Paley-Wiener theorems for the $U_n$-spherical transform on the Heisenberg group

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$U_n$-invariant differential operators on $H_n$

The Heisenberg group $H_n$ is $\mathbb{C}^n \times \mathbb{R}$ with product

$$(z, t)(z', t') = \left( z + z', t + t' - \frac{1}{2} \text{Im} \langle z, z' \rangle \right)$$

with $z = x + iy \in \mathbb{C}^n$ and $t \in \mathbb{R}$.

The unitary group $U_n$ acts on $H_n$ by the automorphisms

$$k \cdot (z, t) = (kz, t)$$

The two commuting operators

$$L = - \sum_{j=1}^{n} (X_j^2 + Y_j^2) \quad \text{(sublaplacian),} \quad T = \partial_t \quad \text{(central derivative)}$$

generate the algebra $\mathbb{D}(H_n)^{U_n}$ of left- and $U_n$-invariant differential operators on $H_n$.

Commutativity of $\mathbb{D}(H_n)^{U_n}$ means that $(U_n \ltimes H_n, U_n)$ is a Gelfand pair.
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Spherical functions

The $U_n$-invariant joint eigenfunctions of $L$ and $T$, normalized by the condition $\varphi(0, 0) = 1$, are the spherical functions.

Each spherical function is uniquely determined by the pair $(\xi, \lambda)$ of its eigenvalues relative to $(L, -iT)$.

For each pair of eigenvalues $(\xi, \lambda) \in \mathbb{C}^2$ there exists the corresponding spherical function $\Phi_{\xi, \lambda}$, given by

$$
\Phi_{\xi, \lambda}(z, t) = e^{i\lambda t} e^{-\lambda \frac{|z|^2}{4}} \left( 1 + \sum_{k=1}^{\infty} \frac{|z|^{2k}}{(n)_k k! 4^k} \prod_{d=0}^{k-1} (\lambda(2d + n) - \xi) \right)
$$

where $(n)_k = n(n + 1) \cdots (n + k - 1)$.
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Characters of the $U_n$-invariant $L^1$-algebra

The algebra $L^1_{\text{rad}}(H_n)$ of $U_n$-invariant functions is commutative. The multiplicative linear functionals of $L^1_{\text{rad}}(H_n)$ are given by integration against the \textit{bounded spherical functions}.

The spherical function $\Phi_{\xi,\lambda}$ is bounded in the following two cases:

- (Bessel type) $\lambda = 0$ and $\xi \geq 0$:
  \[ \Phi_{\xi,0}(z,t) = 2^{n-1}(n-1)|J_{n-1}(\sqrt{\xi}|z|)| \frac{J_{n-1}(\sqrt{\xi}|z|)}{(\sqrt{\xi}|z|)^{n-1}} \]

- (Laguerre type) $\lambda \in \mathbb{R} \setminus \{0\}$ and $\xi = |\lambda|(2d + n)$, $d \in \mathbb{N}$:
  \[ \Phi_{|\lambda|(2d+n),\lambda}(z,t) = e^{i\lambda t} e^{-|\lambda|\frac{|z|^2}{4}} L_d^{(n-1)} \left( |\lambda| \frac{|z|^2}{2} \right) \]
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  \]
The Gelfand spectrum of $L^1_{rad}(H_n)$ can be homeomorphically embedded into $\mathbb{R}^2$ as the set

$$\Sigma = \{(\xi, \lambda) : \Phi_{\xi, \lambda} \text{ is bounded}\}$$
The spherical transform

The spherical transform $\mathcal{G} : L^1_{\text{rad}}(H_n) \longrightarrow C_0(\Sigma)$ is defined as

$$\mathcal{G} f(\xi, \lambda) = \int_{H_n} f(z, t) \Phi_{\xi, \lambda}(-z, -t) \, dz \, dt$$

Plancherel formula:

$$\|f\|_2^2 = c_n \int_{-\infty}^{\infty} \sum_{d=0}^{\infty} \left| \mathcal{G} f(|\lambda|(2d + n), \lambda) \right|^2 |\lambda|^n \, d\lambda = \int_{\Sigma} |\mathcal{G} f(\xi, \lambda)|^2 \, d\mu(\xi, \lambda)$$

Inversion formula:

$$f(z, t) = \int_{\Sigma} \mathcal{G} f(\xi, \lambda) \Phi_{\xi, \lambda}(z, t) \, d\mu(\xi, \lambda)$$
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The Schwartz space

**Theorem (A-DB-R).**

$\mathcal{G}$ is an isomorphism of $S_{\text{rad}}(H_n)$ onto

$$S(\Sigma) = \{ f|_{\Sigma} : f \in S(\mathbb{R}^2) \} \cong S(\mathbb{R}^2) / \{ f : f = 0 \text{ on } \Sigma \}$$

The Schwartz isomorphisms theorem has been extended to all Heisenberg Gelfand pairs $(K \ltimes H_n, K)$ by A-DB-R, and to a large class of nilpotent Gelfand pairs $(K \ltimes N, K)$ by Fisher-R-Yakimova.
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If $f$ has compact support, we do not need boundedness of $\Phi_{\xi,\lambda}$ for the convergence of the integral

$$Gf(\xi, \lambda) = \int_{H_n} f(z, t) \Phi_{\xi, \lambda}(-z, -t) \, dz \, dt,$$

hence $Gf$ can be extended to all of $\mathbb{C}^2$.

The formula

$$\Phi_{\xi,\lambda}(z, t) = e^{i\lambda t} e^{-\lambda \frac{|z|^2}{4}} \left( 1 + \sum_{k=1}^{\infty} \frac{|z|^{2k}}{(n)_k k! 4^k} \prod_{d=0}^{k-1} \left( \lambda(2d + n) - \xi \right) \right)$$

shows that this extension is holomorphic.

Similarly, if $g$ has compact support in $\Sigma$, the inversion formula

$$G^{-1}g = c_n \int_{-\infty}^{\infty} \sum_{d=0}^{\infty} g(|\lambda|(2d + n), \lambda) \Phi_{|\lambda|(2d+n), \lambda} |\lambda|^n \, d\lambda$$

shows that $G^{-1}g$ extends to a homorphic function on $H_n^\mathbb{C} = \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$. 
Holomorphic extensions of spherical transforms

If \( f \) has compact support, we do not need boundedness of \( \Phi_{\xi,\lambda} \) for the convergence of the integral

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Similarly, if \( g \) has compact support in \( \Sigma \), the inversion formula

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shows that \( G^{-1}g \) extends to a homorphic function on \( H_n^\mathbb{C} = \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C} \).
Comparison between Schwartz and holomorphic extensions

Suppose $f \in C_{c,\text{rad}}^\infty(H_n)$. Then $\mathcal{G}f$ admits many Schwartz extensions to $\mathbb{R}^2$ and one holomorphic extension to $\mathbb{C}^2$.

In general, the holomorphic extension is not Schwartz on $\mathbb{R}^2$.

This depends on the fact that $\Phi_{\xi,\lambda}$ grows exponentially, as soon as it is not bounded.
Comparison between Schwartz and holomorphic extensions

Suppose \( f \in C_{c, \text{rad}}(H_n) \). Then \( Gf \) admits many Schwartz extensions to \( \mathbb{R}^2 \) and one holomorphic extension to \( \mathbb{C}^2 \).

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Characterization of Paley-Wiener spaces

The problem is to determine which space of holomorphic functions one gets on $\mathbb{C}^2$ (resp. on $H_n^C$) starting from a given function space of compactly supported radial functions (e.g. $C^\infty_{c,rad}$, $L^2_{c,rad}$, $\mathcal{E}'_{rad}$).

By analogy with Fourier transform, one expects this space to be characterized by growth conditions in non-real directions.

In loose terms, $f$ being supported on the ball $B_r \subset \mathbb{R}^n$ corresponds to the property that $\hat{f}$ extends to an entire function $v$ satisfying

$$|v(\zeta)| \leq Ce^{r|\text{Im}\,\zeta|}$$
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In loose terms, $f$ being supported on the ball $B_r \subset \mathbb{R}^n$ corresponds to the property that $\hat{f}$ extends to an entire function $\nu$ satisfying

$$|\nu(\zeta)| \leq C e^{r|\text{Im}\, \zeta|}$$
Several versions of Paley-Wiener theorems on the Heisenberg group exist, mostly related to the Group Fourier transform:
Ando, Thangavelu, Arnal-Ludwig, Narayanan-Thangavelu

More closely related to this work is an article by Führ of 2010, where compact support is referred to the spectral resolution of the sublaplacian $L$. 

In other contexts, there are Paley-Wiener theorems for the Fourier transform on noncompact symmetric spaces:
Helgason, Gangolli-Varadarajan

and for the inverse spherical transform:
Pasquale, Andersen
Radial functions on \( \mathbb{R}^n \)

Consider first the pair \(( SO_n \ltimes \mathbb{R}^n, SO_n)\), a poor cousin of \(( U_n \ltimes H_n, U_n)\).

The spherical functions are

\[
\Phi_\xi(x) = c_n \frac{J_{n-1}(\sqrt{\xi}|x|)}{(\sqrt{\xi}|x|)^{n-1}}
\]

for \( \xi \in \mathbb{C} \). They are bounded for \( \xi \geq 0 \), i.e., \( \Sigma = [0, +\infty) \)

(the parameter \( \xi \) has been chosen as the \((-\Delta)\)-eigenvalue: \(-\Delta \Phi_\xi = \xi \Phi_\xi\)).

If \( f \) is a radial function with compact support, \( \hat{f} \) is constant on the spheres \( S_r \) centered at 0 and, for \( \xi \in \Sigma \),

\[
Gf(\xi) = \hat{f}(S_{\sqrt{\xi}})
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i.e.,

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\hat{f}(\lambda_1, \ldots, \lambda_n) = Gf(\lambda_1^2 + \cdots + \lambda_n^2)
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Suppose $f$ is supported on $B_r$.

If $u$ is the holomorphic extension of $\mathcal{G}f$ to $\mathbb{C}$ and $v$ the holomorphic extension of $\hat{f}$ to $\mathbb{C}^n$, then

$$v(\zeta_1, \ldots, \zeta_n) = u(\zeta_1^2 + \cdots + \zeta_n^2)$$

Hence

$$|u(\tau)| \leq C \inf_{\zeta_1^2 + \cdots + \zeta_n^2 = \tau} e^{r|\text{Im} \zeta|} = Ce^r \left( \frac{|\tau| - \text{Re} \tau}{2} \right)^{1/2}$$
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Real-variable characterization of the $L^2$-PW space on $\mathbb{R}^n$

We borrow the following statement from N. Andersen, M. deJeu, *Real Paley-Wiener theorems and local spectral radius formulas*, TAMS 2010

**Theorem (Tuan).** $f \in L^2(\mathbb{R}^n)$ has its Fourier transform supported on the ball $B_r$ if and only if

$$\limsup_{k \to \infty} \left\| \Delta^k f \right\|_{L^2(\mathbb{R}^n)}^k \leq r^2$$

Let $D = 4\xi d_\xi^2 + (n + 1)d_\xi$.

**Corollary 1.** A function $f \in L^2_{rad}(\mathbb{R}^n)$ is supported on $B_r$ if and only if

$$\limsup_{k \to \infty} \left\| D^k (f) \right\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^n, \xi_1^n d\xi)}^k \leq r^2$$

**Corollary 2.** A function $f \in L^2_{rad}(\mathbb{R}^n)$ has $Gf$ supported on $[0, r]$ if and only if

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The fan $\Sigma$
PW theorems for \((U_n \ltimes H_n, H_n)\)

In order to establish an analogue of Corollary 1, one must identify a pair

“norm” on \(H_n \leftrightarrow “differential operator” on the fan \(\Sigma\)

and, for Corollary 2, a pair

“differential operator” on \(H_n \leftrightarrow “norm” on \(\Sigma\)

Various choices are possible in principle.

As the first pair we choose:

- the Korányi norm on \(H_n\): 
  \[ N(z, t) = (|z|^4 + t^2/16)^{1/4} \]

- the Benson-Ratcliff operator on the fan:
  \[ Mg(\xi, \lambda) = \frac{1}{\lambda} \left( \lambda \partial_\lambda + \xi \partial_\xi \right) g(\xi, \lambda) - \left( n - \frac{\xi}{\lambda} \right) \frac{g(\xi + 2\lambda, \lambda) - g(\xi, \lambda)}{2\lambda} \]

  characterized by the property

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PW theorems for $G$

**Theorem 1a.** A function $f \in L^2_{\text{rad}}(H^n)$ is supported on the Korányi ball $B_r$ if and only if

$$\limsup_{k \to \infty} \| M^k(Gf) \|_{L^2(\Sigma)}^{\frac{1}{k}} \leq r^2$$

**Theorem 1b.** A function $f \in S_{\text{rad}}(H^n)$ is supported on $B_r$ if and only if for one (and then all) $p \in [1, \infty]$ and one (and then all) $j \in \mathbb{N}$

$$\limsup_{k \to \infty} \| \xi^j M^k(Gf) \|_{L^p(\Sigma)}^{\frac{1}{k}} \leq r^2$$

**Theorem 1c.** A distribution $f \in S'_{\text{rad}}(H^n)$ is supported on $B_r$ if and only if there exists $j \in \mathbb{N}$ such that for one (and then all) $p \in [1, \infty]$

$$\limsup_{k \to \infty} \| (1 + \xi)^{-j} M^k(Gf) \|_{L^p(\Sigma)}^{\frac{1}{k}} \leq r^2$$
PW theorems for $G$

**Theorem 1a.** A function $f \in L^2_{rad}(H^n)$ is supported on the Korányi ball $B_r$ if and only if

$$\limsup_{k \to \infty} \left\| M^k(Gf) \right\|^{1/k}_{L^2(\Sigma)} \leq r^2$$

**Theorem 1b.** A function $f \in S_{rad}(H^n)$ is supported on $B_r$ if and only if for one (and then all) $p \in [1, \infty]$ and one (and then all) $j \in \mathbb{N}$

$$\limsup_{k \to \infty} \left\| \xi^j M^k(Gf) \right\|^{1/k}_{L^p(\Sigma)} \leq r^2$$

**Theorem 1c.** A distribution $f \in S'_{rad}(H^n)$ is supported on $B_r$ if and only if there exists $j \in \mathbb{N}$ such that for one (and then all) $p \in [1, \infty]$

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PW theorems for $\mathcal{G}^{-1}$

For Paley-Wiener theorems in the other direction we choose:

- the sublaplacian $L$ on $H_n$
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**Theorem 2a.** A function $f \in L^2_{rad}(H^n)$ has $\mathcal{G}f$ supported on the subset $\{\xi \leq r\} \subset \Sigma$ if and only if

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Spherical transforms of radial distributions

Theorems 1c and 2c require some definition.

Recall that \( G : S_{\text{rad}}(H_n) \to S(\Sigma) \) is an isomorphism.

Then it induces an isomorphism \( G^* : (S(\Sigma))' \to S_{\text{rad}}'(H_n) \).

Since \( S(\Sigma) = S(\mathbb{R}^2)/\{ f : f|_{\Sigma} = 0 \} \),

\[
(S(\Sigma))' = \{ \psi \in S'(\mathbb{R}^2) : \langle \psi, g \rangle = 0, \forall g = 0 \text{ on } \Sigma \} = S_0'(\Sigma)
\]

This is a space of distributions supported on \( \Sigma \) (synthetizable on \( \Sigma \)).

**Proposition.** A tempered distribution \( \psi \) supported on \( \Sigma \) belongs to \( S_0'(\Sigma) \) if and only for any “Laguerre point” \((\xi_0, \lambda_0) = (|\lambda_0(2d + n)|, \lambda_0) \) with \( \lambda \neq 0 \) there exist a neighborhood \( U \) in \( \mathbb{R}^2 \) and a distribution \( \psi \) on the real line such that, for every \( g \in \mathcal{D}(\mathbb{R}^2) \) supported on \( U \),

\[
\langle \psi, g \rangle_{\mathbb{R}^2} = \langle \psi, g_0 \rangle_{\mathbb{R}}
\]

where \( g_0(\lambda) = g(|\lambda|(2d + n), \lambda) \).
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Since $S(\Sigma) = S(\mathbb{R}^2)/\{f : f|_{\Sigma} = 0\},$

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**Proposition.** A tempered distribution $\psi$ supported on $\Sigma$ belongs to $S'_0(\Sigma)$ if and only for any “Laguerre point” $(\xi_0, \lambda_0) = (|\lambda_0(2d + n), \lambda_0)$ with $\lambda \neq 0$ there exist a neighborhood $U$ in $\mathbb{R}^2$ and a distribution $\psi$ on the real line such that, for every $g \in D(\mathbb{R}^2)$ supported on $U,$

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Inverse spherical transforms of synthetizable distributions

Identifying functions $g \in L^2(\Sigma, \mu)$ with the distributions $g_\mu$, it is quite clear that $G^*$ is an extension of $G^{-1}$. We then set $G^{-1}\psi = G^*\psi$ for $\psi \in S'_0(\Sigma)$.

**Proposition.** If $f \in S'_{rad}(H_n)$ has compact support, then

(*) \[ u(\xi, \lambda) = \langle f, \Phi_{\xi,\lambda} \rangle \]

is an entire function on $\mathbb{C}^2$ and $Gf = u_\mu$.

Theorem 1c must be interpreted in this sense.

In the other direction (Theorem 2c), if $\psi \in S'_0(\Sigma)$,

(**) \[ v(x, y, t) = \langle \psi, \Phi_{.,.}(x, y, t) \rangle \]

gives an entire function on $H^C_n$, whose restriction to $H_n$ is $G^{-1}\psi$.

Notice that (**) defines an entire function for any $\psi \in \mathcal{E}'(\mathbb{R}^2)$, but this does not satisfy the conditions of Theorem 2c (it is not tempered on $H_n$) unless it is supported on $\Sigma$ and synthetizable.
Inverse spherical transforms of synthetizable distributions

Identifying functions \( g \in L^2(\Sigma, \mu) \) with the distributions \( g_\mu \), it is quite clear that \( \mathcal{G}^* \) is an extension of \( \mathcal{G}^{-1} \). We then set \( \mathcal{G}^{-1}\Psi = \mathcal{G}^*\Psi \) for \( \Psi \in S'_0(\Sigma) \).

**Proposition.** If \( f \in S_{\text{rad}}'(H_n) \) has compact support, then

\[
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\]

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