Representations, symmetric products and Hilbert schemes

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Esemplare fuori commercio
Representations, symmetric products and Hilbert schemes

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Abstract  Let $k$ be a commutative ring and let $R$ be a commutative $k$–algebra. The aim of this paper is to define and discuss some connection morphisms between schemes associated to the representation theory of a (non necessarily commutative) $R$–algebra $A$. We focus on the scheme $\text{Rep}_n^A$ of the $n$–dimensional representations of $A$, on the Hilbert scheme $\text{Hilb}_n^A$ parameterizing the left ideals of codimension $n$ of $A$ and on the affine scheme $\text{Spec } \Gamma_n^R(A)^{ab}$ of the abelianization of the divided powers of order $n$ over $A$.

We give a generalization of the Grothendieck-Deligne norm map from $\text{Hilb}_n^A$ to $\text{Spec } \Gamma_n^R(A)^{ab}$ which specializes to the Hilbert Chow morphism on the geometric points when $A$ is commutative and $k$ is an algebraically closed field. Describing the Hilbert scheme as the base of a principal bundle we shall factor this map through the moduli space $\text{Rep}_n^A$ giving a nice description of this Hilbert-Chow morphism, and consequently proving that it is projective.

Keywords  Hilbert-Chow morphism · Hilbert Schemes · Linear Representations · Divided Powers

Mathematics Subject Classification (2000)  14A15 · 14C05 · 16G99

Introduction

Let $k$ be a commutative ring and let $R$ be a commutative $k$–algebra. The aim of this paper is to define and discuss some connection morphisms between schemes associated to the representation theory of a (non necessarily commutative) $R$–algebra $A$. We

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focus on the scheme $\text{Rep}_A^n$ of the $n$-dimensional representations of $A$, on the Hilbert scheme $\text{Hilb}_A^n$ parameterizing the left ideals of codimension $n$ of $A$ and on the affine scheme $\text{Spec } \Gamma^n_R(A)^{ab}$ of the abelianization of the divided powers of order $n$ over $A$.

We give a generalization of the Grothendieck-Deligne norm map from $\text{Hilb}_A^n$ to $\text{Spec } \Gamma^n_R(A)^{ab}$ which specializes to the Hilbert Chow morphism on the geometric points when $A$ is commutative and $k$ is an algebraically closed field. Describing the Hilbert scheme as the base of a principal bundle we shall factor this map through the moduli space $\text{Rep}_A^n/\text{GL}_n$ giving a nice description of this Hilbert-Chow morphism, and consequently proving that it is projective.

The paper goes as follows. In Section 2 we recall the definition of the coarse moduli space $\text{Rep}_A^n/\text{GL}_n$ of the $n$-dimensional representations of $A$, using the algebra of the generic matrices. Then we define an open subscheme $U_A^n$ in $\text{Rep}_A^n \times \text{GL}_n$ and we recall how to construct a principal $\text{GL}_n$-bundle $U_A^n \to U_A^n/\text{GL}_n$. In Section 3 we survey the definition and the main properties of a representable functor of points whose representing scheme $\text{Hilb}_A^n$ is the usual Hilbert scheme of points when $A$ is commutative. In 3.2 we prove that the scheme $U_A^n/\text{GL}_n$ represents $\text{Hilb}_A^n$ and that $U_A^n \to \text{Hilb}_A^n$ is a universal categorical quotient and a $\text{GL}_n$-principal bundle.

In Section 4 we introduce polynomial laws and the representable functor $\Gamma^n_R(A)$ of the divided powers over $A$. When $A$ is flat it is isomorphic to the symmetric tensor functor and actually its representing scheme will play the same role played by the symmetric product in the classical Hilbert-Chow morphism. To connect this object with $\text{Rep}_A^n/\text{GL}_n$ and $\text{Hilb}_A^n$ we consider the affine scheme $\text{Spec } \Gamma^n_R(A)^{ab}$.

Section 5 is devoted to define morphisms which connect the schemes introduced in the previous sections. In 4 we prove that the morphism $p : \text{Hilb}_A^n \to \text{Rep}_A^n/\text{GL}_n$ induced by the "forgetful map" is projective.

Then we define a norm map $hc : \text{Hilb}_A^n \to \text{Spec } \Gamma^n_R(A)^{ab}$ which generalizes the Grothendieck-Deligne norm map and specializes to the classical Hilbert-Chow morphism in the commutative case when the ground field is algebraically closed. The norm map is defined composing the universal map introduced in Section 2 with the determinant and thus factor through $\text{Rep}_A^n/\text{GL}_n$. This point of view allows us to prove in Theorem 5 that the morphism $hc$ is always projective. It follows that the Hilbert-Chow morphism is projective.

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1 Notations

Unless otherwise stated we adopt the following notations:

- $k$ is a fixed commutative ground ring.
- $R$ is a commutative $k$-algebra.
- $B$ is a commutative $R$-algebra.
\[ A \text{ is a not necessarily commutative } R\text{-algebra.} \]
\[ F = k\{x_1, \ldots, x_m\} \text{ denotes the associative free } k\text{-algebra on } m \text{ letters.} \]
\[ \mathcal{N}_-, \mathcal{C}_-, \mathcal{Mod}_- \text{ and } \mathcal{Sets} \text{ denote the categories of } -\text{algebras, commutative } -\text{algebras,} \]
\[ \text{modules and sets, respectively.} \]
\[ \text{we write } A(B, C) := \text{Hom}_A(B, C) \text{ in a category } A \text{ with } B, C \text{ objects in } A. \]

2 The moduli space of representations

2.1 The universal representation

We denotes by \( M_n(B) \) the full ring of \( n \times n \) matrices over \( B \). If \( f : B \to C \) is a ring homomorphism we denote with
\[ M_n(f) : M_n(B) \to M_n(C) \]
the homomorphism induced on matrices.

**Definition 1** By an \textit{n-dimensional representation} of \( A \) over \( B \) we mean a homomorphism of \( R\text{-algebras} \ \rho : A \to M_n(B) \).

The assignment \( B \to \mathcal{N}_R(A, M_n(B)) \) defines a covariant functor \( \mathcal{N}_R \to \mathcal{Sets} \). This functor is represented by a commutative \( R\text{-algebra} \). We report here the proof of this fact to show how this algebra comes up using generic matrices. These objects will be also crucial in the construction of the norm map in Section 5.2.

**Lemma 1** [5, Lemma 1.2.] For all \( A \in \mathcal{N}_R \) there exist a commutative \( R\text{-algebra} \) \( V_n(A) \) and a representation \( \pi_A : A \to M_n(V_n(A)) \) such that \( \rho \mapsto M_n(\rho) \cdot \pi_A \) gives an isomorphism
\[ C_R(V_n(A), B) \cong \mathcal{N}_R(A, M_n(B)) \tag{1} \]
for all \( B \in C_R \).

**Proof** Suppose \( A = R\{x_1, \ldots, x_m\} = R \otimes_k F \) is the free associative \( R\text{-algebra} \) on \( m \) letters. Let \( V_n(A) = R[\xi_{kij}] \) be the polynomial ring in variables \( \{\xi_{kij} : i, j = 1, \ldots, n, k = 1, \ldots, m\} \) over the base ring \( R \). To every \( n\)-dimensional representation of \( A \) over \( B \) it corresponds a unique \( m\)-tuple of \( n \times n \) matrices, namely the images of \( x_1, \ldots, x_m \), hence a unique \( \bar{\rho} \in C_R(R[\xi_{kij}], B) \) such that \( \bar{\rho}(\xi_{kij}) = (\rho(x_k))_{ij} \). Following C.Procesi [21,5] we introduce the generic matrices. Let \( \xi_k = (\xi_{kij}) \) be the \( n \times n \) matrix whose \((i, j)\) entry is \( \xi_{kij} \) for \( i, j = 1, \ldots, n \) and \( k = 1, \ldots, m \). We call \( \xi_1, \ldots, \xi_m \) the generic \( n \times n \) matrices. Consider the map
\[ \pi_A : A \to M_n(V_n(R)), \quad x_k \mapsto \xi_k, \quad k = 1, \ldots, m. \]
It is then clear that the map \( C_R(V_n(A), B) \ni \sigma \mapsto M_n(\sigma) \cdot \pi_A \in \mathcal{N}_R(A, M_n(B)) \) gives the isomorphism (1) in this case.

Let now \( A = R \otimes_k F/C \) be an associative \( R\text{-algebra} \) and write \( \beta : R \otimes_k F \to A \) for the homomorphism such that \( C = \ker \beta \). Suppose \( a_k = \beta(x_k) \), for \( k = 1, \ldots, m \). As before consider \( V_n(R \otimes_k F) = R[\xi_{kij}] \): an \( n\)-dimensional representation \( \rho \) of \( A \) over \( B \) lifts to one of \( R \otimes_k F \) by composition with \( \beta \). This gives a homomorphism \( R[\xi_{kij}] \to B \) that factors through the quotient \( R[\xi_{kij}]/I \), where \( M_n(I) \) is the ideal of \( M_n(R[\xi_{kij}]) \) generated by \( \pi_R \otimes F(I) \).
We set \( V_n(A) = R[\xi_{kij}] / I \) and \( \xi_k^A = (\xi_{kij} + I) = \xi_k + M_n(I) \in M_n(V_n(A)) \) for \( k = 1, \ldots, m \). There is then a homomorphism

\[
\pi_A : A \to M_n(V_n(A))
\]
given by \( \pi_A(a_k) = \xi_k^A \) for \( k = 1, \ldots, m \). To conclude given \( \rho \in \mathcal{N}_R(A, M_n(B)) \) there is a unique homomorphism of commutative \( R \)-algebras

\[
\tilde{\rho} : V_n(A) \to B
\]
for which the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_A} & M_n(V_n(A)) \\
\downarrow{\rho} & & \downarrow{M_n(\tilde{\rho})} \\
M_n(B) & & 
\end{array}
\]
commutes. \( \square \)

**Remark 1** It should be clear that the number of generators \( m \) of \( A \) is immaterial and we can extend the above isomorphism to the not finitely generated case.

**Definition 2** We write \( \text{Rep}_A^n \) to denote \( \text{Spec} V_n(A) \). It is considered as an \( R \)-scheme. The map

\[
\pi_A : A \to M_n(V_n(A)), \quad x_k \mapsto \xi_k.
\]
is called the universal \( n \)-dimensional representation.

Given a representation \( \rho : A \to M_n(B) \) we denote by \( \tilde{\rho} \) its classifying map \( \tilde{\rho} : V_n(A) \to B \) (see (2)).

**Example 1** For the free algebra one has \( \text{Rep}_A^n \cong M_n^n \) the scheme whose \( B \)-points are the \( m \)-tuples of \( n \times n \) with entries in \( B \).

**Example 2** Note that \( \text{Rep}_A^n \) could be quite complicated, as an example, when \( A = \mathbb{C}[x, y] \) we obtain that \( \text{Rep}_A^n \) is the commuting scheme i.e. the couples of commuting matrices and it is not even known (but conjecturally true) if it is reduced or not, see [29].

2.2 An open subscheme

For any \( B \in \mathcal{C}_R \) identify \( B^n \) with \( \mathcal{A}_R^n(B) \), the \( B \)-points of the \( n \)-dimensional affine scheme over \( R \). We introduce another functor that is one of the cornerstones of our construction.

**Definition 3** For each \( B \in \mathcal{C}_R \), let \( U^n_B(B) \) denote the set of \( B \)-points \((\tilde{\rho}, v)\) of \( \text{Rep}_A^n \times_R \mathcal{A}^n_R \) such that \( \rho(A)(Bv) = B^n \), i.e. such that \( v \) generates \( B^n \) as \( A \)-module via \( \rho : A \to M_n(B) \).

**Remark 2** It is easy to check that the assignment \( B \mapsto U^n_B(B) \) is functorial in \( B \). Therefore we get a subfunctor \( U^n_A \) of \( \text{Rep}_A^n \times_R \mathcal{A}^n_R \) that is clearly open and we denote by \( U^n_A \) the open subscheme of \( \text{Rep}_A^n \times_R \mathcal{A}^n_R \) which represents it.
2.3 GL	_n—actions

**Definition 4** We denote by GL	_n the affine group scheme over $R$. Then its $B$—points form the group $GL_n(B)$ of $n \times n$ invertible matrices with entries in $B$, for all $B \in \mathcal{C}_R$.

Define a GL	_n-action on Rep	_n A as follows. For any $\varphi \in \text{Rep}_n^A(B)$, $g \in GL_n(B)$, let $\varphi^g : V_n(A) \to B$ be the $R$—algebra homomorphism corresponding to the representation given by

$$
A \to M_n(B), \\
a \mapsto g(M_n(\varphi) \cdot \pi_A(a))g^{-1}.
$$

(5)

Note that if $\varphi, \varphi'$ are $B$—points of Rep	_n A, then the $A$—module structures induced on $B^n$ by $\varphi$ and $\varphi'$ are isomorphic if and only if there exists $g \in GL_n(B)$ such that $\varphi' = \varphi^g$.

**Definition 5** We denote by Rep	_n A/GL	_n = Spec $V_n(A)^{GL_n(R)}$ the categorical quotient (in the category of $R$—schemes) of Rep	_n A by GL	_n. It is the (coarse) moduli space of $n$—dimensional linear representations of $A$.

We can define an action of GL	_n on Rep	_n A/GL	_n similar to the one on Rep	_n A. Namely for any $B \in \mathcal{C}_R$ let

$$
g(\alpha, v) = (\alpha^g, gv), g \in GL_n(B), \alpha \in \text{Rep}_n^A(B), v \in A_n^R(B).
$$

(6)

It is clear that $U_n^A$ is stable under the above action. Therefore we have that $U_n^A$ is an open GL
	_n—subscheme of Rep	_n A/GL	_n.

**Proposition 1** For the action described above, $U_n^A \to U_n^A/GL_n$ is a locally-trivial principal GL	_n—bundle which is a universal categorical quotient.

**Proof** In [18, Proposition 1] the proposition is proved for the case $A = F$ and $k = R = \mathbb{Z}$. It has been extended in [2, Theorem 7.16] for arbitrary $R$. The statement then follows by observing that Rep	_n A/GL	_n is a closed GL	_n—subscheme of Rep	_n A/GL	_n. □

3 The non commutative Hilbert scheme

For $A \in \mathcal{N}_R$ we recall the definition and the main properties of a representable functor of points whose representing scheme is the usual Hilbert scheme of points when $A$ is commutative. For references see for example [2,18,23,26,27,30].

**Definition 6** For any algebra $B \in \mathcal{C}_R$ we write

$$
\text{Hilb}_n^A(B) := \{ \text{left ideals } I \text{ in } A \otimes_R B \text{ such that } M = A \otimes_R B/I \text{ is projective of rank } n \text{ as a } B\text{-module} \}.
$$

**Proposition 2** The correspondence $\mathcal{C}_R \to \mathbb{Sets}$ induced by $B \mapsto \text{Hilb}_n^A(B)$ gives a covariant functor denoted by $\text{Hilb}_n^A$.

**Proof** Straightforward verification. □

**Proposition 3** [30, Proposition 2] The contravariant functor $R-Schemes \to \mathbb{Sets}$ induced by $\text{Hilb}_n^A$ is representable by an $R$—scheme denoted by $\text{Hilb}_n^A$. 
Proof The functor $\text{Hilb}_A^n$ is a closed subfunctor of the grassmanian functor. \qed

Let now $B$ be a commutative $k$–algebra. Consider triples $(\rho, m, M)$ where $M$ is a projective $B$–module of rank $n$, $\rho : A \to \text{End}_B(M)$ is a $k$–algebra homomorphism such that $\rho(R) \subset B$ and $\rho(A)(Bm) = M$.

**Definition 7** The triples $(\rho, m, M)$ and $(\rho', m', M')$ are equivalent if there exists a $B$–module isomorphism $\alpha : M \to M'$ such that $\alpha(m) = m'$ and $\alpha \rho(a) \alpha^{-1} = \rho'(a)$, for all $a \in A$.

These equivalence classes represent $B$–points of $\text{Hilb}_A^n$ as stated in the following

**Lemma 2** If $B$ is a $k$–algebra, the $B$–points of $\text{Hilb}_A^n$ are in one-one correspondence with equivalence classes of triples $(\rho, m, M)$.

Proof Let $I \in \text{Hilb}_A^n(B)$. Choose $M = A \otimes_R B/I$ and $\rho : A \to \text{End}_B(M)$ given by the composition of the left regular action of $A$ on itself and the $B$–module homomorphism $A \otimes_R B \to M$. Finally let $m = 1_M$.

On the other hand, let $(\rho, m, M)$ be as in the statement, we consider the map

$$A \otimes_R B \to M, \quad a \otimes b \mapsto \rho(a)(bm).$$

(7)

This map is surjective and its kernel $I$ is a $B$–point in $\text{Hilb}_A^n$. If we consider a triple $(\rho', m', M')$ in the same equivalence class of $(\rho, m, M)$, we have that

$$\sum \rho'(a)(ba(m)) = \sum \alpha \rho(a)(ba^{-1} \alpha(m)) = \alpha \sum \rho(a)(bm)$$

so that $I' = I$ and the two triples define the same $B$–point in $\text{Hilb}_A^n$. (See also [26, Lemma 0.1], and [30, Lemma 3]). \qed

**Remark 3** If the triple $(\rho, m, M)$ represents a $B$–point of $\text{Hilb}_A^n$, then the homomorphism $\rho$ induces an $R$–algebra structure on $B$. Moreover a homomorphism $\rho'$ where $(\rho', m', M')$ is in the same equivalence class of $(\rho, m, M)$, induces the same $R$–algebra structure on $B$.

3.1 Examples

3.1.1 Free algebras

Any $A \in \mathcal{N}_R$ is a quotient of an opportune free $R$-algebra $R\{x_m\}$ and in this case it is easy to see that the scheme $\text{Hilb}_A^n$ is a closed subscheme of $\text{Hilb}_R^n\{x_m\}$.

3.1.2 Van den Bergh’s results

In [30] $\text{Hilb}_A^n$ was also defined by M. Van den Bergh in the framework of Brauer-Severi schemes. He proved that the scheme $\text{Hilb}_R^n$ is irreducible and smooth of dimension $n + (m - 1)n^2$ if $k$ is an algebraically closed field (see [30, Theorem 6]).
3.1.3 Commutative case: Hilbert schemes of $n$-points.

Let now $R = k$ be an algebraically closed field and let $A$ be commutative. Let $X = \text{Spec } A$, the $k$-points of $\text{Hilb}_A^n$ parameterize zero-dimensional subschemes $Y \subset X$ of length $n$. It is the simplest case of Hilbert scheme parameterizing closed subschemes of $X$ with fixed Hilbert polynomial $P$, in this case $P$ is the constant polynomial $n$.

The scheme $\text{Hilb}_A^n$ is usually called the Hilbert scheme of the $n$-points of $X$ (see for example [4,14]).

3.2 A principal bundle over the Hilbert scheme

Recall from Proposition 1 the principal bundle $U_A^n \to U_A^n/\text{GL}_n$. We have the following result.

**Theorem 1** The scheme $U_A^n/\text{GL}_n$ represents $\text{Hilb}_A^n$ and $U_A^n \to \text{Hilb}_A^n$ is a universal categorical quotient and a $\text{GL}_n$-principal bundle.

**Proof** In [18, Proposition 1] the proposition is proved for the case $A = F$ and $k = R = \mathbb{Z}$. It has been extended in [2, Theorem 7.16] for arbitrary $R$. The statement then follows by observing that $\text{Rep}_F^n \times_R \mathbb{A}^n_R$ is a closed $\text{GL}_n$-subscheme of $\text{Rep}_F^n \times_R \mathbb{A}^n_R$.

**Remark 4** Madhav Nori gave a direct proof of the representability of $\text{Hilb}_A^n$ for $k = R = \mathbb{Z}$ and $A = F$ in [18]. In this case, since $\mathbb{Z}$ is PID, the Lemma 2 says that $B$-points in $\text{Hilb}_F^n$ are represented by equivalence classes of triples $(\rho, v, k^n)$ where $\rho : F \to M_n(k)$ is an $n$-dimensional representation of the algebra $F$ over $k$ and $v \in k^n$ is such that $\rho(F)v = k^n$. It is then easy to see that $U_A^n \to U_A^n/\text{GL}_n$ is a principal $\text{GL}_n$-bundle and that the above equivalence classes are in bijection with the $\text{GL}_n$-orbits of $U_A^n$ (see the proof of Lemma 2). The isomorphism $U_A^n/\text{GL}_n$ is then obtained by proving that $\text{Hilb}_A^n$ is $\text{Proj } P$ where $P = \oplus_d P_d$ with

$$P_d = \{ f : f(g(a_1, \ldots, a_m, v)) = (\det g)^d f((a_1, \ldots, a_m, v)), \text{ for all } g \in GL_n(k) \}.$$ 

**Example 3** Let $A = \mathbb{C}[x, y]$ then $\text{Hilb}_A^n(\mathbb{C})$ is described by the above Theorem as

$$\{(X, Y, v) : X, Y \in M_n(\mathbb{C}), XY = YX, \mathbb{C}[X, Y]v = \mathbb{C}^n \}.$$ 

This description of the Hilbert scheme of $n$-points of $\mathbb{C}^2$ is one of the key ingredients of the celebrated Haiman’s proof of the $n!$ Theorem [11]. It has also been widely used by H. Nakajima [17].

4 Symmetric products and divided powers

In this part of the paper we introduce a representable functor whose representing scheme will play the same role played by the symmetric product in the classical Hilbert-Chow morphism.
4.1 Polynomial laws

We first recall the definition of polynomial laws between \( k \)-modules. These are mappings that generalize the usual polynomial mappings between free \( k \)-modules. We mostly follow N.Roby (see [24,25]) and we refer the interested reader to these papers for detailed descriptions and proof.

**Definition 8** Let \( M \) and \( N \) be two \( k \)-modules. A polynomial law \( \varphi \) from \( M \) to \( N \) is a family of mappings \( \varphi_A : A \otimes_k M \to A \otimes_k N \), with \( A \in \mathcal{C}_k \) such that the following diagram commutes

\[
\begin{array}{ccc}
A \otimes_k M & \xrightarrow{\varphi_A} & A \otimes_k N \\
\downarrow{f \otimes id_M} & & \downarrow{f \otimes id_N} \\
B \otimes_k M & \xrightarrow{\varphi_B} & B \otimes_k N
\end{array}
\]  

(8)

for all \( A, B \in \mathcal{C}_k \) and all \( f \in \mathcal{C}_k(A,B) \).

**Definition 9** Let \( n \in \mathbb{N} \), if \( \varphi_A(au) = a^n \varphi_A(u) \), for all \( a \in A, u \in A \otimes_k M \) and all \( A \in \mathcal{C}_k \), then \( \varphi \) will be said homogeneous of degree \( n \).

**Definition 10** If \( M \) and \( N \) are two \( k \)-algebras and

\[
\begin{aligned}
\varphi_A(xy) &= \varphi_A(x) \varphi_A(y) \\
\varphi_A(1 \otimes A M) &= 1 \otimes A N
\end{aligned}
\]

for \( A \in \mathcal{C}_k \) and for all \( x, y \in A \otimes_k M \), then \( \varphi \) is called multiplicative.

Let \( A \) and \( B \) be two \( k \)-modules and \( \varphi : A \to B \) be a polynomial law. The following result on polynomial laws is a restatement of Théorème I.1 of [24].

**Theorem 2** Let \( S \) be a set.

1. Let \( L = k[\{x_s\}_{s \in S}] \) and let \( a_s \) be elements of \( A \) such that \( a_s \) is 0 except for a finite number of \( s \in S \), then there exist \( \varphi_\xi((a_s)) \in B, \) with \( \xi \in \mathbb{N}^{(S)} \), such that:

\[
\varphi_L \left( \sum_{s \in S} x_s \otimes a_s \right) = \sum_{\xi \in \mathbb{N}^{(S)}} x^\xi \otimes \varphi_\xi((a_s))
\]

where \( x^\xi = \prod_{s \in S} x_s^{\xi_s} \).

2. Let \( R \) be any commutative \( k \)-algebra and let \( r_s \in R \) for \( s \in S \), then:

\[
\varphi_R \left( \sum_{s \in S} r_s \otimes a_s \right) = \sum_{\xi \in \mathbb{N}^{(S)}} r^\xi \otimes \varphi_\xi((a_s))
\]

where \( r^\xi = \prod_{s \in S} r_s^{\xi_s} \).

3. If \( \varphi \) is homogeneous of degree \( n \), then one has \( \varphi_\xi((a_s)) = 0 \) if \( |\xi| \) is different from \( n \). That is:

\[
\varphi_R \left( \sum_{s \in S} r_s \otimes a_s \right) = \sum_{\xi \in \mathbb{N}^{(S)}, |\xi| = n} r^\xi \otimes \varphi_\xi((a_s))
\]

In particular, if \( \varphi \) is homogeneous of degree 0 or 1, then it is constant or linear, respectively.
Remark 5 The above theorem means that a polynomial law \( \varphi : A \to B \) is completely determined by its coefficients \( \varphi_\xi((a_t)) \), with \( \xi \in \mathbb{N}^\langle T \rangle \).

Remark 6 If \( A \) is a free \( k \)-module and \( \{a_t : t \in T\} \) is a basis of \( A \), then \( \varphi \) is completely determined by its coefficients \( \varphi_\xi((a_t)) \), with \( \xi \in \mathbb{N}^\langle T \rangle \). If also \( B \) is a free \( k \)-module with basis \( \{u \in U\} \), then \( \varphi_\xi((a_t)) = \sum_{u \in U} \lambda_u(\xi)b_u \). Let \( a = \sum_{t \in T} \mu_t a_t \in A \). Since only a finite number of \( \mu_t \) and \( \lambda_u(\xi) \) are different from zero, the following makes sense:

\[
\varphi(a) = \varphi(\sum_{t \in T} \mu_t a_t) = \sum_{\xi \in \mathbb{N}^\langle T \rangle} \mu_\xi \varphi_\xi((a_t)) = \sum_{\xi \in \mathbb{N}^\langle T \rangle} \mu^\xi (\sum_{u \in U} \lambda_u(\xi)b_u) = \sum_{u \in U} (\sum_{\xi \in \mathbb{N}^\langle T \rangle} \lambda_u(\xi)\mu^\xi)b_u.
\]

Hence, if both \( A \) and \( B \) are free \( k \)-modules, a polynomial law \( \varphi : A \to B \) is simply a polynomial map.

Definition 11 Let \( k \) be a commutative ring.

1. For \( M, N \) two \( k \)-modules we set \( \text{Pol}^n_k(M, N) \) for the set of homogeneous polynomial laws \( M \to N \) of degree \( n \).
2. If \( M, N \) are two \( k \)-algebras we set \( \text{Pol}^n_k(M, N) \) for the multiplicative homogeneous polynomial laws \( M \to N \) of degree \( n \).

The assignment \( N \to \text{Pol}^n_k(M, N) \) (resp. \( N \to \text{Pol}^n_k(M, N) \)) determines a functor from \( \text{Mod}_k \) (resp. \( \text{Sets} \)) to \( \text{Sets} \).

Example 4 1. For all \( A, B \in \text{Alg}_k \) it holds \( \text{Pol}^1_k(A, B) = \text{Alg}_k(A, B) \). A linear multiplicative polynomial law is a \( k \)-algebra homomorphism.
2. For \( B \in \text{Comm}_k \) the usual determinant \( \det : M_n(B) \to B \) belongs to \( \text{Pol}^n_k(M_n(B), B) \).
3. For \( B \in \text{Comm}_k \) the mapping \( b \mapsto b^n \) belongs to \( \text{Pol}^n_k(B, B) \).
4. For \( B, C \in \text{Comm}_k \) consider \( p \in \text{Pol}^n_k(B, C) \), since \( p(b) = p(b1) = b^n p(1) = b^n 1_C \) it is clear that in this case raising to the power \( n \) is the unique multiplicative polynomial law homogeneous of degree \( n \).
5. When \( A \in \text{Alg}_k \) is an Azumaya algebra of rank \( n^2 \) over its center \( k \), then its reduced norm \( N \) belongs to \( \text{Pol}^n_k(A, k) \).

4.2 Divided powers

The functors just introduced in Def. 11 are represented by the divided powers which we introduce right now.

Definition 12 For a \( k \)-module \( M \) the divided powers algebra \( \Gamma_k(M) \) (see [24,25]) is an associative and commutative \( k \)-algebra with identity \( 1_k \) and product \( \times \), with generators \( m^{(k)} \), with \( m \in M, k \in \mathbb{Z} \) and relations, for all \( m, n \in M \):

1. \( m^{(i)} = 0, \forall i < 0; \)
2. \( m^{(0)} = 1_k, \forall m \in M; \)
3. \( (am)^{(i)} = a^i m^{(i)} \), \( \forall a \in k, \forall i \in \mathbb{N}; \)
4. \( (m + n)^{(k)} = \sum_{i+j=k} m^{(i)} n^{(j)} \), \( \forall k \in \mathbb{N}; \)
5. \( m^{(i)} \times m^{(j)} = (i+j)^{\binom{i+j}{i}} m^{(i+j)} \), \( \forall i, j \in \mathbb{N}. \)
The $k$-module $\Gamma_k(M)$ is generated by finite products $\times_{i \in I} x_i^{(\alpha_i)}$ of the above generators. The divided powers algebra $\Gamma_k(M)$ is a $\mathbb{N}$-graded algebra with homogeneous components $\Gamma_k^n(M)$, $(n \in \mathbb{N})$, the submodule generated by $\{\times_{i \in I} x_i^{(\alpha_i)} : \mid \alpha \mid = \sum_i \alpha_i = n\}$. One easily checks that $\Gamma_k$ is a functor from $\text{Mod}_k$ to $\mathcal{C}_k$.

4.3 Universal properties

The following properties give the motivation for the introduction of divided powers in our setting.

4.3.1 Functoriality and adjointness

$\Gamma^n_k$ is a covariant functor from $\text{Mod}_k$ to $\text{Mod}_k$ and one can easily check that it preserves surjections. The map $\gamma^n : r \mapsto r^{(n)}$ is a polynomial law $M \to \Gamma_k^n(M)$ homogeneous of degree $n$. We call it the universal map for the following reason: consider another $k$-module $N$ and the set $\text{Mod}_k(\Gamma_k^n(M), N)$ of homomorphisms of $k$-modules between $\Gamma_k^n(M)$ and $N$, we have an isomorphism

$$\text{Mod}_k(\Gamma_k^n(M), N) \cong \text{MP}_k^n(M, N) \quad (12)$$

given by $\varphi \mapsto \varphi \circ \gamma^n$.

4.3.2 The algebra $\Gamma^n_k(A)$

If $A$ is $k$-algebra then $\Gamma_k^n(A)$ inherits a structure of $k$-algebra by $a^{(n)} b^{(m)} = (ab)^{(n+m)}$. The unit in $\Gamma_k^n(A)$ is $1^{(1)}$. It was proved by N.Roby [25] that in this way $R \to \Gamma_k^n(A)$ gives a functor from $k$-algebras to $k$-algebras such that $\gamma^n(a)\gamma^n(b) = \gamma^n(ab)$, $\forall a, b \in A$. Hence

$$\mathcal{N}_k(\Gamma_k^n(A), B) \cong \text{MP}_k^n(A, B) \quad (13)$$

and the map given by $\varphi \mapsto \varphi \circ \gamma^n$ is an isomorphism for all $A, B \in \mathcal{N}_k$. [5, Lemma 1.2.]

4.3.3 Basis change

[24, Thm. III.3, p. 262] For any $R \in \mathcal{C}_k$ and $A \in \text{Mod}_k$ it holds that

$$R \otimes_k \Gamma_k(A) \cong \Gamma_R(R \otimes_k A). \quad (14)$$

When $A$ is a $k$-algebra this gives an isomorphism of $R$–algebras

$$R \otimes_k \Gamma^n_k(A) \cong \Gamma^n_R(R \otimes_k A) \quad (15)$$

for all $n \geq 0$. 


4.4 Symmetric tensors

**Definition 13** Let $M$ be a $k$–module and consider the $n$–fold tensor power $M^\otimes n$. The symmetric group $S_n$ acts on $M^\otimes n$ by permuting the factors and we denote by $\text{TS}_{k}^n(M)$ or simply by $\text{TS}^n(M)$ the $k$–submodule of $M^\otimes n$ of the invariants for this action. The elements of $\text{TS}^n(M)$ are called symmetric tensors of degree $n$ over $M$.

**Remark 7** If $M$ is a $k$–algebra then $S_n$ acts on $M^\otimes n$ as a group of $k$–algebra automorphisms. Hence $\text{TS}_{k}^n(M)$ is a $k$–subalgebra of $M^\otimes n$.

### 4.4.1 Flatness and Symmetric Tensors

Suppose $M \leq N$ (resp. $M \in C_k$). The homogeneous polynomial law $M \to \text{TS}_{k}^n(M)$ given by $x \mapsto x^\otimes n$ gives a morphism $\tau_n : I_k^M(M) \to \text{TS}_{k}^n(M)$ that is an isomorphism when $M$ is flat over $k$. Indeed one can easily prove that $\tau_n$ is an isomorphism in case $M$ is free. The flat case then follows because any flat $k$–module is a direct limit of free modules and $\Gamma$ (resp. $\Gamma^m$) commutes with direct limits.

**Remark 8** Divided powers and symmetric tensor are not always isomorphic. See [16] for counterexamples.

4.5 The abelianization

In this paragraph we introduce the abelianizator and we prove some of its properties. Then we deduce the fact that the abelianization of divided powers commutes with base change.

**Definition 14** Given a $k$–algebra $A$ we denote by $[A]$ the two-sided ideal of $A$ generated by the commutators $[a, b] = ab - ba$ with $a, b \in A$. We write

$$A^{ab} = A/[A]$$

and call it the abelianization of $A$.

We collect some facts regarding this construction.

**Proposition 4** 1. For all $B \in C_k$ the surjective homomorphism $ab_A : A \to A^{ab}$ gives an isomorphism $C_k(A^{ab}, B) \to N_k(A, B)$ via $\rho \mapsto \rho \cdot ab_A$. Equivalently for all $\varphi \in N_k(A, B)$ there is a unique $\overline{\varphi} : A^{ab} \to B$ such that the following diagram commutes

![Diagram](https://via.placeholder.com/150)

2. The assignment $A \to A^{ab}$ induces a covariant functor $N_k \to C_k$ that preserves surjections.

**Proof** The first point is obvious. For $A, B \in N_k$ and $f \in N_k(A, B)$ we have that $f([A]) \subseteq [B]$. This proves the functoriality. □
**Definition 15** We call the just introduced functor the *abelianizator*. For $A, B \in \mathcal{N}_k$ and $f \in \mathcal{N}_k(A, B)$ we denote by $f^{ab} \in \mathcal{C}_k(A^{ab}, B^{ab})$ the abelianization of $f$.

**Proposition 5** The abelianizator preserves surjections.

**Proof** Let $A, B \in \mathcal{N}_k$ and suppose $f \in \mathcal{N}_k(A, B)$ to be surjective. We have that $[B] = f([A])$. □

**Theorem 3** For all $A \in \mathcal{N}_k$ and all $R \in \mathcal{C}_k$ it holds that

$$(R \otimes_k A)^{ab} \cong R \otimes_k A^{ab}$$

**Proof** The homomorphism $id_R \otimes ab_A : R \otimes_k A \to R \otimes_k A^{ab}$ of $R$–algebras induces a unique one $\alpha : (R \otimes_k A)^{ab} \to R \otimes_k A^{ab}$ making the following diagram commutative

$$R \otimes_k A \xleftarrow{id_R \otimes ab_A} R \otimes_k A^{ab} \xrightarrow{\alpha} (R \otimes_k A)^{ab}$$

On the other hand the homomorphism of $k$–algebras given by the composition

$$A \to R \otimes_k A \xrightarrow{ab_{R\otimes_k A}} (R \otimes_k A)^{ab}$$

induces a unique one $\beta : A^{ab} \to (R \otimes_k A)^{ab}$. This $\beta$ extends to a homomorphism of $R$–algebras $\beta : R \otimes_k A^{ab} \to (R \otimes_k A)^{ab}$ that is the inverse of $\alpha$ as can be easily checked. □

**Corollary 1** For all $n \geq 1$

$$\Gamma^n_R(R \otimes_k A)^{ab} \cong R \otimes_k \Gamma^n_k(A)^{ab}$$

**Proof** It follows from (15) and Proposition 3. □

**Remark 9** Corollary 1 remain true by replacing $\Gamma^n$ with $TS^n$ when $A$ is flat. In particular when $R$ is a field.

**Definition 16** We denote by $S^n_A$ the affine $R$–scheme $Spec \, \Gamma^n_R(A)^{ab}$.

**Remark 10** In view of Definition 16 all the results of this section can be rephrased in terms of $R$–schemes. In particular Corollary 1 means that $S^n_A$ base-changes well.

**Remark 11** When $A$ is flat (see paragraph 4.4.1) and commutative one sees that

$$S^n_A \cong X^n/S_n = Spec \, TS^n_R(A)$$

the $n$–fold symmetric product of $X = Spec \, A$.

### 5 Morphisms

In this section we introduce morphisms which connect $Rep^n_A/GL_n$, $\text{Hilb}^n_A$ and $S^n_A$. 
5.1 The forgetful map

Recall from Section 3.2 that we have $U^n_A \to U^n_A/\text{GL}_n \cong \text{Hilb}_A^n$. We have then a commutative diagram

$$
\begin{array}{ccc}
\text{Rep}_A^n \times_R A^n_R & \to & \text{Rep}_A^n \\
\downarrow & & \downarrow \\
\text{Hilb}_A^n & \to & \text{Rep}_A^n/\text{GL}_n \\
\downarrow & & \downarrow \\
\text{Rep}_A^n \times_R A^n_R/\text{GL}_n & \to & 
\end{array}
$$

(16)

**Theorem 4** The morphism

$$p : \text{Hilb}_A^n \to \text{Rep}_A^n/\text{GL}_n$$

in (16) is projective.

**Proof** By Theorem 7.16 and Remark 7.17 in [2] this is true for $A = F$. The result follows since Hilb$_A^n$ is a closed subscheme of Hilb$_F^n$ and Rep$_A^n/\text{GL}_n$ is affine. \qed

The fibers of the map $p$ are very difficult to study. Some results is known for the case $A = F$ and $k$ algebraically closed field [15].

5.2 The norm map

Let $\rho \in \mathcal{N}_R(A, B)$ be a representation of $A$ over $B \in \mathcal{C}_R$. The composition $\det \cdot \rho$ is a multiplicative polynomial law homogeneous of degree $n$ belonging to $\text{MP}_R^n(A, B)$. We denote by $\det_\rho$ the unique homomorphism in $\mathcal{C}_R(N_R^R(A)^{ab}, B)$ such that $\det_\rho = \det_\rho \cdot \gamma^n$, see (13). The correspondence $\rho \mapsto \det_\rho$ is clearly functorial in $B$ giving then a morphism $N_R^R(A)^{ab} \to V_n(A)$ by universality and henceforth a morphism of $R$–schemes

$$
\text{det} : \text{Rep}_A^n \to S_A^n
$$

(17)

Let us look a little bit deeper into the nature of this map. Let $\tilde{\rho} : V_n(A) \to B$ be the unique $B$–point of Rep$_A^n$ such that $\rho = M_n(\tilde{\rho}) \cdot \pi_A$, see Definition 2. It is easy to check that

$$
\tilde{\rho} \cdot \det \cdot \pi_A = \det \cdot M_n(\tilde{\rho}) \cdot \pi_A = \det \cdot \rho
$$

so that the following commutes

$$
\begin{array}{ccc}
A & \xrightarrow{\pi_A} & M_n(V_n(A)) & \xrightarrow{\text{det}} & V_n(A) \\
\| & & \| & \| & \\
A & \xrightarrow{\rho} & M_n(B) & \xrightarrow{\text{det}} & B
\end{array}
$$

(18)
It follows that $\det$ is the affine morphism corresponding to the composition of the universal map $\pi_A$ introduced in (2) with the determinant i.e. to the top horizontal arrows of diagram (18).

The determinant is invariant under basis changes and $\text{Rep}_A^n / \text{GL}_n$ is a categorical quotient. Hence there exists a unique morphism $\det : \text{Rep}_A^n / \text{GL}_n \rightarrow S^n_A$ such that the following commutes

\[
\begin{array}{ccc}
\text{Rep}_A^n & \xrightarrow{\det} & S^n_A \\
\downarrow & & \downarrow \text{det} \\
\text{Rep}_A^n / \text{GL}_n & \xrightarrow{\det} & S^n_A 
\end{array}
\]  

The Hilbert scheme enters the scene again.

**Definition 17** We denote by $hc$ the morphism given by

\[
\begin{array}{ccc}
\text{Hilb}^n_A & \xrightarrow{p} & \text{Rep}_A^n / \text{GL}_n \\
\text{Hilb}^n_A & \xrightarrow{hc} & S^n_A 
\end{array}
\]

There is the following

**Theorem 5** The morphism $hc$ is projective.

**Proof** We know $p$ to be projective by Proposition 4. Since $\det$ is affine and hence separated the result follows. \qed

5.2.1 Commutative case: the Hilbert-Chow morphism and the Grothendieck-Deligne norm.

In this paragraph $A$ is a commutative $R$-algebra and let $X = \text{Spec} A$. Let $R = k$ be a
an algebraically closed field of arbitrary characteristic. In this case $A$ is flat over $k$ and we have an isomorphism $S^n_A \cong X^n / S_n$, the $n$-fold symmetric product of $X$. There is a natural set-theoretic map

\[
\text{Hilb}^n_A \rightarrow X^n / S_n
\]

mapping a zero-dimensional subscheme $Z$ in $\text{Hilb}^n_A$ to the 0-cycle of degree $n$:

\[
[Z] = \sum_{P \in Z} \dim_k (O_Z)_P [P].
\]  

This is indeed a morphism of schemes, the *Hilbert-Chow morphism* (see for example [4,9,12,13]). If $X$ is a non singular curve, this map is an isomorphism (see [6,10]). If $X$ is a non singular surface, the Hilbert-Chow morphism is a resolution of singularities of $X^n / S_n$ (see [9]), but this is no longer true in higher dimensions.

Suppose now $k$ is again a commutative ring and $R \in C_R$. The Grothendieck-Deligne norm map was firstly introduced in [14] as a natural transformation

\[
n_A : \text{Hilb}^n_A \rightarrow \text{Rep}_A^n
\]

generalizing previous works of A.Grothendieck [10] and P.Deligne 6.3.8 [6]. It works this way: given a triple $(\rho, m, M)$ representing a $B-$point of $\text{Hilb}^n_A$ (see Lemma 2) we can consider the composition $\det \cdot \rho$ and this induces the natural transformation $n_A$. 
Remark 12 The Grothendieck-Deligne norm map was the starting point and the inspiration for this work.

Remark 13 It should be clear to the reader that in the preceding hypotheses $hc$ is the same as $n_A$.

Proposition 6 (Par.4.5[8]) Let $k$ be an algebraically closed field. The Grothendieck-Deligne norm and the Hilbert-Chow morphism coincide on the $k$-points.

Theorem 6 The Hilbert-Chow morphism is projective

Proof It follows by Theorem 5 and the Proposition above. □

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References