A MULTiresolution APPROACH TO THE ELECTRIC FIELD
INTEGRAL EQUATION IN ANTENNA PROBLEMS

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Abstract. This paper deals with a multiresolution approach to the finite-element solution of
the Electric Field Integral Equation (EFIE) formulation of the boundary value problem for Maxwell
equations. After defining a multiresolution set of discretized spaces, each of them is first separated
into solenoidal and non-solenoidal complementary spaces. The possibility of obtaining these two
spaces with a scalar-to-vector space mappings is used to consider first two separate scalar wavelet
deformations, and then to transform them properly into the two desired vector finite element sets,
finally joined to obtain the complete multiresolution basis. Numerical results, obtained for real life
antennas, are presented to verify the actual sparsity of the system matrix and to show the fast
convergence behaviour of iterative solvers.

Key words. Multiresolution analysis, Wavelets, Integral Equations, Method of Moments, Tri-
angular meshes, Antennas.

1. Introduction. The problem of finding the electromagnetic field radiated by
an antenna, and its network parameters (impedance, scattering matrix) is of key
importance in applied electromagnetics, and the related numerical simulation tools
are part of all practical antenna design. The typical antenna structure is constituted of
metal conductors (typically assumed ideal), and of dielectric bodies that are piecewise
homogeneous; the related analysis thus amounts to the solution of Maxwell equations
with boundary conditions to be enforced on the surface of material discontinuity; it
is to be noted that a fully three-dimensional electromagnetic problem is intrinsically
vector-valued, and in general cannot be transformed in a straightforward manner to
multiple scalar problems. Various numerical methods have been devised and applied
to antenna analysis over the past decades, either based on the differential formulation
of the boundary-value problem, or on its transformation into an integral equation; for
a review, one can e.g. refer to [14].

In the following, we will explicitly consider antennas made of metal structures
embedded in a homogeneous medium (air, vacuum, etc.) called “free space” in the
following; the metals will be considered perfect electric conductors (PEC). Although
not directly addressed in the presented results, the method proposed here is directly
applicable to the case of conductors embedded in a layered dielectric medium that can
be considered infinite in two directions (as is commonly assumed in the analysis of
“printed” antennas and circuits). We will consider the integral-equation formulation
of the boundary-value problem for time-harmonic dependence, which is very well
suited for this class of structures.

In our setting, we will consider antennas that can be modelled by a two dimen-
sional surface \( \Gamma \), that need not be close. The Electric-Field Integral Equation (EFIE)
for the boundary-value problem is obtained (see e.g. [14]) by invoking the Equiv-
alance theorem (“Huygens principle”) to substitute the PEC surface with unknown
equivalent surface currents \( \mathbf{J} \) that radiate in free space; the total electric field \( \mathbf{E} \)
is then separated into a “primary” (or “incident”) field \( \mathbf{E}_{\text{in}} \) produced in free space by
the independent sources, and a secondary (or “scattered’) field \( \mathbf{E}_{\text{sc}} \) produced by the
equivalent currents, \( \mathbf{E} = \mathbf{E}_{\text{in}} + \mathbf{E}_{\text{sc}} \); the boundary conditions on the conductor surface
requires that the total tangent field be zero, whose enforcement leads to the integral
equation for the unknown current $J$:

$$
-i \omega \mu \int_\Gamma \frac{e^{ik|r-r'|}}{4\pi |r-r'|} J(r') \, dS(r') + \frac{1}{\varepsilon} \nabla_s \int_\Gamma \frac{e^{ik|r-r'|}}{4\pi |r-r'|} \nabla_s \cdot J(r') \, dS(r') - E^\text{tan}_\text{in}(r) = 0
$$

(1.1)

where a time dependence $\exp(-i\omega t)$ has been assumed and suppressed; $\epsilon$ and $\mu$ denote the permeability and the permittivity of the free space and $k = \omega \sqrt{\mu/\varepsilon}$ is the wavenumber; $\nabla_s$ indicates the surface divergence. The operator setting of the EFIE (1.1) and the solution spaces are discussed in [9, 5]; despite its apparent first-kind nature, the EFIE is a well-posed problem because of its hypersingular kernel [5, 24]. The discussion of the above issues is however outside the scopes of the present paper, as it will be explained later on.

The parameters of practical interest (radiated field, impedance, etc.) are obtained from operations on the surface current $J$. In scattering problems, the source term $E^\text{in}$ is typically an incident plane (or spherical) wave radiated by a distant source; the connection between the source term and the physical problem is more involved for antennas, where the feeding region is part of the problem. The typical choice is to relate the primary field $E^\text{in}$ to the so-called “voltage-gap” between the antenna (circuit) terminals (e.g. [14]); a more satisfactory model involves magnetic-current sources [3, 6, 11], and it can be shown that in this case the impedance is a stationary (variational) functional of the current $J$; however, implementation of this source modelling is clumsier than the widespread voltage-gap approximation.

The numerical solution of the problem is customarily obtained via finite-element approximation of the unknown $J$ and weighted-residual (weak) enforcement of the EFIE (1.1); summarizing the basic steps of the procedure, one begins with choosing an approximating space $X$ of finite dimension $d$, spanned by a suitable basis $x_i$, $i = 1 \ldots d$, yet to be discussed, and accordingly expressing the approximate sought-for solution via

$$
J = J_X + e, \quad J_X = \sum_i I_i x_i
$$

(1.2)

where $J_X$ is the approximate solution, obtained by projection onto $X$, and $e$ the approximation error; when (1.2) is inserted into (1.1), the approximation error gives rise to a residual term. The equation (1.1) is then weakly enforced by requiring that the residual term be zero upon projection onto a specified finite-dimensional test space $T$; the most common choice is the Galerkin choice, in which $T = X$. This gives rise to the finite-dimensional linear system

$$
[Z^A + Z^\phi][I] = [V]
$$

(1.3)

where

$$
[Z^A]_{p,q} = i \omega \mu \int_\Gamma x_p(r) \cdot \int_\Gamma \frac{e^{ik|r-r'|}}{4\pi |r-r'|} x_q(r') \, dS(r') \, dS(r)
$$

(1.4)

$$
[Z^\phi]_{p,q} = -\frac{i}{\omega \varepsilon} \int_\Gamma \nabla_s \cdot x_p(r) \cdot \int_\Gamma \frac{e^{ik|r-r'|}}{4\pi |r-r'|} \nabla_s \cdot x_q(r') \, dS(r') \, dS(r)
$$

(1.5)

$$
[V]_i = -\int_\Gamma x_i(r) \cdot E^\text{tan}_\text{in}(r) \, dS(r)
$$

(1.6)

$$
[I]_i = I_i
$$

(1.7)
The overall approach is a boundary-element method, but in computational electromagnetics the above approach is widely known with the name of “Method of Moments” (MoM). For a general shape conductor, the construction of the approximation space \( X \) has two steps: a) the construction of a mesh on the surface \( \Gamma \), or surface discretization, and b) the construction of an appropriate finite-element basis over the cells of the mesh. In the following, we will assume the most-employed such basis, constructed upon a triangular-facet approximation of the actual geometry; the basis was first introduced in computational electromagnetics by Rao, Wilton and Glisson in [16], and since then the basis functions are called RWG; for a discussion on the history of this basis see [14]. The RWG functions are edge-defined (e.g. there is one RWG function per inner edge of the mesh) and are linear-tangential continuous-normal (LTCN) with respect to the defining edge; they are div-conforming (refer to Section 2 for the formal definition). The system matrix \( [Z] = [Z^A] + [Z^b] \) is fully-populated, with poor conditioning and poor convergence performance of iterative solvers for practical large problems; furthermore, the condition number of the problem, and the speed of convergence degrade with increasing density of the mesh (e.g. [24]).

Among the several attempts to overcome the difficulties of the standard approach, the properties of multi-resolution representations have attracted researchers of computational electromagnetics, and mostly their property of having, under certain conditions, vanishing moments; this is usually considered the key to the sparsification of the MoM matrix – i.e. the clipping of entries below a given threshold. The earlier works (e.g. [19, 8]), and many of those that use more structured wavelets, are confined to the analysis of two-dimensional (i.e. scalar) problems, or to wire-type problems in which the current direction is one-dimensional: for the sake of brevity we will not attempt to list them here, since this work specifically concentrates on the issue of vector electromagnetic problems for surfaces. Due to the difficulties of extending wavelets to this latter class of problems, applications in this sense are more recent; in [1, 17, 10] the analysis of planar circuits on rectangular meshes is addressed, yet taking advantage of the separability of the current into two cartesian components. A wavelet-related approach has been applied to scattering from plates in [20], again employing the separability of the problem, and to triangular grids in [21] separating the current directions into local cartesian and diagonal components. In [13], Coifman Intervallic Wavelets are applied to scattering problems from closed smooth bodies, for which the surface current can be conveniently decomposed along two locally orthogonal directions (azimuth and elevation components on spheres and spheroids).

In our approach, we want to keep the geometric flexibility of the RWG approach, yet constructing a multi-resolution (MR) basis for \( X \); our procedure will generate a hierarchical sequence of successively refined meshes over which the MR functions will be defined, but in the end we will be able to express all our MR basis as a linear combination of the RWG basis for the finest mesh. This has the important consequence that the present method allows the re-use of existing MoM codes for the RWG basis.

Another relevant feature of our approach is the use of the MR basis as a preconditioner; i.e., in addition to the more usual emphasis on the sparsifying potential of the MR bases, we will also focus on the effect of the MR transformation on the spectrum of the system matrix, as it results from the improved convergence properties of iterative solvers.

In our approach, we employ first a mapping between the approximate current \( J_X \) and two isotropically scalar quantities [23], that can be viewed as projection of
the Helmholtz decomposition onto the RWG space $X$. This allows the use of scalar multiresolution functions on triangular grids, and this issue has already been addressed in the literature [22]. The vector-to-scalar transformation will be done with a proper operator mapping, that ensures the preservation of the necessary wavelet properties. The employed mapping to scalars is based on the separation of the current $J_X$ into a solenoidal component and a quasi-irrotational remainder. The side benefit of this decomposition is that it prevents the low-frequency problem of the EFIE, i.e. the instability that appears when the cell size is much smaller than the wavelength, as it often happens when the size of the mesh cells is dictated by frequency-independent fine geometrical details and not by the operating wavelength.

The present approach follows the strategy outlined in [15], and applied there to planar and quasi-planar antennas and circuits on rectangular grids. Here, we extend that approach to a triangular grid; preliminary results of the present method have been presented in conference papers [26, 27]. The emphasis of the present paper is on the mathematical properties of the developed multi-resolution vector-valued basis; conversely, a detailed account of the algorithmic issues and its application is given elsewhere [28, 25].

The paper is organized as follows: in Section 2 we describe the Multiresolution Analysis setting. All the computation and justification are postponed to the Appendices in order to avoid technicalities at a first reading. Finally in Section 3 we present our numerical results and we compare them with the previous approaches.

2. Multiresolution Analysis. We start defining projection spaces $X_j$ that will be used in the following. In doing this we need a triangular mesh on $\Gamma$. For sake of simplicity we will assume that $\Gamma$ is a faceted structure i.e. the triangulization can be obtained using planar triangles (examples can be found in Figures 3.1, 3.2 and 3.3). If this is not verified, a continuous bijection is needed in addition to the following construction [29]. In antenna problems, a widely used finite element space is the one introduced in [12] and applied to computational electromagnetics in [16], spanned by the so called Rao-Wilton-Glisson (RWG) functions. These are defined on the inner edges of the mesh and, referring to Figure 2.1, their expression in local coordinates is

\[
f_p = \begin{cases} 
\frac{1}{2A_p} \rho_p^+ & r \in T_p^+ \\
\frac{1}{2A_p} \rho_p^- & r \in T_p^- \\
0 & \text{otherwise}
\end{cases}
\]  

(2.1)

Let us define

\[X_j^{RWG} = \text{span} \{ f_1, \ldots, f_{n_{ed}^j} \}\]

where $j$ labels a particular mesh and $n_{ed}^j$ is the number of inner edges relative to the mesh $j$.

Considering the antenna surface $\Gamma$ and a mesh at level $j$ defined on it, we can obtain a finer mesh bisecting each edge (refer to Figure 2.2). The obtained mesh will be labelled $j + 1$. Thus on each mesh one can define a RWG space, obtaining spaces $X_j^{RWG}, X_{j+1}^{RWG}, X_{j+2}^{RWG}, \ldots$ which satisfy the inclusion relationship

\[X_j^{RWG} \subset X_{j+1}^{RWG}\]  

(2.2)
Fig. 2.1. Rao-Wilton-Glisson function

Different level meshes (see Appendix B).

Relationship (2.2) makes spaces $X_{j}^{RWG}$ eligible for a wavelet construction, according to the formal setting described in Appendix A.1. The coarsest mesh will be labelled $j = 0$. Referring to notation introduced in Appendix A.1, it is a natural choice to set $V_j = X_j^{RWG}$. Now we look for spaces $W_j^{RWG}$ that satisfy the decomposition relationship

$$X_{j+1}^{RWG} = X_j^{RWG} \oplus W_j^{RWG}$$

or equivalently

$$X_l^{RWG} = X_0^{RWG} \oplus \bigoplus_{j=0}^{l} W_j^{RWG}$$

(2.3)

where $X_0^{RWG}$ is the space spanned by the $RWG$ functions defined on the coarsest level mesh.

The space $W_j^{RWG}$ will be decomposed in two parts denoted $W_j^T$ and $W_j^B$ respect-
tively. The space $W_j^T$ is obtained working separately on each cell. More precisely

$$W_j^T = \bigoplus_{i=1}^{n_{cell}} W_{j,i}^T$$

(2.4)

where the spaces $W_{j,i}^T$ are suitable detail spaces (described in the following) of functions with domains included in each of the single coarsest level cell $i$. The space $W_j^B$ is instead a connection space in the sense that

$$W_j^{RWG} = W_j^T \oplus W_j^B$$

(2.5)

and take into account functions with domain between two cells of the coarsest level.

In choosing the basis functions, it is convenient to separate the basis element belonging to the kernel of operator $\nabla_s^r$, the so called solenoidal part (see [23]). For this reason the spaces $W_{j,i}^T$ will be obtained maintaining this separation, in other words

$$W_{j,i}^T = W_{j,i}^{TE} \oplus W_{j,i}^{TM}$$

(2.6)

with

$$(\nabla_s^r) (W_{j,i}^{TE}) = 0$$

(2.7)

2.1. The solenoidal spaces $W_{j,i}^{TE}$. We recall that when $i$ is fixed, we are looking for a decomposition starting from a single triangular cell. The space spanned by $RWG$ functions defined on the mesh obtained by bisecting $j$ times the triangular cell $i$ will be called $T_{j,i}^{RWG}$, and clearly $T_{j,i}^{RWG} \subset X_j^{RWG}$. A basis for the kernel of $\nabla_s^r$ acting on $T_{j,i}^{RWG}$ (with fixed $j$) can be found considering the loop functions (introduced in [23]):

$$L_n(r) = \nabla_s \times \hat{n} \Lambda_n^j(r) \quad n = 1, \ldots, n_{nodes}$$

(2.8)

where $\hat{n}$ is the direction normal to the antenna surface, $\Lambda_n^j$ is the scalar linear Lagrange nodal interpolating function defined on inner node $n$ (equal to one at the node and linearly going to zero on all neighboring nodes) and $n_{nodes}$ is the number of inner nodes of the mesh $j$ (refer to Figure 2.3).

Observe that the elements of the space $\nabla_s^r \cdot (T_{j,i}^{RWG})$ are the functions piecewise constant on each cell at level $j$ with zero mean value. Indeed, referring to notation used in Figure 2.1,

$$\nabla_s^r \cdot f_p = \begin{cases} l/A_p^+ & \text{when } r \in T_p^+ \\ -l/A_p^- & \text{when } r \in T_p^- \\ 0 & \text{elsewhere} \end{cases}$$

(2.9)

Moreover, the dimension of $\nabla_s^r \cdot (T_{j,i}^{RWG})$ is equal to $n_{cell}^j - 1$, where $n_{cell}^j$ is the number of cells at level $j$. In fact from the Euler theorem

$$n_{cell}^j + n_{nodes}^j = n_{ed}^j + 1$$

thus

$$\dim \ker(\nabla_s^r |_{T_{j,i}^{RWG}}) = \dim T_j^{RWG} - \dim \nabla_s^r \cdot (T_j^{RWG}) = n_{ed}^j - n_{cell}^j + 1 = n_{nodes}^j$$
Therefore the functions $U_n(r)$ belongs to the kernel and are linearly independent.

Calling $V^A_{j,i}$ the space obtained by the span of Lagrange functions $\Lambda_n$ on inner nodes

$$\ker(\nabla_s|_{T_{j,i}RWG}) = \nabla_s \times \hat{\Lambda}_{j,i}$$

it is possible to obtain a wavelet construction for the spaces $V_{j,i}^A$ (refer, for example, to [29]). A basis of $V_{j+1,i}^A$ is obtained adding to a basis of $V_{j,i}^A$ the functions $\Lambda_n^{j+1}$ where $n \in N_{j+1,i}^{new}$, the set of new nodes of level $j+1$. In other words, we define

$$W_{j,i}^A = \text{span}\left\{\Lambda_n^{j+1}, n \in N_{j+1,i}^{new}\right\}$$

so that

$$V_{j+1,i}^A = V_{j,i}^A \oplus W_{j,i}^A$$

According to the construction presented in Section A.2 relative to the operator $A = \nabla_s \times$, we obtain

$$W_{j,i}^{TE} = \text{span}\left\{U_n^{j+1}, n \in N_{j+1,i}^{new}\right\} \quad (2.10)$$

and

$$\ker(\nabla_s|_{T_{j,i}RWG}) = \bigoplus_{y=0}^{j} W_{y,i}^{TE} \quad (2.11)$$

### 2.2. The complementary spaces $W_{j,i}^{qTM}$

We recall that $W_{j,i}^{qTM}$ is a space so that

$$T_{j+1,i}^{RWG} = T_{j,i}^{RWG} \oplus W_{j,i}^{TE} \oplus W_{j,i}^{qTM} \quad (2.12)$$

Since we are working with finite dimension spaces it is ensured that the decomposition

$$T_{j,i}^{RWG} = \ker(\nabla_s|_{T_{j,i}RWG}) \oplus \ker(\nabla_s|_{T_{j,i}RWG})^c \quad (2.13)$$
holds. Taking into account the relationship \((2.11)\), the equality \((2.12)\) would be satisfied if

\[
\ker(\nabla_s|_{T^\text{RWG}_j})^c = \bigoplus_{y=0}^j W^y_{y,i}^{TM}
\]

in other words it is sufficient to find a decomposition for \(\ker(\nabla_s|_{T^\text{RWG}_j})^c\) to satisfy \((2.12)\). We will use again the approach explained in Appendix A.2 starting first from a scalar construction. The space \(\nabla_s \cdot (T^\text{RWG}_j)_i\) considered before can be decomposed using a generalized Haar basis \([2]\). This is obtained, referring to Figure 2.4, considering in each cell of level \(j\) the three functions

\[
\psi_{1,k}^{j+1,i}(r) = \begin{cases} 
1 & r \in T_4 \\
-1 & r \in T_1 \\
0 & \text{otherwise}
\end{cases} \tag{2.14}
\]

\[
\psi_{2,k}^{j+1,i}(r) = \begin{cases} 
1 & r \in T_4 \\
-1 & r \in T_2 \\
0 & \text{otherwise}
\end{cases} \tag{2.15}
\]

\[
\psi_{3,k}^{j+1,i}(r) = \begin{cases} 
1 & r \in T_4 \\
-1 & r \in T_3 \\
0 & \text{otherwise}
\end{cases} \tag{2.16}
\]

In this way letting \(n_{cell}^{j,i}\) be the number of cells at level \(j\) obtained bisecting \(j\) times edges of coarsest level cell \(i\) and defining the space

\[
W^H_{j,i} = \text{span} \left\{ \psi_{q,k}^{j+1,i}, k = 1 \ldots, n_{cell}^{j,i}, q = 1, 2, 3 \right\}
\]

we have

\[
\nabla_s \cdot (T^\text{RWG}_j) = \bigoplus_{y=0}^j W^H_{y,i} \tag{2.17}
\]
Considering the operator
\[ A = \left( \nabla_s |_{\ker(\nabla_s |_{T_{j,i}^{RWG}})} \right)^{-1} \]  
and observing that
\[ A : \nabla_s \cdot (T_{j,i}^{RWG}) \mapsto \ker(\nabla_s |_{T_{j,i}^{RWG}})^c \]
using relationships (A.11), (2.17) and (2.18) we obtain
\[ W_{q,j,i}^{TM} = \text{span} \left\{ A \left( \psi_{q,k}^{j+1,i} \right), k = 1, \ldots, n_{cell}^j, q = 1, 2, 3 \right\} \]
One possible restriction of the divergence operator to the complementary kernel is the mapping that leaves unchanged the function domains. In other words, the restriction is such that \( A(\psi_{q}^{k,j+1}) \) is the RWG with the same domain of starting function \( \psi_{q}^{k,j+1} \).
A further modification of the mapping is necessary in order to obtain better moment properties for the basis: on each cell \( k \) of level \( j \), we will consider the three functions
\[ \begin{pmatrix} g_{1,k}^{j+1,i}(r) \\ g_{2,k}^{j+1,i}(r) \\ g_{3,k}^{j+1,i}(r) \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A(\psi_{1}^{k,j+1}) \\ A(\psi_{2}^{k,j+1}) \\ A(\psi_{3}^{k,j+1}) \end{pmatrix} \]
Finally
\[ W_{q,j,i}^{TM} = \text{span} \left\{ g_{q}^{k,j+1}, k = 1, \ldots, n_{cell}^j, q = 1, 2, 3 \right\} \]
and
\[ \ker(\nabla_s |_{T_{j,i}^{RWG}})^c = \bigoplus_{y=0}^{j} W_{y,j,i}^{TM} \]

2.3. The connection spaces \( W_{j}^{B} \). Since spaces \( T_{j,i}^{RWG} \) are spanned by the RWG of level \( j \) completely included in the coarsest level cell \( i \), it is clear that the spaces
\[ T_{j}^{RWG} = \bigoplus_{i=1}^{n_{cell}^0} T_{j,i}^{RWG} \]
are spanned by RWG functions of level \( j \) completely included in one of the coarsest level cells. Moreover, from relationships (2.20), (2.11), (2.13), (2.6), (2.4), (2.5) and (2.3), we obtain
\[ X_{j}^{RWG} = T_{j}^{RWG} \oplus X_{0}^{RWG} \oplus \bigoplus_{y=0}^{j} W_{y}^{B} \]
in other words, \( X_{j}^{RWG} \oplus \bigoplus_{y=0}^{j} W_{y}^{B} \) is the space of RWG of level \( j \) defined between two cells of coarsest level. Referring to the notation introduced in Appendix A.2,
starting from $X_0^{RWG}$ each $RWG$ between two cell of a level $j$ have to be substituted, at level $j+1$, by the two corresponding $RWGs$ labelled $I$ and $II$ (refer to Figure B.1). For the relationship shown between the $RWG$ function and the eight sub $RWGs$ in its domain, instead of taking the two $j + 1$ level $RWG$ it is possible to maintain the $RWG$ of level $j$ and adding a linear combination of the two $j + 1$ level functions $I$ and $II$. The most symmetric choice is to take the loop spanned by functions $I$, $III$, $IV$, $II$, $VII$ and $VI$. The linear independence of the chosen element is ensured by the fact that the two functions $I$ and $II$ have to be taken in the same direction to span the whole $RWG$ and in opposite directions to span the loop. From these consideration it follows that

$$W_j^B = \text{span} \{ l_n^{j+1}, n \in N_{j+1,\text{edg}}^{\text{new}} \}$$

(2.21)

where $N_{j+1,\text{edg}}^{\text{new}}$ is the set of new nodes of level $j + 1$ insisting on coarsest level edges. This is exemplified in Figure 2.5. With this last space, the decomposition of $X_j^{RWG}$ is complete. We recall here the completeness property of the decomposition is ensured, according to (2.3), (2.5) and (2.6), since we have proved the relationships (2.11), (2.20) and (2.21). Summarizing the combination of the basis sets (2.10), (2.19) for each cell $i$ and (2.21) represent an alternative basis for the space $X_j^{RWG}$ that now will be used instead of $RWG$ functions in (1.4), (1.5) and (1.6).

3. Numerical Results. The surfaces $\Gamma$ that will be considered represent some antenna structures actually usable in applications. The first structure is a wide band dipole known as Bowtie antenna. In Figure 3.1 the structure is presented with a mesh that is relative to the coarsest level $j = 0$. Two levels ($j=0$, 1, 2) of the wavelets presented in this work have been used for this structure, so each edge of the coarsest level presented in Figure 3.1 has been bisected twice. The second structure is a sub-array of four printed patches (Figure 3.2), a frequent configuration in the microstrip technology. Also for this structure the figure represents the coarsest level mesh, and two level of wavelets have been used. The last structure is a so called Log-Periodic antenna (Figure 3.3), which is a well known wide band antenna; for this structure one level of wavelets has been used.

Prior to the solution, a diagonal preconditioning is always effected, which is an integral constituent of the method (i.e. the basis allows a diagonal preconditioner to
Fig. 3.1. Γ: Bowtie

Fig. 3.2. Γ: Patch Array

Fig. 3.3. Γ: Log-Periodic
actually condition the system). In addition, to enhance preconditioner performances, the functions of level $j=0$ are made orthogonal using the Singular Value Decomposition method.

To solve the linear system (1.3) obtained by the numerical method, the modified Laczos conjugate gradient presented in [18] has been used. Before solving the system a preconditioning operation is carried out. A diagonal preconditioning is used and, to enhance preconditioner’s performances, the functions of level $j=0$ are made orthogonal using the Singular Value Decomposition method. The performances of the new basis will be tested considering the number of iterations needed by the conjugate gradient to obtain a fixed precision ($10^{-10}$) on the solution. In Figure 3.4, we present the results for the Bowtie antenna; the number of iterations of the modified conjugate gradient needed to solve the system obtained with the usual Rao-Wilton-Glisson basis is compared with the number of iterations needed with the basis presented in this work. This comparison is made for several frequencies $f = \frac{\omega}{2\pi}$, where $\omega$ is the constant which appears in (1.1). The same analysis is performed for the other two structures and the results are presented in Figures 3.5 and 3.6. As already underlined in the introduction, one of the usual achievements of a wavelet basis is that the matrix of the system can be sparsified with a low error on the solution by a thresholding operation, i.e. the setting to zero of all matrix entries below (in absolute value) a properly chosen threshold. For the Bowtie antenna this property is tested in Figure 3.7, where increasing sparsification percentages (number of zero elements over the total number of elements), due to the increasing threshold levels, are plotted versus the obtained relative error on the system solution. The test is performed considering $f = 800$ MHz since this is a resonance frequency for the structure considered. With the same criterion, sparsification performances are tested for the other two structures in Figures 3.8 and 3.9, respectively.

Appendix A. Multilevel decompositions.

We will recall how to define a multilevel decomposition of a separable Banach space $V$, with norm denoted by $\| \cdot \|$. For more details and proofs we refer, among the others, to [4, 7].

A.1. Abstract setting. Let $\{V_j\}_{j \in \mathbb{N}}$ be a family of closed subspaces of $V$ such that $V_j \subset V_{j+1}$, $\forall j \in \mathbb{N}$. For all $j \in \mathbb{N}$, let $P_j : V \rightarrow V_j$ be a continuous linear
operator satisfying the following properties:

\[ \| P_j \|_{L(V, V_j)} \leq C \quad \text{(independent of } j), \]  
\[ P_j v = v, \quad \forall v \in V_j, \]  
\[ P_j \circ P_{j+1} = P_j. \]

Observe that (A.1) and (A.2) imply

\[ \| v - P_j v \| \leq C \inf_{u \in V_j} \| v - u \|, \quad \forall v \in V, \]

where \( C \) is a constant independent of \( j \). Through \( P_j \), we define another set of operators \( Q_j : V \to V_{j+1} \) by

\[ Q_j v = P_{j+1} v - P_j v, \quad \forall v \in V, \forall j \in \mathbb{N}, \]

and the detail spaces

\[ W_j := \text{Im}

\[ Q_j, \quad \forall j \in \mathbb{N}. \]
Thanks to the assumptions (A.1)-(A.3), every $Q_j$ is a continuous linear projection on $W_j$, the sequence $\{Q_j\}_{j \in \mathbb{N}}$ is uniformly bounded in $\mathcal{L}(V, V_{j+1})$ and each space $W_j$ is a complement space of $V_j$ in $V_{j+1}$, i.e.,

$$V_{j+1} = V_j \oplus W_j.$$  \hfill (A.4)

By iterating the decomposition (A.4), we get for any integer $j_n$,

$$V_{j_n} = V_0 \oplus \bigoplus_{j=0}^{j_n-1} W_j,$$ \hfill (A.5)

so that every element in $V_{j_n}$ can be viewed as a rough approximation of itself on a coarse level plus a sum of refinement details. For each $j \in \mathbb{N}$, let us fix a basis for the subspaces $V_j$

$$\Phi_j = \{ \varphi_{jk} : k \in \mathcal{K}_j \},$$ \hfill (A.6)

and for the subspaces $W_j$

$$\Psi_j = \{ \psi_{jk} : k \in \mathcal{K}_j \},$$ \hfill (A.7)
with $\mathcal{K}_j$ and $\mathcal{K}_j$ suitable sets of indices.

We can represent the operators $P_j$ and $Q_j$ in the form
\[ P_jv = \sum_{k \in \mathcal{K}_j} \hat{v}_{jk} \varphi_{jk}, \quad Q_jv = \sum_{k \in \mathcal{K}_j} \hat{v}_{jk} \psi_{jk}, \quad \forall v \in V. \quad (A.8) \]

Functions $\varphi_{jk}$ are often referred to as scaling functions while $\psi_{jk}$ as wavelet functions.

**A.2. Decomposition with a linear operator.** We start with a wavelet decomposition of a space $V$ as described before. We consider a linear, bounded, surjective operator
\[ A : V \mapsto X \]

mapping $V$ onto a linear space $X$. Starting from a decomposition in $V$ we will find, through $A$, a decomposition in $X$. Assume first that $A$ is invertible. Defining the subspaces $X_j = A(V_j)$ we immediately obtain the inclusion relationship
\[ X_{j+1} \subset X_j \quad (A.9) \]

Moreover defining
\[ P_j^X = AP_jA^{-1} \quad (A.10) \]

it is not difficult to show that $P_j^X$ satisfy conditions (A.1)-(A.3).

Similarly, the operators
\[ Q_j^X = AQ_jA^{-1} = P_j^X - P_{j+1}^X \]

allows us to define the subspaces $W_j^X$ as
\[ W_j^X = \text{Im} Q_j. \]

Since $A$ is invertible, it maps linearly independent vectors into linear independent ones, in other words starting from bases of $V_j$ and $W_j$, we obtain bases of $V_j^X$ and $W_j^X$ through the relations
\[ \varphi_{jk}^X = A\varphi_{jk} \quad \text{and} \quad \psi_{jk}^X = A\psi_{jk}. \quad (A.11) \]
We can now extend the construction to a non invertible operator $A$. We may assume a decomposition of $V$ of the form

$$V = \ker A \oplus (\ker A)^c$$

where $(\ker A)^c$ is a complementary space to the kernel. The existence of at least one of these spaces $(\ker A)^c$ is ensured when $V$ is an Hilbert space, in fact since $A$ is bounded, decomposition

$$V = \ker A \oplus (\ker A)^\perp$$

exists. Under this hypothesis

$$\dim (\ker A)^c = \dim (\ker A)^\perp = \dim (\text{Im } A)$$

it follows that $A$ restricted to $(\ker A)^c$ is invertible. In other words, we can apply the previous construction to the operator $A' = A|_{(\ker A)^c}$

**Appendix B. Inclusion relationship between spaces $X^\text{RWG}_j$.** We will make explicit the coefficients that allow to express a RWG function of a given level $j$ as linear combination of RWG functions of level $j + 1$. Referring to B.1 the following conventions are used: RWG functions are identified by the edge which they insist on, the positive sign on a cell indicate that it is a $T^+$ cell (according to conventions used in Figure B.1)) for each if the RWG functions on it and, similarly, the minus sign in a cell indicate that it is of the kind $T^-$. We shall use the following conventions. The greek letter $\alpha$ will indicate coefficients of linear combinations and their index will indicate a particular edge; similarly, the letter $l$ corresponds to the length of an edge, $A$ to the area of a cell. Vertices are labelled with capitals letters, and they are thought as vectors; an independent vector variable is denoted by $x$. We will proceed considering a single cell at a time. We start from cell number 1. On this cell we have to enforce, according to (2.1)

$$\alpha_I l_I x - D \frac{x - D}{2A_1} + \alpha_{II} l_{II} \frac{x - A}{2A_1} = l_{Tot} \frac{x - F}{2A_{Tot}^+}, \quad x \in 1$$

where $A_{Tot}^+ = A_1 + A_2 + A_3 + A_4$ and $l_{Tot} = l_I + l_{II}$. This is equivalent to the two conditions

$$\left\{ \begin{array}{l}
\alpha_I l_I \frac{x - D}{A_1} + \alpha_{II} l_{II} \frac{x - A}{A_1} = l_{Tot} \frac{x - F}{A_{Tot}^+} \\
\alpha_I l_I \frac{x - D}{A_1} + \alpha_{II} l_{II} \frac{x - A}{A_1} = l_{Tot} \frac{x - F}{A_{Tot}^+} 
\end{array} \right. \quad (B.1)$$

thus

$$\alpha_I \frac{l_I}{A_1} (D - A) = \frac{l_{Tot}}{A_{Tot}^+} (F - A)$$

Since the two vectors $(A-D)$ and $(F-D)$ have the same direction (refer to figure B.1), this is actually a scalar condition, obtained taking vectors’ magnitude

$$\alpha_I \frac{l_I}{A_1} |A-D| = \frac{l_{Tot}}{A_{Tot}^+} |A-F| \quad (B.2)$$
Combining with B.1, we get
\[ \alpha_I = \frac{l_{\text{Tot}}}{l_I} \frac{A_1}{A_{\text{Tot}}^+} |A - F| \]  \hspace{1cm} (B.3)
\[ \alpha_{III} = \frac{l_{\text{Tot}}}{l_{\text{III}}} \frac{A_1}{A_{\text{Tot}}^+} \left( 1 - \frac{|A - F|}{|A - D|} \right) \]  \hspace{1cm} (B.4)

For symmetry reasons,
\[ \alpha_{II} = \frac{l_{\text{Tot}}}{l_{\text{II}}} \frac{A_3}{A_{\text{Tot}}^+} \frac{|C - F|}{|C - E|} \]  \hspace{1cm} (B.5)
\[ \alpha_{IV} = \frac{l_{\text{Tot}}}{l_{\text{IV}}} \frac{A_3}{A_{\text{Tot}}^+} \left( 1 - \frac{|C - F|}{|C - E|} \right) \]  \hspace{1cm} (B.6)

Consider now cell number 4; we have
\[ \alpha_V(x - F) = \frac{l_{\text{IV}}}{2A_4} (x - F) \frac{l_{\text{Tot}}}{2A_{\text{Tot}}} \]  \hspace{1cm} \( x \in 4 \)
so that
\[ \alpha_V = \frac{A_4}{A_{\text{Tot}}^+} \frac{l_{\text{Tot}}}{l_V} \]  \hspace{1cm} (B.7)

Now we have to verify that everything is coherent also on cell number 2. We get
\[ \alpha_{III} \frac{l_{\text{III}}}{A_2} (E - x) + \alpha_{IV} \frac{l_{\text{IV}}}{A_2} (D - x) + \alpha_V \frac{l_V}{A_2} (B - x) = \frac{l_{\text{Tot}}}{A_{\text{Tot}}} (x - F) \]
which is equivalent to
\[ \left\{ \begin{array}{ll}
\alpha_{III} \frac{l_{\text{III}}}{A_2} E + \alpha_{IV} \frac{l_{\text{IV}}}{A_2} D + \alpha_V \frac{l_V}{A_2} B & = -\frac{l_{\text{Tot}}}{A_{\text{Tot}}} F \\
\alpha_{III} \frac{l_{\text{III}}}{A_2} E + \alpha_{IV} \frac{l_{\text{IV}}}{A_2} D + \alpha_V \frac{l_V}{A_2} B & = -\frac{l_{\text{Tot}}}{A_{\text{Tot}}} F 
\end{array} \right. \]  \hspace{1cm} (B.8)
Substituting (B.4) (B.6) and (B.7) in (B.8), we get from the first equation the constraint

\[ A_{Tot} = A_1 \frac{|A - F|}{|A - D|} + A_3 \frac{|C - F|}{|C - E|} \]  

(B.9)

This relation is verified when the triangle edges are bisected, as it is in our case. In fact \( \frac{|A - F|}{|A - D|} = 2 \) and \( A_{Tot} = 2A_1 + 2A_3 \) holds since \( A_1 \) and \( A_3 \) are both equal to \( A_{Tot}/4 \). From the second equation we get

\[ A_1 \left( 1 - \frac{|A - F|}{|A - D|} \right) E + A_3 \left( 1 - \frac{|C - F|}{|C - E|} \right) D + A_4 B = -FA_2 \]  

(B.10)

Again this holds in our case since the \( A_i \) are all equal and \( \frac{|A - F|}{|A - D|} = \frac{|C - F|}{|C - E|} = 2 \), so that

\[ B - D = E - F \]

As regards to the remaining coefficients they are easily obtained by symmetry considerations in the following way.

\[ \alpha_I^- = \frac{l_{Tot}}{l_I} \frac{A_5}{A_{Tot}^-} \frac{|A - G|}{|A - H|} \]  

(B.11)

\[ \alpha_{II}^- = \frac{l_{Tot}}{l_{II}} \frac{A_7}{A_{Tot}^-} \frac{|C - G|}{|C - I|} \]  

(B.12)

\[ \alpha_{VI} = \frac{l_{Tot}}{l_{VI}} \frac{A_5}{A_{Tot}^-} \left( 1 - \frac{|A - G|}{|A - H|} \right) \]  

(B.13)

\[ \alpha_{VII} = \frac{l_{Tot}}{l_{VII}} \frac{A_7}{A_{Tot}^-} \left( 1 - \frac{|C - G|}{|C - I|} \right) \]  

(B.14)

\[ \alpha_{VIII} = A_8 \frac{l_{Tot}}{A_{Tot}^-} \frac{l_{Tot}}{l_{VIII}} \]  

(B.15)

Moreover \( \alpha_I^- = \alpha_I \) and \( \alpha_{II}^- = \alpha_{II} \) since the sides are bisected.

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