The Lusin area function and admissible convergence of harmonic functions on non-homogeneous trees

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We prove admissible convergence to the boundary of functions that are harmonic on subsets of a non-homogeneous tree equipped with a transition operator that satisfies uniform bounds suitable for transience. The approach is based on a discrete Green formula, suitable estimates for the Green and Poisson kernel and an analogue of the Lusin area function. Estimates of the area function by means of the nontangential maximal function were used by F.Di Biase and this speaker (Zeitschrift 1995) to deal with the $H^p$ theory on trees, with an approach based on good lambda inequalities: what we introduce here is an alternative, more differential approach that yields the admissible convergence.
To start, let us review the continuous case: the Lusin area theorem in the half-plane (or equivalently the disc). Let $f$ be harmonic in the half-plane $\mathbb{H}^+$, for instance. $\omega \in \mathbb{R}$ generic point in the real axis. $\Gamma_\alpha(\omega)$ cone of width $\alpha$ in $\mathbb{H}^+$ with vertex in $\omega$ (note: in the hyperbolic distance of $\mathbb{H}^+$ this is a tube with axis $\{\omega + i\tau, \tau > 0\}$)

Area function:

$$A_\alpha(\omega) := \int_{\Gamma_\alpha(\omega)} \|\nabla f\|^2 \, dx \, dy$$
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**Theorem (Lusin area theorem in \( \mathbb{H}^+ \))**

Let \( E \subset \mathbb{R} \) measurable, \( f \) harmonic on \( \mathbb{H}^+ \). Then the following are equivalent:

(i) \( f \) is non-tangentially bounded at a.e. \( \omega \in E \): \( \exists \alpha = \alpha(\omega) \) such that, \( E \)–almost everywhere, \( f \) is bounded on \( \Gamma_\alpha(\omega) \);

(ii) \( f \) has non-tangential limit at almost every \( \omega \in E \);

(iii) \( \exists \alpha = \alpha(\omega) \) such that \( A_{\alpha, \omega} f(\omega) < \infty \) for almost every \( \omega \in E \);

(iv) for every fixed \( \alpha \geq 0 \), \( A_\alpha f(\omega) < \infty \) for a.e. in \( E \);

(v) \( \forall \) fixed \( \alpha \geq 0 \), \( f \) is bounded on \( \Gamma_\alpha(\omega) \) \( \omega \)–a.e. on \( E \).
Moreover, $\|A_\alpha f\|_p \sim \|N_\beta f\|_p$, for all $p < \infty$ and all $\alpha, \beta > 0$, where $N_\beta f(\omega) = \sup_{\Gamma(\omega)} |f|$ is the non-tangential maximal function.
Sketch of proof. Up to negligible measure $E$ is a union of intervals. For simplicity let us assume $E$ is an interval and prove $(i) \Rightarrow (iv)$. We are assuming that $f$ is bounded on almost all cones with vertex in $E$. For the moment, assume more: that $f \in L^\infty$. Also, for the time being assume $\alpha$ constant.

Consider the conical extension $W_E$ of $E$. Truncate at heights $\tau^- < \tau^+$ ($\tau^-$ near the boundary) to get a trapeze $W^\tau_E$. 
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The Lusin area function and admissible convergence of harmonic functions.
Now for $\omega \in E$ and $\tau = [\tau^-, \tau^+]$ consider the truncated cone $\Gamma^\tau_\omega$ and compute the integral

$$I := \int_E \int_{\Gamma^\tau_\omega} \|\nabla f\|^2 \, dx \, dy \, d\omega$$

When the vertices of two cones are translated along $E$, their sections at height $y$ overlap by a quantity $w(y)$ proportional to $y$:

$$w(y) = |\{\omega : y \in \Gamma(\omega)\}| = c_\alpha y$$

But so, $w$ is harmonic on $\mathbb{H}^+$!

Since also $f$ is harmonic,

$$\|\nabla f\|^2 = \|\nabla f\|^2 + 2f \Delta f = \Delta (f^2)$$

so we can now compute the integral $I$ via the Green identities and the Green formula.
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\[ l = \int_{\mathcal{W}_E^T} w \triangledown (f^2) = \int_{\mathcal{W}_E^T} w \triangledown (f^2) - f^2 \triangledown w \]
\[ = \int_{\partial \mathcal{W}_E^T} w \frac{\partial}{\partial n} f^2 - f^2 \frac{\partial}{\partial n} w \]

Now remember that \( f \in L^\infty \) and observe the following:
\[ \left| \frac{\partial}{\partial n} f^2 \right| = 2 \left| f \frac{\partial}{\partial n} f \right| \leq C |f|^2 \]

by elliptic estimates (or simply because for a bounded harmonic function \( \frac{\partial}{\partial n} f \sim f \) by Harnack’s inequality...)

- \( w = c_\alpha y \) is proportional to the Green function of \( \mathbb{H}^+ \), hence \( \frac{\partial}{\partial n} w \sim w \sim \) harmonic measure on boundary. Hence:

- \( \int_{\partial W_E \cap \{y=\tau^+\}} w \) is bounded, and

- independently of \( \tau^- \), \( \int_{\partial W_E \cap \{y=\tau^-\}} w \sim \nu(E) \)

(\( \nu \) is the harmonic measure, i.e., Lebesgue measure on \( \mathbb{R} \))
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- \( \int_{\partial W^T_E \cap \{ y = \tau^+ \}} w \) is bounded, and
- independently of \( \tau^- \), \( \int_{\partial W^T_E \cap \{ y = \tau^- \}} w \sim \nu(E) \)
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independently of \( \tau^- \), \[ \int_{\partial \mathcal{W}_E} \mathcal{E} \cap \{ y = \tau^- \} w \sim \nu(E) \]

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Moreover, the integral of $w$ on the side facets is bounded. So, by letting $\tau^- \to 0^+$ and $\tau^+ \to \infty$, we have

$$\lim |I| = \left| \int_{W_E} w \Delta (f^2) \right|$$

$$= \left| \int_{\partial W^T_E} w \frac{\partial}{\partial n} f^2 - f^2 \frac{\partial}{\partial n} w \right| \leq c \| f \|_\infty^2$$

hence, for a.a. $\omega$,

$$|A_\alpha(\omega)|^2 \leq c \| f \|_\infty^2$$

This proves the theorem if $f$ is bounded. If not, uniformization:
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Assume $f$ unbounded, but bounded on almost all cones $\Gamma_\alpha(\omega)$. 

Claim: we can choose $E^N \subset E$ such that $f$ is bounded on the conical extension $W_{E^N}$.

Indeed,

$$E_k := \{\omega : |f| < k \text{ on } \Gamma_\alpha(\omega)\}$$

$$E^N := \bigcup_{k \leq N} E_k$$

Since $f$ is bounded on almost each cone, for $N$ large we have

$$\nu(E \setminus E^N) < \varepsilon$$

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Finally, let us uniformize again to remove the extra assumption that $\alpha$ be constant (this step will not be necessary in a tree). Choose $j \in \mathbb{N}$, let $\beta = \min\{\alpha(\omega), j\}$, $\Gamma_j(\omega) := \Gamma_\beta(\omega)$ and

$$E_{j,k} := \{\omega : |f| < k \text{ on } \Gamma_j(\omega)\}$$

Now let

$$E_{M,N} := \bigcup_{j \leq M} \bigcup_{k \leq N} E_{j,k}$$

For every $\varepsilon$, $\nu(E \setminus E_{M,N}) < \varepsilon$ for large $M$, $N$, and $|f| < N$ on $E_{M,N}$. □
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Definition (Notation)

A tree $T$ is a connected, simply connected, locally finite graph. We shall also write $T$ for its set of vertices. We do not assume that $T$ is homogeneous (but our forthcoming assumptions imply that the number of edges at each vertex is bounded).

Natural distance $d(x, y)$: length of the direct path from $x$ to $y$. Write $x \sim y$ if $x, y$ are neighbors: distance 1. We fix a reference vertex $o \in T$ and call it the origin. The choice of $o$ induces a partial ordering in $T$: $x \leq y$ if $x$ belongs to the geodesic from $o$ to $y$. Length: $|x| = d(o, x)$.

For any vertex $x$ and $k \leq |x|$, $x_k$ is the vertex of length $k$ in the geodesic $[o, x]$. 
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Definition (Boundary)

Let $\Omega$ be the set of infinite geodesics starting at $o$. $\omega_n$ is the vertex of length $n$ in the geodesic $\omega$. For $x \in T$ the interval $U(x) \subset \Omega$, generated by $x$, is the set $U(x) = \{\omega \in \Omega : x = \omega|_{\xi}\}$. The sets $U(\omega_n)$, $n \in \mathbb{N}$, form an open basis at $\omega \in \Omega$. Equipped with this topology $\Omega$ is compact and totally disconnected. For every vertex $x$, the set of vertices $v > x$ form the sector $S(x)$. Call closed sector the set $\overline{S(x)} := S(x) \cup U(x) \subset T \cup \Omega$. The closed sectors induce on $T \cup \Omega$ a compact topology.
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On the vertices of $T$ we consider operators $P$ with transition probabilities $p(u, v)$ that satisfy:

**Definition (Very regular transition operators)**

1. **(H1)** $P$ is a nearest neighbor operator, that is, $p(u, v) = 0$ unless the vertices $u$ and $v$ are neighbors;
2. **(H2)** for some constant $\delta$, with $0 < \delta < \frac{1}{2}$ and all neighbors $u$ and $v$ the following inequality holds:

\[
\delta \leq p(u, v) \leq \frac{1}{2} - \delta.
\]
\[ p(u, v) \geq \delta \Rightarrow \text{number of neighbors} \leq \frac{1}{\delta}. \]

\[ p(u, v) \leq \frac{1}{2} - \delta \Rightarrow \text{probability of moving forward larger than} \]
\[ \text{probability of moving backwards} \Rightarrow \text{transience}. \]

The sets \( \{U(x) : |x| = n\} \) generate a \( \sigma \)-algebra on \( \Omega \). On this \( \sigma \)-algebra, balayage theory yields the harmonic measure, that is the measure that reconstructs the value of harmonic functions at 0 from their boundary values (see later):

**Definition (Harmonic measure)**

\[ \nu(E) = \Pr[X_\infty \in E]. \]
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Definition (Laplace operator)
The Laplace operator is $\Delta = P - \mathbb{I}$.

If $p^*(x, y) = p(y, x)$, then the conjugate Laplacian is the transpose operator $\Delta^* = P^* - \mathbb{I}$.

Definition (Harmonic functions)
$f : T \to \mathbb{R}$ is harmonic at $x \in T$ if $Pf(x) \equiv \sum_{y \sim x} p(x, y) f(y) = f(x)$. $f$ harmonic on $T \iff \Delta f = 0$.

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\( f \) conjugate harmonic if \( \Delta^* f = 0 \).
For $\sigma \in \Lambda$ denote by $b(\sigma)$ the beginning vertex of $\sigma$ and by $e(\sigma)$ the ending vertex: $\sigma = [b(\sigma), e(\sigma)]$.

**Definition (Gradient)**

For any function $f : T \to \mathbb{R}$, the gradient $\nabla f : \Lambda \to \mathbb{R}$ is

$$\nabla f(\sigma) = f(e(\sigma)) - f(b(\sigma)).$$

For $x \in T$,

$$\|\nabla f(x)\|^2 = \sum_{y \sim x} p([x, y])|\nabla f([x, y])|^2.$$
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$$\|\nabla f(x)\|^2 \equiv \sum_{y \sim x} p([x, y])|\nabla f([x, y])|^2.$$
$\forall x \in T, \omega \in \Omega$, set \(d(x, \omega) = \min_{j \in \mathbb{N}} d(x, \omega_j)\).

**Definition**

Let \(\alpha \geq 0\) be an integer. The tube \(\Gamma_\alpha(\omega)\) around the geodesic \(\omega \in \Omega\) is

\[
\Gamma_\alpha(\omega) = \{x \in T : d(x, \omega) \leq \alpha\}.
\]

**Definition (Area function)**

\[
A_\alpha f(\omega) = \left( \sum_{x \in \Gamma_\alpha(\omega)} \|\nabla f(x)\|^2 \right)^{\frac{1}{2}}.
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\forall x \in T, \omega \in \Omega, \text{ set } d(x, \omega) = \min_{j \in \mathbb{N}} d(x, \omega_j).

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A_\alpha f(\omega) = \left( \sum_{x \in \Gamma_\alpha(\omega)} \| \nabla f(x) \|^2 \right)^{\frac{1}{2}}.
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Observe that, if $f \in L^1(T)$, $A_\alpha f(\omega) < \infty$ for every $\alpha, \omega$.

**Definition (Non-tangential limits and boundedness)**

A function $f$ on $T$ has non-tangential limit at $\omega \in \Omega$ if, for every integer $\alpha \geq 0$, $\lim f(x)$ exists as $|x| \to \infty$ and $x \in \Gamma_\alpha(\omega)$.

We say that $f$ has non-tangential limit up to width $\beta$ if the above limit exists for all $0 \leq \alpha \leq \beta$.

A function $f$ on $T$ is non-tangentially bounded at $\omega \in \Omega$ if, for some $M > 0$, one has $|f(x)| \leq M$ for $x \in \Gamma_0(\omega)$. 
Observe that, if \( f \in L^1(T) \), \( A_\alpha f(\omega) < \infty \) for every \( \alpha, \omega \).

**Definition (Non-tangential limits and boundedness)**

A function \( f \) on \( T \) has non-tangential limit at \( \omega \in \Omega \) if, for every integer \( \alpha \geq 0 \), \( \lim_{x \to \infty} f(x) \) exists as \( |x| \to \infty \) and \( x \in \Gamma_\alpha(\omega) \).

We say that \( f \) has non-tangential limit up to width \( \beta \) if the above limit exists for all \( 0 \leq \alpha \leq \beta \).

A function \( f \) on \( T \) is non-tangentially bounded at \( \omega \in \Omega \) if, for some \( M > 0 \), one has \( |f(x)| \leq M \) for \( x \in \Gamma_0(\omega) \).
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**Definition (Non-tangential limits and boundedness)**

A function $f$ on $T$ has non-tangential limit at $\omega \in \Omega$ if, for every integer $\alpha \geq 0$, $\lim f(x)$ exists as $|x| \to \infty$ and $x \in \Gamma_\alpha(\omega)$.

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A function $f$ on $T$ is non-tangentially bounded at $\omega \in \Omega$ if, for some $M > 0$, one has $|f(x)| \leq M$ for $x \in \Gamma_0(\omega)$. 

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The Lusin area function and admissible convergence of harmonic functions
Definition (**Conical** - better, **tubular** - extension of a boundary set)

The “cone” (actually, tube) \( W_\alpha(E) \) over a measurable subset \( E \) of \( \Omega \) is

\[
W_\alpha(E) = \bigcup_{\omega \in E} \Gamma_\alpha(\omega).
\]

For any integer \( s > 0 \) let \( W_\alpha^s(E) = \bigcup_{\omega \in E} \Gamma_\alpha(\omega) \cap [s; \infty) \).

The main goal of this paper is the following extension of the Lusin area theorem to non-homogeneous trees. This theorem has been proved for homogeneous trees By L. Atanasi and this speaker (TAMS 2008).
Main Theorem (The Lusin Area Theorem)

Let $E$ be a measurable subset of $\Omega$ and $f$ a harmonic function on $T$. Then the following are equivalent:

(i) $f$ is non-tangentially bounded at almost every $\omega \in E$;
(ii) $f$ has non-tangential limit at almost every $\omega \in E$;
(iii) $A_0f(\omega) < \infty$ for almost every $\omega \in E$;
(iv) for every fixed $\alpha \geq 0$, $A_\alpha f(\omega) < \infty$ for almost every $\omega \in E$. 
The same statement holds if $f$ is harmonic on a connected subset of $T$ whose boundary contains $E$, or more precisely on some tube $W_\beta(E)$, provided that $\alpha \leq \beta$ in (iv) and, in (ii), $f$ is assumed to be non-tangentially bounded up to width $\beta$.

Observe that the definition of non-tangential boundedness is equivalent to say that, for some non-negative integer $\alpha = \alpha(\omega)$, $f$ is bounded on a tube of width $\alpha$ around the geodesic $\Gamma_0(\omega)$ (by a different constant depending on $\alpha$), and, in this discrete setting, condition (iii) in the theorem is equivalent to the more familiar statement that for almost every $\omega$ there is some integer $\alpha$ such that $A_\alpha f$ is finite at $\omega$. 

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Hitting probability from $u$ to $v$:

$$F(u, v) := \Pr \left[ \exists n > 0 : X_n = v, X_j \neq v \ \forall j < n \mid X_0 = u \right].$$

Nearest neighbor random walk $\Rightarrow$ stopping time argument $\Rightarrow$ multiplicativity rule:

$$F(u_0, u_n) = \prod_{i=1}^{n} F(u_{i-1}, u_i). \quad (1)$$

**Definition (Green kernel)**

The Green kernel $G(u, v)$ is the expected number of visits to $v$ of the random walk starting at $u$:

$$G(u, v) = \sum_{n=0}^{\infty} P^n(u, v). \quad (2)$$

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\( G(u, v) = F(u, v) G(v, v) \). \hspace{1cm} (3)

For each \( v \in T \), the Green kernel \( G(x, v) \) is harmonic with respect to \( x \) at every vertex \( x \neq v \), and conjugate harmonic with respect to \( v \) at every \( v \neq x \).
Given two positive functions $f$ and $g$, we write $f \approx g$ if $f < Cg$ and $g < Cf$ for some constant $C$.


If $P$ is very regular, then for $x \in T$

$$\nu(U(x)) \approx F(o, x) \approx G(o, x).$$
Corollary (Uniform estimate for harmonic measure)

If the transition operator $P$ is very regular, with $\delta$ as in (H2), then for all vertices $u$, $v$ at distance $n$,

$$\delta^n < F(u, v) < \left( \frac{\delta}{\frac{1}{2} + \delta} \right)^n.$$ 

Therefore, for some $0 < \varepsilon < 1$ and for all vertices $x$ in $T$ and $v$ in the sector $S(x)$ at distance $d(v, x)$ from $x$,

$$\nu(U(v)) < (1 - \varepsilon)^{d(v, x)} \nu(U(x)).$$
Definition (Poisson kernel)

For every $x, \nu \in T$ the Martin kernel $K(x, \nu)$ is defined as

$$K(x, \nu) \equiv \frac{G(x, \nu)}{G(o, \nu)} = \frac{F(x, \nu)}{F(o, \nu)} .$$

For every $x \in T, \omega \in \Omega$ the Poisson kernel $K(x, \omega)$ is defined as

$$K(x, \omega) \equiv \lim_{\nu \to \infty} \frac{G(x, \nu)}{G(o, \nu)} = \lim_{\nu \to \infty} \frac{F(x, \nu)}{F(o, \nu)} .$$

For every $\omega \in \Omega$, $K(\cdot, \omega)$ is harmonic on $T$. 
Definition (Poisson kernel)

For every $x, v \in T$ the *Martin kernel* $K(x, v)$ is defined as

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For every $\omega \in \Omega$, $K(\cdot, \omega)$ is harmonic on $T$. 
Corollary (Estimates for the Poisson kernel)

(i) If $\omega \in U(x)$ then

$$\delta^{-|x|} \left( \frac{1}{2} + \delta \right)^{|x|} < K(x, \nu) < \delta^{-|x|}.$$

The same inequalities are satisfied by $K(u, \omega)$ if $\omega \in U(x)$.

(ii) If $x < y$ and $\omega \notin U(x)$ then

$$\frac{K(y, \omega)}{K(x, \omega)} < \left( \frac{1}{2} + \delta \right)^{d(x, y)}.$$
Remarks (Poisson integral representation)

The Poisson kernel, being a locally constant function on $\Omega$, belongs to $L^p(\Omega)$ for every $x \in T$, for $1 \leq p \leq \infty$. The Poisson integral of a function $h$ in $L^1(\Omega)$ is defined by

$$K_h(x) = \int_\Omega h(\omega)K(x, \omega)d\nu.$$
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$$K h(x) = \int_\Omega h(\omega) K(x, \omega) d\nu.$$
Definition (Variants of the gradient)

∀σ = [u, v] edge, set p(σ) = p(u, v). Gradient on functions on vertices:

∇f(σ) = (f ◦ e − f ◦ b)(σ).

Gradient on functions h on edges:

∇f(σ) = f(σ∗) − f(σ),

where σ∗ = [e(σ), b(σ)] (opposite orientation). Moreover,

∂f(σ) = p(σ)∇f(σ),

Df(σ) = (Pf ◦ e − f ◦ b)(σ),

D†f = P∗f ◦ e − f ◦ b.
The Green formula, well known in the continuous setup, has been extended to the discrete context of trees by F. Di Biase and this speaker (Zeitschrift, 1995).

**Proposition (The Green formula)**

∀f and h functions on \( T \), \( Q \subset T \) finite,

\[
\sum_{Q} (h \Delta f - f \Delta^* h) = \sum_{\partial Q} (h \circ b Df - f \circ b D^\dagger h)
\]

\[
= \sum_{\partial Q} (h \circ b \partial f - f \circ b \partial^* h - h \circ b f \circ b \nabla P).
\]

**Proposition (The Green identity)**

\[
\Delta(f^2)(x) = \|\nabla f(x)\|^2 + 2f(x)\Delta f(x).
\]
Let us see a sketch of the same part of the proof that we gave for the half-plane. For simplicity, let us deal with a *homogeneous tree with isotropic transition operator*: say, homogeneity degree \( q \), that is \( q + 1 \) neighbors at each vertex. Up to a perfectly analogous uniformization procedure, we need to show that boundedness of \( f \) on cones implies \( A_\alpha f < \infty \) for some \( \alpha \): since \( A_0 f \leq A_\alpha f \) this is equivalent to show:

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f \in L^\infty(T) \Rightarrow \|Af\|_\infty < C\|f\|_\infty
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### Theorem (Lusin theorem, $(i) \Rightarrow (iii)$)

$$f \in L^\infty(T) \Rightarrow \|Af\|_\infty < C\|f\|_\infty$$
Sketch of proof. $o$ reference vertex, $y^-$ predecessor of $y \in T$, $(y \neq o)$.

$$|A_0 f(\omega)|^2 = \sum_{\text{y vertex in } \omega} |f(y) - f(y^-)|^2$$

$K \subset \text{Aut}(T)$ stability subgroup of $o$, compact. $K$ can be regarded as a group of automorphisms of $\Omega$.

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\( K \subset \text{Aut}(T) \) stability subgroup of \( o \), compact. \( K \) can be regarded as a group of automorphisms of \( \Omega \).

\( \nu \) \( K \)-invariant measure on \( \Omega \).
Choose reference point $\overline{\omega} \in \Omega$. Then

$$
\int_{\Omega} |A_0 f(\omega)|^2 \, d\nu = \int_{K} \sum_{y \in k\overline{\omega}} |f(y) - f(y^-)|^2 \, dk
$$

$$
= \sum_{n>0} \sum_{|y|=n} |f(y) - f(y^-)|^2 \left| \{k \in K : k\overline{\omega}_n = y \} \right|
$$

$$
:= \sum_{n>0} \sum_{|y|=n} |f(y) - f(y^-)|^2 w(n)
$$

$K_n :=$ stabilizer of $\overline{\omega}_n$ in $K$. The group $K_n$ fixes the chain $o \rightarrow \overline{\omega}_1 \rightarrow \overline{\omega}_2 \rightarrow \cdots \rightarrow \overline{\omega}_n$. 

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Choose reference point \( \bar{\omega} \in \Omega \). Then

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\( o \rightarrow \bar{\omega}_1 \rightarrow \bar{\omega}_2 \rightarrow \cdots \rightarrow \bar{\omega}_n \).
But $K$ acts transitively: so

$$w(n) = [K : K_n] = \frac{1}{\{y | |y| = n\}} = \frac{1}{(q + 1)q^{n-1}}$$

(remember that $q$ is the homogeneity degree of $T$). Now, this $w$ is exactly the Green function of the homogeneous tree with singularity at $o$: $Pw = w$ off $o$. Indeed, the average of $w$ at any vertex $y \neq o$ is again $w$: 

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Indeed, the average of $w$ at any vertex $y \neq o$ is again $w$:
$|y|=n,$
$w(y)=w(n)$

$|y^-|=n-1,$
$w(y) = q w(n)$

$|y^+|=n+1,$
$w(y) = w(n)/q$

Figure: Harmonicity away from 0 of the isotropic Green function $w$: $w(y^-) + \sum_{y^+} w(y^+) = qw(y) + qw(y)/q = (q + 1)w(y)$
So \( w \) is harmonic away from \( o \). The rest of the proof is now exactly as in the continuous case, via the discrete Green formula and Green identities.
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