LeLing3: Inhomogeneous linear systems.

Contents:
- Inhomogeneous linear systems.
- Gauß-Jordan for inhomogeneous systems.
- The general solution.
- General solution as sum of the ‘homogeneous’ solution plus a particular solution.
- Geometric meaning of solutions.

Recommended exercises: GeoLing 3,5.

1 Inhomogeneous linear systems.

A system of equations of the form:

\[
S = \begin{cases} 
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
  a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3 \\
  \cdots \cdots \cdots \cdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m 
\end{cases}
\]

with at least one \(b_j\) different from zero is called inhomogeneous or non-homogeneous system of linear equations in \(n\) unknowns and \(m\) equations.

The zero column \(\mathbf{0} = \begin{pmatrix} 
0 \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix} \) is never a solution of an inhomogeneous system.

Example 1.1. Two examples:

\[
A = \begin{cases} 
  x - y = 1 \\
  x + y = 0
\end{cases} \quad B = \begin{cases} 
  3x_1 - x_2 + x_4 = 3 \\
  x_5 - x_6 = -2
\end{cases}
\]

The system is not hard to remember if one uses the augmented matrix
1.1 The notion of solution.

The notion of solution.

Geometry at Lingotto.

The matrix $A$ is the coefficient matrix:

$$
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    a_{31} & a_{32} & \cdots & a_{3n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    \vdots \\
    b_m
\end{pmatrix}
$$

The matrix $A$ is the coefficient matrix:

$$
\begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    a_{31} & a_{32} & \cdots & a_{3n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    b_2 \\
    b_3 \\
    \vdots \\
    b_m
\end{pmatrix}
$$

and the last column of the augmented matrix is denoted with $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$.

The symbol $(A|B)$ is used to indicate the augmented matrix of an inhomogeneous system having $A$ as coefficient matrix and $B$ as the above mentioned column.

**Example 1.2.** The augmented matrices of the previous examples are:

$$
\begin{pmatrix}
    1 & -1 & 1 \\
    1 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
    3 & -1 & 0 & 1 & 0 & 0 & 3 \\
    0 & 0 & 0 & 0 & 1 & -1 & -2
\end{pmatrix}
$$

Note the relation between the unknown $x_1$ and the first column of $A$, $x_2$ and the second column of $A$, etcetera, and at last, the relation between the equations of $S$ and the rows of $A$.

**Remark 1.3.** Be extremely careful when writing the matrix associated to a system. The matrix of \( \begin{pmatrix} 3x + y = 0 \\ 3y + x = 2 \end{pmatrix} \) is not \( \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \).

1.1 The notion of solution.

The most striking difference between homogeneous and inhomogeneous systems concerns the existence of a solution (the zero column $0$ is not a solution in the inhomogeneous
case, by definition!!). That is to say, inhomogeneous system might admit no solution at all, like in this example:

$$\mathcal{I} = \begin{cases} x = 0 \\ x = 1 \end{cases} .$$

It is a system on 1 equation in 1 unknown that clearly has no solution. Its matrix is:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

**Definition 1.4.** A linear system is **inconsistent** if it has no solution. Otherwise the system is called **consistent**.

Later we will explain how to decide whether a system is consistent or not.

As for the homogeneous systems, a solution to a system $S$ is a column $R = (r_i)$ whose elements satisfy each equation of the system $S$ when substituted to the unknowns $x_i$. Solutions are thus written as **columns**.

### 1.2 Equivalent systems and EROs.

As for homogeneous systems, $S$ and $S'$ are said equivalent if they admit the same solutions.

The elementary row operations $\text{ERO1}, \text{ERO2}, \text{ERO3}$ can be used to generate equivalent systems starting from a given system. Once again one works on the system’s matrix.

The idea is to simplify the system using EROs.

### 1.3 Gauß-Jordan elimination.

The Gauß-Jordan method can be adapted to solve inhomogeneous systems as well. We simply act on the augmented matrix as if dealing with a homogeneous system, using the Gaußstep only on the coefficient matrix $A$. 

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Example 1.5. We show how to solve the system with matrix \( \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \). We find

\[
\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \xrightarrow{R_2/2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \end{pmatrix}
\]

Here ends the Gauß step and begins the Jordan step. Then

\[
\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1/2 \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \end{pmatrix}
\]

At the end \( x = 1/2 \) and \( y = -1/2 \), so the solution is the column \( \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \).

The vertical line separating the matrix coefficient from the column \( B \) tells where to end the Gauß step.

It may happen to have to solve many systems simultaneously. If so, we may just juxtapose the columns \( B \) of the various systems after the vertical line and proceed as before. For instance:

Example 1.6. The systems

\[
\begin{cases}
3x_1 - x_2 + x_4 + 3x_5 = 1 \\
x_5 - x_6 = -1
\end{cases}
\quad
\begin{cases}
3x_1 - x_2 + x_4 + 3x_5 = 4 \\
x_5 - x_6 = 7
\end{cases}
\]

have a common coefficient matrix and can therefore be solved at the same time by considering:

\[
\begin{pmatrix} 3 & -1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{R_1/3} \begin{pmatrix} 1 & -1/3 & 0 & 1/3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{1/3} \begin{pmatrix} 1 & 4/3 \\ -1 & 7 \end{pmatrix}
\]

With \( R_1/3 \) we get \( \begin{pmatrix} 1 & -1/3 & 0 & 1/3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{1/3} \begin{pmatrix} 1 & 4/3 \\ -1 & 7 \end{pmatrix} \)

and ends Gauß. By \( R_1 - R_2 \) we find

\[
\begin{pmatrix} 1 & -1/3 & 0 & 1/3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \xrightarrow{4/3} \begin{pmatrix} 4/3 & -17/3 \\ -1 & 7 \end{pmatrix}
\]

and finish the Jordan step. The equivalent systems obtained are:

\[
\begin{cases}
x_1 - 1/3x_2 + 1/3x_4 + x_6 = 4/3 \\
x_5 - x_6 = -1
\end{cases}
\quad
\begin{cases}
x_1 - 1/3x_2 + 1/3x_4 + x_6 = -17/3 \\
x_5 - x_6 = 7
\end{cases}
\]
1.4 General solution

The solutions come from assigning to \( x_2, x_3, x_4, x_6 \) arbitrary values and finding \( x_1, x_5 \) from the system

\[
\begin{pmatrix}
  \frac{x_2}{3} - \frac{x_4}{3} - x_6 + \frac{4}{3} \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_6 - 1 \\
  x_6
\end{pmatrix}
\]

for the first system and

\[
\begin{pmatrix}
  \frac{x_2}{3} - \frac{x_4}{3} - x_6 + \frac{17}{3} \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_6 + 7 \\
  x_6
\end{pmatrix}
\]

for the second.

1.4 General solution

The solutions of

\[
\begin{align*}
3x_1 - x_2 + x_4 + 3x_5 &= 1 \\
x_5 - x_6 &= -1
\end{align*}
\]

are

\[
\begin{pmatrix}
  \frac{x_2}{3} - \frac{x_4}{3} - x_6 + \frac{4}{3} \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_6 - 1 \\
  x_6
\end{pmatrix}
\]

This column is a sum of two

\[
\begin{pmatrix}
  \frac{x_2}{3} - \frac{x_4}{3} - x_6 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_6 \\
  x_6
\end{pmatrix}
\] + \[
\begin{pmatrix}
  4/3 \\
  0 \\
  0 \\
  0 \\
  -1 \\
  0
\end{pmatrix}
\]

where the former involves the parameters, while the latter does not depend on the unknowns and is called particular solution.

To a non-homogeneous system \( S \) with matrix \((A|B)\) one can associate a homogeneous system: just take \( A \) as matrix of a homogeneous system.

**Theorem 1.7.** Let \( S \) be an inhomogeneous system with matrix \((A|B)\). Call \( X_0 \) an arbitrary solution of \( S \). Then every solution of \( S \) can be expressed as:

\[ X_0 + Y \]

where \( Y \) is the solution of the associated homogeneous system.

So the knowledge of an inhomogeneous solution allows to find all solutions by solving the associated homogeneous system.
Example 1.8. The column \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) solves the system

\[
S = \begin{cases} 
    x + y + z &= 6 \\
    2x - y - 3z &= -9 
\end{cases}
\]

The solutions of \( S \) can be written as: \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) with

\[
\begin{cases} 
    x + y + z &= 0 \\
    2x - y - 3z &= 0 
\end{cases}
\]
1.5 Geometric interpretation.

A linear equation $ax = b$ represents the point $x = \frac{b}{a}$ of the real line, provided $a \neq 0$.

**Example 1.9.** The system \( \begin{cases} 3x = 3 \end{cases} \) has $x = 1$ as solution, a point of the real line.

Similarly, a linear equation $ax + by = c$ in two unknowns $x, y$ represents a straight line in the plane, if $a \neq 0$ or $b \neq 0$.

**Example 1.10.** The system \( \begin{cases} 3x + 2y = 3 \end{cases} \) has as solution a straight line in the plane.

Any equation of a system with two variables can be understood geometrically as a line in the plane. If these lines intersect at a common point the system is consistent, otherwise it will be inconsistent.

The solution of \( \begin{cases} y - x = 1 \\ y + 2x = 4 \end{cases} \) is \( \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \), hence the intersection point of the two lines.

Take the inconsistent system \( \begin{cases} y - x = 1 \\ y - x = -1 \end{cases} \). The lines are parallel, hence do not meet.
1.5 Geometric interpretation.

The equation \( x + y + 3z = 0 \), with three variables, is a plane in space.

The inconsistent system \( \begin{cases} z = 1 \\ z = -1 \end{cases} \) has no solution:

At last, here is the line solution of a consistent system, seen as intersection of two planes in space: