\ . \tag{4}$$

If Σ is not complete, for each $x\in X$ we denote by $\tilde\Sigma(x)$ the set formed by σ and all the driving signals $\tilde\sigma$ defined in (4), for each $\sigma\in\Sigma(x)$ and each $\tau>0$. Clearly, the set valued map $\tilde\Sigma$ is complete, and $\Sigma(x)\subseteq\tilde\Sigma(x)\subseteq\mathcal{U}$ for each $x\in X$.

We say that a set valued map of admissible driving signals Σ has the *concatenation property* if for each $x\in X$, for each $T\geq 0$ and each $\sigma\in\Sigma(x)$ we also have $\hat\sigma\in\Sigma(\Phi_\sigma^T(x))$, where

$$\hat\sigma(t)=\sigma(t+T)\quad\text{for}\quad t\geq 0\ . \tag{5}$$

If Σ does not have the concatenation property, for each $y\in X$ we denote by $\hat\Sigma(y)$ the set formed by all the driving signals $\hat\sigma$ defined in (5), for each $x\in X$, each $\sigma\in\Sigma(x)$ and each $T\geq 0$ such that $\Phi_\sigma^T(x)=y$. Clearly, the set valued map $\hat\Sigma$ has the concatenation property, and $\Sigma(x)\subseteq\hat\Sigma(x)\subseteq\mathcal{U}$ for each $x\in X$.

Definition 4 *A family* $\{V_u(x)\}_{u\in U}$ *is called a* uniform family of Liapunov functions *for a family of dynamical systems* $\mathcal{S}$ *if the following conditions hold:*

(i) for each $u \in U$, $x \mapsto V_u(x) : X \to \mathbf{R}$ *is continuous and positive definite at* x_e*;*

(ii) for each $u \in U$ *and each* $x \in X$, $t \mapsto V_u(\phi_u^t(x))$ *is non-increasing on* $[0, +\infty)$*;*

(iii) for each $\varepsilon > 0$ *there exists* $\eta > 0$ *such that if* $x \in B(x_e, \eta)$, *then* $V_u(x) < \mu = \inf_{v\in U} \min_{y\in S(x_e,\varepsilon)} V_v(y)$ *for each* $u \in U$.

Note that (iii) implicitly means that $\mu > 0$ for each $\varepsilon > 0$. Next Proposition is straightforward.

Proposition 1 *In Definition 4, if* U *is finite then (i) implies (iii).*

Moreover, we have:

Proposition 2 *Let* U *be a compact metric space, and let* $V_u(x)$ *be continuous with respect to both variables* x *and* u. *Then, in Definition 4, (i) implies (iii).*

Proof Since $V_u(x)$ is continuous with respect to both variables and positive definite with respect to x for each $u \in U$, there exists

$$m = \min\{V_v(y) : \ v \in U \text{ and } y \in S(x_e, \varepsilon)\}$$

and $m > 0$. For each $u_* \in U$ there exist $\delta(u_*) > 0$, $\eta(u_*) > 0$ such that

$$y \in B(x_e, \delta(u_*)),\ u \in B(u_*, \eta(u_*)),\ \Longrightarrow\ 0 = V_{u_*}(x_e) \le V_u(y) < m\ .$$

Using the compactness argument, we can find a finite number of elements $u_1, \ldots, u_N$ such that the open balls $B(u_i, \eta(u_i))$ cover U. Let $\delta = \min\{\delta(u_i)\,,\ i = 1, \ldots, N\}$. Then, for $x \in B(x_e, \delta)$ and any $u \in U$, we have

$$V_u(x) < m$$

as required.

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3 Sufficient conditions for stability

We give two types of results. The first one is more general, the second applies when $\Sigma(x) = \mathcal{U}$ for each $x \in X$.

3.1 Polysystems with possible driving constraints

Let $(\mathcal{S}, \Sigma)$ be a given dynamical polysystem and let $\{V_u(x)\}_{u\in U}$ be a uniform family of Liapunov functions for $\mathcal{S}$. Below, we will use the following compatibility condition:

(C1) $\forall x \in X$, $\forall \sigma \in \Sigma(x)$, $\forall u \in U$, we have

$$t', t'' \in L_\sigma(u)\ ,\quad 0 \le t' < t'' \quad \Longrightarrow \quad V_u(\Phi_\sigma^{t'}(x)) \ge V_u(\Phi_\sigma^{t''}(x))\ .$$

Proposition 3 *If the polysystem* $(\mathcal{S}, \Sigma)$ *satisfies Condition (***C1***), then also the polysystem* $(\mathcal{S}, \tilde{\Sigma})$ *satisfies Condition (***C1***).*

Proof Let $\tilde{\sigma} \in \tilde{\Sigma}(x)\backslash\Sigma(x)$ for some $x \in X$. Then by construction there exists $\sigma \in \Sigma(x)$ and $\tau > 0$ such that $\tilde{\sigma}$ has the form (4). It is sufficient to notice that for each $u \in U$, $L_{\tilde{\sigma}}(u) \subseteq L_\sigma(u)$.

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Proposition 4 *If the polysystem* $(\mathcal{S}, \Sigma)$ *satisfies Condition (***C1***), then also the polysystem* $(\mathcal{S}, \hat{\Sigma})$ *satisfies Condition (***C1***).*

Proof It is sufficient to remark that if $y = \Phi_\sigma^T(x)$, then $\Phi_{\hat{\sigma}}^t(y) = \Phi_\sigma^{t+T}(x)$.

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Let $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbf{N}_* = \mathbf{N} \setminus \{0\}$. For $\sigma \in \mathcal{U}$, let us denote by $\sharp\sigma$ the cardinality of $\sigma([0, +\infty))$. By the definition of driving signal, $\sharp\sigma$ is finite or countable for any σ. For $N \in \mathbf{N}_*$, let $\mathcal{U}^N$ be the set of all $\sigma \in \mathcal{U}$ such that $\sharp\sigma \le N$, and let $\Sigma^N(x) = \Sigma(x) \cap \mathcal{U}^N$.

Theorem 1 *Let the dynamical polysystem* $(\mathcal{S}, \Sigma)$ *be given. Assume that there exists a uniform family of Liapunov functions* $\{V_u(x)\}_{u\in U}$ *for* $\mathcal{S}$*. Assume also that Condition (***C1***) holds. Then, for each* $N \in \mathbf{N}_*$*,* x_e *is stable for the polysystem* $(\mathcal{S}, \Sigma^N)$*.*

Proof By Proposition 3, Condition (**C1**) holds for $(\mathcal{S}, \tilde{\Sigma})$, as well. In fact, we will prove that for each $N \in \mathbf{N}_*$, x_e is stable for $(\mathcal{S}, \tilde{\Sigma}^N)$; the theorem follows, since $\Sigma^N(x) \subseteq \tilde{\Sigma}^N(x)$ for each $x \in X$. The proof exploits the mathematical induction principle.

When $N = 1$, only constant driving signals are admissible. The proof that x_e is stable for $(\mathcal{S}, \tilde{\Sigma}^1)$ can be carried out as in the classical first Liapunov Theorem, taking into account the uniformity condition (iii) of Definition 4.

Now assume the conclusion valid for $(\mathcal{S}, \tilde{\Sigma}^{N-1})$. Fix $\varepsilon > 0$ and let $\eta > 0$ be as in (iii), Definition 4. Then we can find $\delta > 0$ such that

$$\bar{x} \in B(x_e, \delta) \Longrightarrow \Phi_\sigma^t(\bar{x}) \in B(x_e, \eta) \quad \forall t \ge 0 \tag{6}$$

for each $\sigma \in \tilde{\Sigma}^{N-1}(\bar{x})$.

In order to prove that x_e is stable also for $(\mathcal{S}, \tilde{\Sigma}^N)$, we argue by contradiction, assuming that for some $\bar{x} \in B(x_e, \delta)$, some $\sigma \in \tilde{\Sigma}^N(\bar{x})$ and some $T > 0$, $\Phi_\sigma^T(\bar{x}) \in S(x_e, \varepsilon)$. The set $\cup_{u \in U} L_\sigma(u) \cap (0, T)$ is finite; let us denote by $t_1 < t_2 \ldots < t_{k-1}$ its elements, and set by uniformity $t_0 = 0$, $t_k = T$. We also set $u^* = \sigma(t)$ for $t \in [t_{k-1}, t_k)$.

Clearly, $\Phi_\sigma^{t_{k-1}}(\bar{x}) \notin B(x_e, \eta)$; otherwise, we should have

$$V_{u^*}(\Phi_\sigma^{t_{k-1}}(\bar{x})) < \mu \quad \text{and} \quad V_{u^*}(\Phi_\sigma^{t_k}(\bar{x}))V_{u^*}(\phi_{u^*}^{t_k - t_{k-1}}(\Phi_\sigma^{t_{k-1}}(\bar{x}))) \geq \mu \ ,$$

which is impossible by (ii) of Definition 4.

We claim that there exists at least one index $j^* < k-1$ such that

$$u^* = \sigma(t) \quad \text{for} \quad t \in [t_{j^*-1}, t_{j^*}) \ . \tag{7}$$

If this were not true, $\sharp\sigma([t_0, t_{k-1})) \leq N-1$ (recall that $\sigma \in \Sigma^N(\bar{x})$, which implies $\sharp\sigma([t_0, +\infty)) \leq N$). Let

$$\tilde{\sigma}(t) = \begin{cases} \sigma(t) & \text{if } t < t_{k-2} \\ \sigma(t_{k-2}) & \text{if } t \geq t_{k-2} \end{cases} .$$

Clearly $\tilde{\sigma} \in \tilde{\Sigma}^{N-1}(\bar{x})$, hence we must have $\Phi_\sigma^{t_{k-1}}(\bar{x}) \in B(x_e, \eta)$, and this is a contradiction to what established above.

Let $\bar{\jmath}$ be the minimal index with the property (7). By repeating the previous argument, we conclude that $\Phi_\sigma^{t_{\bar{\jmath}}}(\bar{x}) \in B(x_e, \eta)$. Now we invoke Condition (**C1**); we obtain

$$V_{u^*}(\Phi_\sigma^{t_{\bar{\jmath}}}(\bar{x})) \geq V_{u^*}(\Phi_\sigma^{t_{k-1}}(\bar{x}))$$

which in turn implies

$$V_{u^*}(\Phi_\sigma^{t_{k-1}}(\bar{x})) < \mu \ . \tag{8}$$

Finally, by (ii) of Definition 4, from (8) we deduce

$$V_{u^*}(\Phi_\sigma^{t_k}(\bar{x})) < \mu \ . \tag{9}$$

On the other hand, $\Phi_\sigma^{t_k}(\bar{x}) \in S(x_e, \varepsilon)$, so that

$$V_{u^*}(\Phi_\sigma^{t_k}(\bar{x})) \geq \mu \ . \tag{10}$$

Inequalities (9) and (10) contradict each other. We have thus proven that the continuous curve $\Phi_\sigma^t(\bar{x})$ cannot cross the sphere $S(x_e, \varepsilon)$, so that it remains inside $B(x_e, \varepsilon)$ for each $t \geq 0$.

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Remark 5 *By virtue of Propositions 1 and 2, we point out that Theorem 1 contains as particular cases Theorems 2.3 (continuous time case) and 2.7 of Branicky's paper [9].*

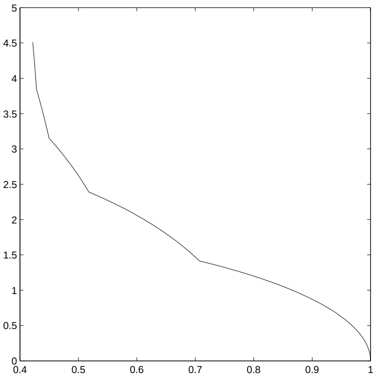

Figure 1: Polysystem with divergent evolution

Under the assumption of Theorem 1, the point x_e is not stable, in general, for the dynamical polysystem $(\mathcal{S}, \Sigma)$. As an example, we can take $X = \mathbf{R}^2$, $U = \mathbf{N}$, and the family of stable dynamical systems defined by the differential equations

$$\begin{cases} \dot{x} = -\frac{y}{4^n} \\ \dot{y} = x \end{cases} \qquad n \in U \ .$$

Let σ be defined by the sequence of updating times

$$t_0 = 0,\ t_n = \frac{n\pi}{2}$$

and the input sequence $u_n = n+1$. Assume that $\Sigma(x_0, y_0) = \{\sigma\}$ for each $(x_0, y_0) \in \mathbf{R}^2$. The assumptions of Theorem 1 are met, with $V_n(x,y) = 4^n x^2 + y^2$; in particular, the compatibility condition (**C1**) is empty, since the driving signal does not take a same value twice. However, the evolution corresponding to the initial state $x = 1, y = 0$ has a divergent norm (see Figure 1).

3.2 Polysystems without driving constraints

Consider a dynamical polysystem $(\mathcal{S},\mathcal{U})$ i.e., with $\Sigma(x) = \mathcal{U}$ for each $x \in X$. This means that all the curves generated by arbitrary switching among the dynamical systems of $\mathcal{S}$ are admissible evolutions. Consider also the following condition:

(**C2**) there exists a continuous function $V : X \to [0, +\infty)$ which is positive definite at x_e, and there exists an index $u^* \in U$ such that: $\forall x \in X$ we have,

$$0 \le t' < t'' \implies V(\phi_{u^*}^{t'}(x)) \ge V(\phi_{u^*}^{t''}(x)) \tag{11}$$

and, $\forall x \in X$, $\forall \sigma \in \mathcal{U}$,

$$t', t'' \in L_\sigma(u^*),\ \ t' < t'' \implies V(\Phi_\sigma^{t'}(x)) \ge V(\Phi_\sigma^{t''}(x)) \ . \tag{12}$$

Theorem 2 *Assume that Condition (***C2***) holds. Then, x_e is stable for the polysystem $(\mathcal{S},\mathcal{U})$.*

Proof Let us suppose that x_e is not stable. Then there exists $\varepsilon > 0$ such that for each $\delta > 0$ there exist $\bar{x} \in B(x_e, \delta)$, $\sigma \in \mathcal{U}$ and $T > 0$ such that

$$d(\Phi^T_\sigma(\bar{x}), x_e) \geq \varepsilon \ . \tag{13}$$

Let us set

$$m = \min_{\frac{\varepsilon}{2} \leq d(x, x_e) \leq 2\varepsilon} V(x)$$

and let us choose $\delta > 0$ so that $V(x) < m$ for each $x \in B(x_e, \delta)$. Let $\bar{x} \in B(x_e, \delta)$, $\sigma \in \mathcal{U}$ and $T > 0$ such that (13) holds. Since the map $t \mapsto \Phi^t_\sigma(\bar{x})$ is continuous, without loss of generality we can assume in addition that $d(\Phi^T_\sigma(\bar{x}), x_e) \leq 2\varepsilon$.

Under these conditions, we want to show that it is always possible to construct a driving signal $\bar{\sigma} \in \mathcal{U}$ that violates the decreasing condition (12).

Several cases are possible. We summarize them in the following scheme.

Case 1. $\exists k_1 \in \mathbf{N}$ such that $t_{k_1} \in L_\sigma(u^*)$ and $d(\Phi^{t_{k_1}}_\sigma(\bar{x}), x_e) < \delta$.

- Case 1.A. $\exists k_2 \in \mathbf{N}$ such that $t_{k_2} \in L_\sigma(u^*)$, $t_{k_1} < t_{k_2}$ and $\frac{\varepsilon}{2} < d(\Phi^{t_{k_2}}_\sigma(\bar{x}), x_e) < 2\varepsilon$.
- Case 1.B. Case 1.A does not hold but $\exists \tau \in I_\sigma(u^*) \setminus L_\sigma(u^*)$ with $\tau > t_{k_1}$ such that $\frac{\varepsilon}{2} < d(\Phi^\tau_\sigma(\bar{x}), x_e) < 2\varepsilon$.
- Case 1.C. Cases 1.A and 1.B do not hold.

Case 2. Case 1 does not hold, that is $d(\Phi^t_\sigma(\bar{x}), x_e) \geq \delta$ for all $t \in L_\sigma(u^*)$.

Continuation of the proof in Case 1.A. Since $d(\Phi^{t_{k_1}}_\sigma(\bar{x}), x_e) < \delta$ and $\frac{\varepsilon}{2} < d(\Phi^{t_{k_2}}_\sigma(\bar{x}), x_e) < 2\varepsilon$, we have

$$V(\Phi^{t_{k_1}}_\sigma(\bar{x})) < m \quad \text{and} \quad V(\Phi^{t_{k_2}}_\sigma(\bar{x})) \geq m$$

with $t_{k_1} < t_{k_2}$, a contradiction to (12).

Continuation of the proof in Case 1.B. If $\tau \in I_\sigma(u^*) \setminus L_\sigma(u^*)$, there exists an updating time $t_i \in L_\sigma(u^*)$, $t_i \geq t_{k_1}$ such that $\sigma(t) = u^*$ for each $t \in [t_i, \tau]$. By assumption (11) we have $V(\Phi^{t_i}_\sigma(\bar{x})) \geq V(\Phi^\tau_\sigma(\bar{x}))) \geq m$ while $V(\Phi^{t_{k_1}}_\sigma(\bar{x})) < m$, a contradiction to (12).

Continuation of the proof in Case 1.C. Using again the continuity of the map $t \mapsto \Phi^t_\sigma(\bar{x})$, we pick $\bar{T}$ such that $\Phi^{\bar{T}}_\sigma(\bar{x}) \in S(x_e, \varepsilon)$. There exist $\theta > 0$ such that

$$\frac{\varepsilon}{2} < d(\phi^s_{u^*}(\Phi^{\bar{T}}_\sigma(\bar{x})), x_e) < 2\varepsilon$$

for each $s \in [0, \theta)$. Define the new driving signal

$$\tilde{\sigma}(t) = \begin{cases} \sigma(t) & \text{if } t \in [0, \bar{T}) \\ u^* & \text{if } t \geq \bar{T} \ . \end{cases}$$

Then $\Phi^t_{\tilde{\sigma}}(\bar{x}) = \Phi^t_\sigma(\bar{x})$ for each $t \in [0, \bar{T})$, and $t_{k_1}, \bar{T} \in L_{\tilde{\sigma}}(u^*)$.

Since $\tilde{\sigma}$ satisfies case 1.A, we get a contradiction.
Continuation of the proof in Case 2. By continuity, there exists $\theta > 0$ such that $\phi^t_{u^*}(\bar{x}) \in B(x_e, \delta)$ for each $t \in [-\theta, 0]$. Let $\bar{y} = \phi^{-\theta}_{u^*}(\bar{x})$. We define a driving signal

$$\bar{\sigma}(s) = \begin{cases} u^* & \text{if } s \in [0, \theta) \\ \sigma(s-\theta) & \text{if } s \geq \theta \ . \end{cases}$$

Let us observe that

- $\Phi^{\theta}_{\bar{\sigma}} = \Phi^0_{\sigma}(\bar{x}) = \bar{x}$;
- $\Phi^{s}_{\bar{\sigma}}(\bar{y}) = \Phi^{s-\theta}_{\sigma}(\bar{x})$ for all $s \geq \theta$;
- $0 \in L_{\bar{\sigma}}(u^*)$ and $\Phi^0_{\bar{\sigma}}(\bar{y}) \in B(x_e, \delta)$.

By replacing σ by $\bar{\sigma}$, we see that the conditions of Case 1 are met. Then we may apply the same procedure as above to get a contradiction.

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Remark 6 *At a first glance, Theorem 2 may look very surprising. It requires the existence of a single Liapunov function V, which is non-increasing along the trajectories of only one of the dynamical systems of $\mathcal{S}$ (note that V is* <u>*not*</u> *a common Liapunov function, in general). The point is that (12) in Condition (***C2***) is required to hold for* <u>*each*</u> *$\sigma \in \mathcal{S}$. To this respect, (***C2***) is stronger than (***C1***), as the following Corollary shows.*

Corollary 1 *If ($\mathcal{S}$,$\mathcal{U}$) satisfies Condition (***C2***) for all $\sigma \in \mathcal{U}$, then x_e is stable for any ($\mathcal{S}, \Sigma$), with $\Sigma \subseteq \mathcal{U}$.*

4 Sufficient conditions for asymptotic stability

In this section we study asymptotic stability of dynamical polysystems. To this purpose, we need to strengthen condition (ii) of Definition 4 and the compatibility conditions (**C1**) and (**C2**).

4.1 Polysystems with possible driving constraints

According to [17], we denote by $\mathcal{K}$ the class of continuous, strictly increasing maps $\rho : [0, +\infty) \to [0, +\infty)$, such that $\rho(0) = 0$. The following result is related to Theorem 3.1 of [4]: the proof is similar, but a more careful insight enables us to improve two aspects of it. First, as in the previous section, we allow families of dynamical systems with infinitely many members; second, the gain function ρ is allowed to be dependent on $u \in U$.

Theorem 3 *Let the polysystem $(\mathcal{S}, \Sigma)$ be given. Assume that there exists a uniform family of Liapunov function for $\mathcal{S}$, with (ii) replaced by* *(ii') for each $u \in U$ and each $x \in X$, $t \mapsto V_u(\phi_u^t(x))$ is strictly decreasing on $[0, +\infty)$.*

Assume also the following compatibility condition:

*(**D1**) For all $u \in U$ there exists $\rho_u \in \mathcal{K}$ such that $\forall x \in X$, $\forall \sigma \in \Sigma(x)$, we have*

$$t', t'' \in L_\sigma(u) \ , \quad 0 \le t' < t'' \quad \Longrightarrow \quad V_u(\Phi_\sigma^{t'}(x)) \ge V_u(\Phi_\sigma^{t''}(x)) + \rho_u(d(\Phi_\sigma^{t'}(x), x_e)) \ .$$

Then, for each $N \in \mathbf{N}_*$, x_e is asymptotically stable for the polysystem $(\mathcal{S}, \Sigma^N)$.

Proof Let $N \in \mathbf{N}_*$ be fixed. Since $(\mathbf{D1}) \Longrightarrow (\mathbf{C1})$, by Theorem 1 the equilibrium x_e is stable for $(\mathcal{S}, \Sigma^N)$. Thus, there exists $\delta_0 > 0$ such that for each $\bar{x} \in B(x_e, \delta_0)$, for each $\sigma \in \Sigma^N(\bar{x})$ and for each $t \ge 0$ we have $\Phi_\sigma^t(\bar{x}) \in B(x_e, 1)$.

Let $\bar{x} \in B(x_e, \delta_0)$ and, for each $\sigma \in \Sigma^N(\bar{x})$, let $L_\sigma = \cup_{u \in U} L_\sigma(u)$ be the set of points where σ is not continuous. We distinguish two cases.

First case. L_σ is empty or finite. Thus, there exist $T \ge 0$ and $\bar{u} \in U$ such that $\Phi_\sigma^t(\bar{x}) = \phi_{\bar{u}}^{t-T}(\bar{y})$ for each $t \ge T$, where $\bar{y} = \Phi_\sigma^T(\bar{x})$. The remaining part of the reasoning exploits a classical argument, to conclude that $\lim_{s \to +\infty} \phi_{\bar{u}}^s(\bar{y}) = x_e$. We sketch it for reader's convenience.

Let $\Omega_{\bar{u}}(\bar{y})$ be the limit set of $\bar{y}$ with respect to the dynamical system $\phi_{\bar{u}}$. From (ii') it follows that the limit

$$\lim_{s \to +\infty} V_{\bar{u}}(\phi_{\bar{u}}^s(\bar{y})) = l$$

exists, and $V_{\bar{u}}(y) = l \ge 0$ for each $y \in \Omega_{\bar{u}}(\bar{y})$. Since $\Omega_{\bar{u}}(\bar{y})$ is invariant with respect to $\phi_{\bar{u}}$, we finally see that $\Omega_{\bar{u}}(\bar{y}) = \{x_e\}$, otherwise we would find a contradiction to (ii'). Now it is not difficult to get the desired conclusion.

Second case. L_σ is countable. Since $\sigma \in \Sigma^N(\bar{x})$, there exists $\bar{u} \in U$ such that $L_\sigma(\bar{u})$ is infinite. Let $t_1, t_2, \ldots$ be a strictly increasing, divergent sequence in $L_\sigma(\bar{u})$. The sequence

$$\{V_{\bar{u}}(\Phi_\sigma^{t_i}(\bar{x}))\}_{i=1,2,\ldots}$$

is decreasing, because of $(\mathbf{D1})$, and non-negative. Let $l = \lim_i V_{\bar{u}}(\Phi_\sigma^{t_i}(\bar{x}))$. Since

$$V_{\bar{u}}(\Phi_\sigma^{t_{i+1}}(\bar{x})) - V_{\bar{u}}(\Phi_\sigma^{t_i}(\bar{x})) \le \rho_{\bar{u}}(d(\Phi_\sigma^{t_i}(\bar{x}), x_e)) \le 0$$

and

$$\lim_{t \to +\infty} V_{\bar{u}}(\Phi_\sigma^{t_{i+1}}(\bar{x})) = \lim_{t \to +\infty} V_{\bar{u}}(\Phi_\sigma^{t_i}(\bar{x})),$$

we have that

$$\lim_i \rho_{\bar{u}}(d(\Phi_\sigma^{t_i}(\bar{x}), x_e)) = 0 \ .$$

This implies $\lim_i \Phi_\sigma^{t_i}(\bar{x}) = x_e$. Now let $\eta > 0$. We already noticed that $(\mathbf{D1}) \Longrightarrow (\mathbf{C1})$. Therefore, by Proposition 4, Condition $(\mathbf{C1})$ holds for the polysystem $(\mathcal{S}, \hat{\Sigma})$, as well. Invoking again Theorem 1, we have that x_e is stable for $(\mathcal{S}, \hat{\Sigma}^N)$, and consequently we can find $0 < \delta < \delta_0$ such that

$$y \in B(x_e, \delta) \Longrightarrow \Phi^t_\varsigma(y) \in B(x_e, \eta)$$

for each $t \geq 0$ and each $\varsigma \in \hat{\Sigma}^N(y)$.

Let i so large that $\Phi^{t_i}_\sigma(\bar{x}) \in B(x_e, \delta)$, and let $\hat{\sigma}(t) = \sigma(t + t_i)$. Clearly, $\hat{\sigma} \in \hat{\Sigma}^N(\Phi^{t_i}_\sigma(\bar{x}))$, which implies that $\Phi^t_\sigma(\bar{x}) \in B(x_e, \eta)$ for each $t \geq t_i$.

In conclusion, we have shown that for each $\eta > 0$ there exists $T = t_i$ such that $\Phi^t_\sigma(\bar{x}) \in B(x_e, \eta)$ for each $t \geq T$. This means that $\lim_{t\to+\infty} \Phi^t_\sigma(\bar{x}) = x_e$, as desired.

▮

4.2 Polysystems without driving constraints

Consider the condition:

(**D2**) there exist a function $\rho \in \mathcal{K}$, a continuous function $V : X \to [0, +\infty)$ which is positive definite at x_e, and an index $u^* \in U$ such that: $\forall x \in X$, we have

$$0 \leq t' < t'' \implies V(\phi^{t'}_{u^*}(x)) > V(\phi^{t''}_{u^*}(x))$$

and $\forall x \in X$, $\forall \sigma \in \mathcal{U}$

$$t', t'' \in L_\sigma(u^*)\ ,\ t' < t'' \implies V(\Phi^{t'}_\sigma(x)) \geq V(\Phi^{t''}_\sigma(x)) + \rho(d(\Phi^{t'}_\sigma(x), x_e))\ .$$

Theorem 4 *Assume that Condition (*__D2__*) holds. Then, x_e is asymptotically stable for the polysystem $(\mathcal{S}, \mathcal{U})$.*

Proof Since (**D2**) implies (**C2**), by Theorem 2 there exists δ such that $\Phi^t_\sigma(\bar{x}) \in B(x_e, 1)$ for each $\bar{x} \in B(x_e, \delta)$, each $\sigma \in \mathcal{U}$ and each $t \geq 0$.

We will prove that there exists an increasing, positively divergent sequence $\{s_k\}_{k\in\mathbf{N}}$ such that $\lim_k \Phi^{s_k}_\sigma(\bar{x}) = x_e$. By contradiction, assume that $\Phi^{s_k}_\sigma(\bar{x})$ does not converge to x_e, for any sequence $\{s_k\}$ with the required properties. Since $\Phi^{s_k}_\sigma(\bar{x})$ remains bounded, we can extract a subsequence, still denoted by $\{s_k\}$, such that

$$\lim_k \Phi^{s_k}_\sigma(\bar{x}) = \bar{y} \neq x_e\ . \tag{14}$$

Without loss of generality, we can assume $\Phi^{s_k}_\sigma(\bar{x}) \neq x_e$ for each $k \in \mathbf{N}$. Let r, R be positive numbers such that $r < d(\bar{y}, x_e) < R$. Since $\lim_{k\to+\infty} \Phi^{s_k}_\sigma(\bar{x}) = \bar{y}$ and $\Phi^{s_k}_\sigma(\bar{x})$ is bounded, there exists $\bar{k}$ such that if $k \geq \bar{k}$, then

$$r < d(\Phi^{s_k}_\sigma(\bar{x}), x_e) < R.$$

By renaming the indices, we may say that the last inequality holds for any $k \geq 0$.

Let $m = \min_{r\leq d(x,x_e)\leq R} V(x)$, $M = \max_{r\leq d(x,x_e)\leq R} V(x)$, $0 < b < \frac{r}{2}$. Let K be an integer greater than $\frac{M-m}{\rho(b)}$, and let finally $a > 0$ be such that $B(\bar{y}, 2a) \subset \{x : r < d(x, x_e) < R\}$. Because of (14), there exists ν such that $k \geq \nu \Longrightarrow d(\Phi^{s_k}_\sigma(\bar{x}), \bar{y}) < a$. It is not restrictive to assume $\nu > K$.

We are now ready to construct a new driving signal $\bar{\sigma}$, by some suitable modifications of σ. After each instant s_k (with $k = 0, \dots, \nu - 1$), we insert an interval of length l on which $\bar{\sigma}$ takes the value u^*. Let $T = s_\nu + \nu l$.

By continuity, we can take l small enough, so that

$$\Phi_{\bar{\sigma}}^{s_1+l}(\bar{x}), \Phi_{\bar{\sigma}}^{s_2+2l}(\bar{x}), \dots, \Phi_{\bar{\sigma}}^{s_{\nu-1}+(\nu-1)l}(\bar{x})$$

remain outside the ball $B(b, x_e)$. By possibly taking a smaller l, we can further assume that $d(\Phi_{\bar{\sigma}}^T(\bar{x}), \bar{y}) < 2a$. This implies in particular that $d(\Phi_{\bar{\sigma}}^T(\bar{x}), x_e) \geq r$.

On the other hand,

$$V(\Phi_{\bar{\sigma}}^{s_0}(\bar{x})) - V(\Phi_{\bar{\sigma}}^T(\bar{x})) \geq \sum_{k=0}^{\nu-1} \rho(d(\Phi_{\bar{\sigma}}^{s_k+kl}(\bar{x})), x_e) \geq \nu\rho(b) \geq K\rho(b) \ .$$

This yields $M - m \geq K\rho(b)$, which contradicts our choice of K. From now on, the proof can be carried out as the proof of Theorem 3.

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