Implicit numerical methods for the BGK equation

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Abstract

In this work a new class of numerical methods for the BGK model of kinetic equations is introduced. The schemes proposed are implicit with respect to the distribution function, while the macroscopic moments are evolved explicitly. In this fashion, the stability condition on the time step coincides with a macroscopic CFL, evaluated using estimated values for the macroscopic velocity and sound speed. Thus the stability restriction does not depend on the relaxation time and it does not depend on the microscopic velocity of energetic particles. With the technique proposed here, the updating of the distribution function requires the solution of a linear system of equations, even though the BGK model is highly non linear. Thus the proposed schemes are particularly effective for high or moderate Mach numbers, where the macroscopic CFL condition is comparable to accuracy requirements. We show results for schemes of order 1 and 2, and the generalization to higher order is sketched.

Keywords: BGK model, kinetic equations, implicit schemes, hybrid methods.

AMS Subject Classifications: 65M06, 76P05, 82C80.

1 Introduction

This paper presents a new class of numerical schemes for the integration of the BGK model of kinetic equations. The schemes proposed here evolve implicitly the distribution function, which is the main variable in kinetic models, while computing an explicit evolution of macroscopic variables. Thus the schemes are explicit in the space time domain, while being implicit in phase space. With this technique, the Maxwellian equilibrium distribution is evolved explicitly, under a macroscopic CFL stability restriction, thus eliminating the main non linearity of the BGK model. Next, the evolution of the distribution function, containing non equilibrium information, is computed implicitly, solving an algebraic linear system of equations, which turns out to be well conditioned. In the following, we illustrate the motivation and the background of the present work.

The BGK model [7] is an approximation to Boltzmann’s equation which is the main tool for modeling rarefied gas regimes, characterized by a fluid behavior far from equilibrium conditions. In Boltzmann model, the fluid can be interpreted as a set of particles interacting through collisions. In the rarefied regime, the Knudsen number Kn, defined as the ratio between the mean free path \( \lambda \) of the particles and the characteristic dimensions of the problem \( L \) (\( Kn = \lambda/L \)), is relatively large. Since equilibrium is approached through collisions, a large Knudsen number indicates a slow relaxation towards equilibrium. Traditionally, the main field of application of Boltzmann equation has consisted in rarefied regimes, where the Knudsen number is large because the mean free path \( \lambda \gg 1 \), as in the upper layers of the atmosphere.

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More recently, rarefied regimes have attracted attention, because kinetic effects cannot be disregarded when the dimensions of the problem are so small that $L \sim \lambda$, even for a gas in standard conditions, as in micro and nano devices, see for instance the recent book [22].

The main numerical method to integrate Boltzmann equation is the Direct Simulation Monte Carlo (DSMC) method, see for instance the classical reference [8]; a more recent review can be found in [31]. The DSMC scheme is based on the computation of collisions of a large number of sample particles, and it is exceedingly slow close to hydrodynamic regimes, when the number of collisions becomes very large. Moreover its results are polluted by stochastic noise and therefore lack smoothness. Interest has also focused on deterministic methods, see for instance [30]. A recent review on numerical methods for Boltzmann equation is [32], see also references therein.

From a numerical point of view, the BGK [7, 10] model approximating Boltzmann equation for moderate Knudsen numbers is particularly attractive, because the collision integral is simplified. Infact, only precollisional microscopic velocities are involved, since the postcollisional distribution is known to be Maxwellian. Thus, only a precollisional velocity grid must be defined. It has a strong theoretical background, see for instance [33]. For the Boltzmann equation, the Chapman-Enskog expansion yields the system of compressible Euler equations for Kn → 0 and Navier Stokes equations for moderate Kn [11]. The same asymptotic procedure can be carried out for the BGK equations [38], yielding again the compressible Euler and the compressible Navier-Stokes equations, although with an incorrect Prandtl number. If the more sophisticated BGK-ES model [21] is used, the thermal conductivity coefficients can be recovered, [3, 27]. See also [36] for a recent work on the incompressible Navier Stokes limit of the BGK model and [39] for numerical applications, translating the BGK kinetic framework in the construction of reliable numerical schemes for Navier Stokes equations. Finally, Lattice Boltzmann schemes for Navier Stokes equations are based on a simplified BGK model, [37].

Extensive numerical computations have tested the potential of the BGK model to approximate Boltzmann solutions for moderate Kn and Euler solutions for Kn ≪ 1, see [13] and [40]. The BGK model has also been used to evaluate several flows of physical interest, as gas mixtures, [2], reacting gas mixtures [28, 9], or phase change in kinetic regimes, [18, 5]. We also mention an application of BGK-like ideas to the development of models for the behaviour of fluids in nanostructures [20].

The importance of the BGK model in applications has prompted a parallel development of numerical methods tailored to the particular structure of BGK equations. We start mentioning the first order numerical scheme proposed in [13], and the linear second order scheme in [4, 5]. We also mention the third order in space scheme appearing in [40]. The schemes considered so far are not exactly conservative. This issue has been addressed in [26] and [27]. Exact conservation is obtained computing equilibrium at the discrete level. This construction requires the solution of a non linear system of 5 equations for the BGK model and 10 equations for the ES-BGK model at each grid point in space, even for explicit integration in time.

All schemes described so far are either explicit or fully implicit. In the first case, the time step can be severely restricted by a small collision time, close to the hydrodynamic regime. In the second case, the fully implicit BGK equations result in a highly non linear large system of equations.

In [34] we proposed a high order scheme in both space and time to solve the BGK equations, based on an Implicit-Explicit Runge-Kutta scheme, see also [35]. In particular, the convective term is treated explicitly, unlike most numerical schemes for the BGK model in the stiff regime, while the source term is integrated implicitly, thus resolving the stiffness in the hydrodynamic regime. In the case of the BGK model, the implicit treatment of the collision term is very simple, because the local Maxwellian can be evaluated explicitly, see also [13]. The main drawback of IMEX schemes for the BGK model is the fact that the stability condition depends on the fastest microscopic velocities of the grid, the high energy modes of [14]. This paper addresses this convective stability restriction.

In the methods we propose here, we evolve explicitly in time the macroscopic variables, with a CFL stability restriction linked to the macroscopic velocity and local sound speed. The fluxes of the macro-
scopic equations are kinetic, but the fast microscopic modes are weighted with the corresponding values of the distribution function, and therefore their influence is limited, as in [14]. This allows to obtain the local Maxwellian explicitly, thus removing the non linearity of the BGK model. Once the Maxwellian is known, the BGK equation is linear in the distribution function and can be integrated implicitly in time easily, at least for a first order scheme. At second order, even though the BGK equation is linear in $f$, the numerical flux functions are non linear to prevent the onset of spurious oscillations, see [25]. Here we describe a technique to obtain semi-linear numerical fluxes, which avoids the need to solve non linear systems of equations, as in current non oscillatory implicit schemes for conservation laws, see [17].

In many flows of physical interest, kinetic and hydrodynamic regimes co-exist, and recent multiscale approaches seek to solve the whole problem with a domain decomposition technique, where a hydrodynamic solver is used in equilibrium regions, while a kinetic solver is switched on in non equilibrium regimes. In [15] and [16] the kinetic model is given by the BGK equations, while the scheme for the hydrodynamic region can be any Euler solver. In these cases, the present MiMe schemes provide the correct match for the hydrodynamic solver, since they permit to use the same time step all over the computational domain.

Finally, our MiMe schemes present similarities with the Micro-Macro approach of [24]. In that work the kinetic BGK system is split into an equilibrium (Maxwellian) and a non equilibrium equation, through the introduction of a projection operator on equilibrium and non equilibrium contributions. In our case, this projection is not needed, because the whole kinetic BGK equation is evolved, rather than computing and evolving the non equilibrium part.

The paper is organized as follows. In Section 2 the BGK model is reviewed. The IMEX schemes of [34] are recalled in Section 3. The new MiMe schemes are described in Section 4, with details for the first and the second order case. Numerical results obtained with MiMe schemes are presented in Section 5, with a discussion on the entropic behavior of the schemes and the condition number of the implicit system. Finally, we end with a summary and perspectives for future work.

2 The BGK model

In this section, we introduce the equations defining the BGK model, recalling their main properties. For simplicity, we only consider the classical BGK model introduced in [7], using the notation of [13] and [26]. The scheme can be easily extended to more general BGK models.

We consider the initial value problem:

\begin{align}
\frac{\partial f}{\partial t}(x, v, t) + v \cdot \nabla_x f(x, v, t) &= \frac{1}{\tau} (M_f(x, v, t) - f(x, v, t)) \quad t \geq 0, \quad x \in \mathbb{R}^d, v \in \mathbb{R}^N \quad (1) \\
M_f(x, v, t) &= \frac{\rho(x, t)}{(2\pi RT(x, t))^{N/2}} \exp \left( - \frac{\|v - u(x, t)\|^2}{2RT(x, t)} \right).
\end{align}

We consider a monoatomic gas, which, in the general case, gives $d = 3$ and $N = 3$. In the 1D case, $d = 1$ and $N = 3$ and $\nabla_x = (\partial_x, 0, 0)$. In (1) $M_f$ is the Maxwellian obtained from the moments of $f$, namely:

\begin{align}
M_f(x, v, t) &= \frac{\rho(x, t)}{(2\pi RT(x, t))^{N/2}} \exp \left( - \frac{\|v - u(x, t)\|^2}{2RT(x, t)} \right).
\end{align}

The quantities $\rho$, $u$ and $T$ are respectively the macroscopic density, velocity and temperature of the gas, and they are obtained from the moments of $f$, defined below. Given any function $g : \mathbb{R}^N \mapsto \mathbb{R}$, let $\langle g \rangle$ be the quantity $\int_{\mathbb{R}^N} g(v) dv$; if $g : \mathbb{R}^N \mapsto \mathbb{R}^p$, $p > 1$, we still denote by $\langle g \rangle \in \mathbb{R}^p$ the vector whose components are given by $\langle g_i \rangle$. The moments of $f$ are defined by

\begin{align}
\begin{pmatrix}
\rho \\
m \\
E
\end{pmatrix}(x, t) &= \langle f(x, v, t) \phi(v) \rangle, \quad \text{with} \quad \phi(v) = \begin{pmatrix}
1 \\
v \\
\frac{1}{2} \|v\|^2
\end{pmatrix}.
\end{align}

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Here $m$ is momentum, so that the macroscopic velocity is simply $u = m/\rho$, while $E$ is the total energy, and the temperature is obtained from the internal energy $e$, through the relations: $pe = E - \frac{1}{2} \rho ||u||^2$, $e = NRT/2$. In many applications, $N = 3$ which corresponds to a monoatomic gas with three translational degrees of freedom. In our tests, for simplicity, we will choose instead $N = 1$, as in [13]. This corresponds to a gas with a single degree of freedom, so that $e = RT/2$. The only difference with respect to a physical monoatomic gas appears in this rescaling of the temperature, which results, at equilibrium, in a $\gamma$-law gas, with $\gamma = 3$, where $\gamma$ is the ratio of the specific heats. With this choice, all velocity integrals will be evaluated in $\mathbb{R}$ instead of $\mathbb{R}^3$. Another approach to reduce the computational complexity of the velocity integrals, while maintaining the physical properties of the gas, has been introduced in [12] and used, among others, in [5] and [40].

The parameter $\tau$ is the relaxation time, and it is a macroscopic quantity, i.e. $\tau = \tau(x,t)$. The collision frequency is $\tau^{-1}$. In [5], $\tau^{-1} = \lambda_k \rho$, where $\lambda_k$ is a given constant. In [26] the relaxation time is given by $\tau^{-1} = C \rho T^{1-\omega}$ where $\omega$ is the exponent of the viscosity law of the gas (for example, for argon one has $\omega = 0.81$).

In our tests, we will consider the adimensional case, for which:

$$\tau^{-1} = \frac{C}{Kn}, \quad (3)$$

and we will take $C = 1$ as in [13]. However, we will write the scheme allowing for a dependence of $\tau$ on macroscopic variables.

The first macroscopic moments of $f$ are conserved, in the sense that:

$$\partial_t \langle f \rangle + \nabla_x \cdot \langle f v \rangle = 0, \quad (4a)$$

$$\partial_t \langle f v \rangle + \left( v \otimes v f \right) = 0, \quad (4b)$$

$$\partial_t \left( \frac{1}{2} ||v||^2 f \right) + \nabla_x \cdot \left( \frac{1}{2} ||v||^2 v f \right) = 0. \quad (4c)$$

Moreover, it is well known that for the BGK model an entropy principle holds, namely:

$$\partial_t \langle f \log f \rangle + \nabla_x \langle v f \log f \rangle \leq 0, \quad \forall f \geq 0, \quad (5)$$

the equality holding only for $f = M_f$.

A numerical scheme for (1) should be able not only to yield an accurate solution to equation (1), but also to satisfy the conservation equations and the entropy principle in some discretized form. Moreover, it is important that the scheme preserves the asymptotic limits of the BGK model. For $\tau \to 0$, $f \to M_f$, and the conservation laws (4) decay to the closed system of the compressible Euler equations of gas-dynamics. For small values of $\tau$, non equilibrium effects are still present, but a Chapman Enskog expansion shows that the asymptotic limit in this case coincides with the Compressible Navier-Stokes equations, although the standard BGK model does not provide the correct Prandtl number. The correct asymptotic ratio between viscous and thermal effects is recovered by the ES-BGK model, see [3]. We will show that the scheme we propose naturally preserves the Euler limit of the BGK model, see also [6].

The BGK equation (1) is stiff if the relaxation time $\tau$ is small. Thus an explicit time integration of (1) would require very small time steps close to the hydrodynamic limit. On the other hand, the equation is highly non linear, so that an implicit solver might be computationally expensive. In [34] we have proposed an effective way to circumvent the stiffness of the collision term computing the Maxwellian explicitly and thus reducing source term in (1) to a linear operator on $f$ that can therefore be easily treated implicitly. In the next section we will review the algorithm in [34], which will give the background for the time marching algorithm of the scheme proposed here.

Another source of stiffness in (1) is due to the presence of high velocity modes in the convective term. The present work wishes to address both the convective and the relaxation stiffness of (1), without loosing the computational efficiency of [34].
3 IMEX scheme for the BGK equation

The application of Implicit-Explicit (IMEX) schemes to the BGK equation is quite natural if the main stiffness of the problem is due to the relaxation term. This is the case of small relaxation times, when the flow is close to the hydrodynamic regime. Here we consider IMEX Runge-Kutta schemes, see for instance [23] or [29] and references therein. To set notation, consider an autonomous ordinary differential equation of the form:

\[ y'(t) = f(y) + \frac{1}{\varepsilon} g(y). \]

Here, \( \varepsilon \) is a small parameter, emphasizing that the term containing \( g \) is stiff. A Runge-Kutta IMEX scheme consists of a double Butcher tableau, coupling an explicit Runge-Kutta scheme (ERK) with a diagonally implicit Runge-Kutta (DIRK). We will label the coefficients of the tableau for the diagonally implicit DIRK scheme with \( b_i, a_{i,j}, \) while \( \bar{b}_i, \bar{a}_{i,j} \) will be the coefficients of the explicit ERK scheme in the IMEX pair. With this notation, for a \( \nu \) stage IMEX-RK scheme, the advancement of the solution is given by a sequence of “predictor” steps:

\[
\begin{align*}
    y^{(1)} &= y^n + \Delta t \frac{1}{\varepsilon} \sum_{i=1}^\nu \bar{a}_{1,i} g(y^{(1)}) \\
    y^{(i)} &= y^n + \Delta t \sum_{l=1}^\nu \bar{a}_{i,l} f(y^{(l)}) + \Delta t \frac{1}{\varepsilon} \sum_{l=1}^\nu a_{i,l} g(y^{(l)}), \quad i = 2, \ldots, \nu
\end{align*}
\]

and a final corrector step:

\[
y^{n+1} = y^n + \Delta t \left( \sum_{i=1}^\nu \bar{b}_i f(y^{(i)}) + \sum_{i=1}^\nu b_i \frac{1}{\varepsilon} g(y^{(i)}) \right).
\]

At each predictor step, a nonlinear equation must be solved for \( y^{(i)} \), if \( g \) is non linear, while the corrector step involves only previously computed data. We apply the same framework to the BGK equation, integrating only in time. Let \( f^n(x, v) = f(x, v, n \Delta t) \). We drop the dependence on \((x, v)\) for simplicity. Then the predictor steps can be written as:

\[
\begin{align*}
    f^{(1)} &= f^n + \Delta t \sum_{i=1}^\nu a_{1,i} (M_f^{(i)} - f^{(1)}) , \quad (6a) \\
    f^{(i)} &= f^n - \Delta t \sum_{l=1}^\nu \bar{a}_{i,l} v \partial_x f^{(l)} + \Delta t \sum_{l=1}^\nu \frac{a_{i,l}}{\tau^{(l)}} (M_f^{(l)} - f^{(l)}), \quad (6b)
\end{align*}
\]

while the corrector step is:

\[
f^{n+1} = f^n - \Delta t \sum_{i=1}^\nu \bar{b}_i v \partial_x f^{(i)} + \Delta t \sum_{i=1}^\nu \frac{b_i}{\tau^{(i)}} (M_f^{(i)} - f^{(i)}), \quad (7)
\]

where \( \tau^{(i)} = \tau^{(i)}(x) \) and \( M_f^{(i)} = M_f^{(i)}(x, v), i = 1, \ldots, \nu \), are the relaxation times and the Maxwellian functions obtained from the moments of the intermediate stages \( f^{(i)}(x, v) \). The predictor steps are non linear, due to the complex dependence of \( M_f \) on the distribution \( f \). However, exploiting the fact that \( f \) and \( M_f \) have the same first moments, it is easy to rewrite the predictor step in such a way to compute \( f \) easily, solving a linear equation, see [34]. In fact, at each stage \( \left( M_f^{(i)} - f^{(i)} \right) \phi = 0 \). Thus the moments at the \( i \)-th stage are given by:

\[
\left< f^{(i)} \phi \right> = \left< f^n \phi \right> - \Delta t \sum_{l=1}^{i-1} \bar{a}_{il} \left< v \partial_x f^{(l)} \phi \right>.
\]
where, on the RHS, only previously computed quantities appear. Then the Maxwellian \( M_f^{(i)} \) can be computed
\[
M_f^{(i)}(x, v, t) = \frac{\rho^{(i)}(x, t)}{(2\pi RT^{(i)}(x, t))^{N/2}} \exp \left( -\frac{||v - u^{(i)}(x, t)||^2}{2RT^{(i)}(x, t)} \right).
\]
Now let:
\[
B^{(i)} = f^n - \Delta t \sum_{l=1}^{i-1} a_{il} v \partial_x f^{(l)} + \Delta t \sum_{l=1}^{i-1} \frac{a_{il}}{\tau^{(i)}} (M_f^{(l)} - f^{(l)}),
\]
that is \( B^{(i)} \) contains all information coming from the previously computed stages. The equation for \( f^{(i)} \) reduces to:
\[
f^{(i)} = B^{(i)} + \frac{\Delta t}{\tau^{(i)}} a_{ii} \left( M_f^{(i)} - f^{(i)} \right)
\]
which is now linear in \( f^{(i)} \). Note that if \( \tau^{(i)} \ll 1 \), \( f^{(i)} \) immediately relaxes on the local Maxwellian \( M_f^{(i)} \).

For more details on the space and velocity discretization, see [34, 1].

The scheme outlined above derives its time step restriction from the explicit part of the IMEX pair. Since the explicit part solves a linear convection problem, the time step restriction is given by the fastest modes in the convective terms, and these are given by the fastest microscopic velocity in the velocity grid. On the other hand, the fastest modes correspond to small values for \( f \), thus it is quite natural to suppose that the macroscopic \( \nu \) will not depend strongly on the fast velocity modes, see also [14]. For this reason, we would like to have a less severe CFL restriction, linked only to the macroscopic scales of the BGK equation. Furthermore, we want to keep the simplicity of the previous scheme in the evaluation of the Maxwellian.

### 4 MiMe implicit schemes

The system (4) forms a non closed system of conservation laws, because the fluxes cannot be written as functions of the conserved variables. For the simple case of \( N = 1 \) (one degree of freedom in velocity space), the conservation laws can be written as:
\[
\begin{align*}
\partial_t \rho + \nabla_x \cdot m &= 0, \quad (8a) \\
\partial_t m + \nabla_x \cdot (2E) &= 0, \quad (8b) \\
\partial_t E + \nabla_x \cdot \left( \frac{1}{2} ||v||^2 v f \right) &= 0. \quad (8c)
\end{align*}
\]
These equations are not closed, because the energy flux depends on the unknown distribution \( f \). For the case \( N > 1 \), even the momentum flux cannot be written as a function of conserved variables, but still the structure of the system is similar. Note that the dependence on the high velocity modes is weighted by the distribution \( f \), which decays fast for large values of \( |v| \). Thus we expect that the evolution of the macroscopic variables depends only weakly on the fast velocity modes. We rewrite the system above as:
\[
\partial_t U + \nabla_x F = 0, \quad U = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, \quad F = \begin{pmatrix} m \\ 2E \\ \frac{1}{2} ||v||^2 v f \end{pmatrix}.
\]
Since \( U \) and \( f \) are known at the beginning of each time step, we can integrate (9) using any available scheme for conservation laws, as long as we are able to estimate a stability condition and write a consistent numerical flux.

To achieve this goal, we note that the eigenvalues of the Jacobian of this system of equations will converge to the eigenvalues of the Euler equation, as \( \tau \to 0 \). To quantify this statement, we write \( f \) as a
micro-macro decomposition, namely \( f = M_f + \varepsilon g \), where \( M_f \) is as usual the Maxwellian corresponding to \( f \), \( \varepsilon \) is a parameter, measuring the size of the relaxation time, and thus \( \varepsilon g \) is the non equilibrium correction to \( M_f \), which is not necessarily small, see for instance [6]. Introducing the peculiar velocity \( c \), such that \( v = u + c \), the last component of the flux can be written as:

\[
\left\langle \frac{1}{2} \|v\|^2 v_f \right\rangle = uE + \rho uRT + \left\langle \frac{1}{2} ||c||^2 c_f \right\rangle = uE + \rho uRT + \varepsilon \left\langle \frac{1}{2} ||c||^2 c g \right\rangle,
\]

since the third order moment of the Maxwellian is zero. It follows that the flux can be written as:

\[
F = \begin{pmatrix} m \\ 2E \\ uE + \rho uRT \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ 0 \\ \left\langle \frac{1}{2} ||c||^2 c g \right\rangle \end{pmatrix}
\]

Thus the eigenvalues of the Jacobian can be computed as in the compressible Euler equations, within an approximation of order \( \varepsilon \), as:

\[
\lambda = u, u \pm C, \quad \text{where} \quad C = \sqrt{\gamma RT}
\]

is the local speed of sound, where in our simplified case \( \gamma = 3 \). Thus the approximation to the macroscopic CFL will be given by \( \alpha = \max_x (|u| + C) \).

To construct a numerical flux function for equation (9), we need details on the solution of the Riemann problem for our kinetic flux function. A simpler approach is to use a more diffusive numerical flux, as the local Lax Friedrichs numerical flux, which is simply based on a decomposition of the flux into positive and negative parts, and does not require any other detail of the solution to the Riemann problem:

\[
F^+(U) = \frac{1}{2} (F(U) + \alpha U), \quad F^-(U) = \frac{1}{2} (F(U) - \alpha U).
\]

In this fashion, the simple estimate \( \alpha \) for the local CFL will also provide the means to write a numerical flux function for the kinetic flux. Here, \( \alpha \) can be computed as a function of \( x \) and \( t \) as \( \alpha(x,t) = |u(x,t)| + C(x,t) \), giving the Local Lax Friedrichs flux splitting, or we can choose a global value for \( \alpha \) as \( \alpha = \max_{x,t} (|u(x,t)| + C(x,t)) \), yielding the Global Lax Friedrichs flux splitting, which will be used in our tests. Once the macroscopic equation has been integrated, the updated values of the moments are available. With these, we can compute the Maxwellian at the new time level. Note that in this fashion, the Maxwellian at the time \( t^{n+1} \) is computed with the macroscopic CFL, without solving non linear equations.

We now turn to the equation for \( f \). We consider a generic implicit numerical scheme, such that the evolution of \( f \) will be computed as:

\[
f^{n+1} = \mathcal{H}_{\Delta t}(f^n, M_f^n, f^{n+1}, M_f^{n+1}). \tag{11}
\]

We note that we have already computed \( M_f^{n+1} \). For a first order scheme, this is enough to provide a linear dependence of \( \mathcal{H}_{\Delta t} \) on \( f^{n+1} \). For a higher order scheme, \( \mathcal{H}_{\Delta t} \) can depend non-linearly on \( f^{n+1} \), but more details will follow. For the time being, we are ready to outline the structure of MiMe schemes for the BGK equation:

1. Solve the macroscopic equation (9) with any explicit Runge-Kutta scheme, using the stability estimate \( \alpha \) for the CFL. Obtain the macroscopic moments \( U \) at the new time \( t^{n+1} \).
2. Compute the updated Maxwellian \( M_f^{n+1} \), using the new moments \( U^{n+1} \).
3. Solve for \( f^{n+1} \) the time-discretized equation for \( f \), (11).
We start from the first order MiMe scheme. The integration of (9) is carried out with the explicit Euler scheme in time, and the Lax-Friedrichs splitting for the numerical flux. Thus:

\[ \mathbf{U}_{j}^{n+1} = \mathbf{U}_{j}^{n} - \lambda \left[ \mathcal{F}_{j+1/2} - \mathcal{F}_{j-1/2} \right], \]

where \( \mathcal{F}_{j+1/2} \) is the numerical flux, which is given by:

\[ \mathcal{F}_{j+1/2} = \mathcal{F}(\mathbf{U}_{j+1}, U_{j}) = \mathbf{F}^{+}(\mathbf{U}_{j}) + \mathbf{F}^{-}(\mathbf{U}_{j+1}). \]

Here, \( \mathbf{F}^{+} \) and \( \mathbf{F}^{-} \) are the positive and negative parts of the flux \( \mathbf{F} \) defined in (10). Note that the macroscopic moments computed in (12) are exactly conserved, because they are computed with a conservative scheme. So, provided that the numerical solution converges under grid refinement, then the limit solution is a weak solution of (9) by the Lax-Wendroff theorem.

Once the updated macroscopic moments are available, we compute a discrete Maxwellian as:

\[ (\phi M_{v})_{kj}^{n+1} = M(\mathbf{U}_{j}^{n+1}|(x_{j}, v_{k}, t^{n+1})) = \frac{\rho_{k}^{n+1}}{(2\pi RT_{j}^{n+1})^{N/2}} \exp \left( -\frac{\|v_{k} - u_{k}^{n+1}\|^{2}}{2RT_{j}^{n+1}} \right). \]

This is an approximate Maxwellian in the sense that \( (\phi M_{v})_{N_{v}} \approx \mathbf{U} \), with an error depending on the accuracy of the quadrature rule. To achieve the equality sign, it is necessary to compute an exact discrete Maxwellian as in [26]. However, in general the difference between the approximate and the exact discrete Maxwellians are much below the truncation error of the scheme, especially when the moments are computed with Gauss-Hermite quadrature, see also [1]. From the updated moments, we also compute the new relaxation time \( \tau_{j}^{n+1} = \tau(\mathbf{U}_{j}^{n+1}) \).

We can now integrate the equation for \( f \). We obtain a system of \( N_{v} \) partial differential equations. Note that only the equations corresponding to high velocity nodes, i.e., values of \( v_{k} \) such that \( |v_{k}| > \alpha \) need to be integrated implicitly. Accordingly, let \( \theta_{k} = 0 \) if \( |v_{k}| \leq \alpha \) while \( \theta_{k} = 1 \) if \( |v_{k}| > \alpha \). The discrete (in time and velocity) equation for \( f \) is:

\[ f_{k}^{n+1}(x) = f_{k}^{n}(x) - \Delta t \theta_{k} \partial_{x}(v_{k} f_{k}^{n+1}) - \Delta t (1 - \theta_{k}) \partial_{v}(v_{k} f_{k}^{n}) + \frac{\Delta t}{\tau_{j}^{n+1}(x)} (M_{k}^{n+1}(x) - f_{k}^{n+1}(x)). \]
Finally, the space discretization is given by first order upwinding. Let \( v^+ = \max(v_k, 0) \) and \( v^- = \min(v_k, 0) \). Then the first order scheme for \( f \) takes the form:

\[
f^n_{kj}^{+1} = f^n_{kj} - \lambda \theta_k (v^+ (f^n_{kj+1} - f^n_{kj}) + v^- (f^n_{kj} - f^n_{kj-1})) + \lambda (1 - \theta_k) (v^+ (f^n_{kj} - f^n_{kj-1}) + v^- (f^n_{kj} - f^n_{kj})).
\]

The implicit scheme is linear in \( f^{n+1} \) and can be rewritten as:

\[
(1 + \frac{\Delta t}{\tau_j} + \lambda \theta_k |v_k|) f^n_{kj}^{+1} - \lambda \theta_k v^+_k f^n_{kj-1} + \lambda \theta_k v^-_k f^n_{kj+1} =
\]

\[
f^n_{kj} - \lambda (1 - \theta_k) (v^+_k (f^n_{kj} - f^n_{kj-1}) + v^-_k (f^n_{kj} - f^n_{kj})) + \frac{\Delta t}{\tau_j} M^{n+1}_{kj}.
\]

Thus the coefficient matrix of the algebraic system is tridiagonal. Its structure will be more complex in the general \( N > 1 \) case, but it still enjoys a high degree of sparsity. Moreover, we will see in the numerical results section that its condition number is small, so that an iterative scheme will converge in a few iterations. Note also that the implicit equations will be solved only for the fast microscopic velocities, for the first order scheme.

**Second order MiMe scheme**

To extend these ideas to higher order schemes, we will illustrate mainly the second order case, where most of the difficulties already appear. The time integration will be implicit for all velocity nodes. The integration of the macroscopic equation (9) will be carried out with a \( \nu \) stages explicit Runge-Kutta scheme:

\[
U^{n+1}_j = U^n_j - \lambda \sum_{i=1}^{\nu} b_i \left[ F^{(i)}_{j+1/2} - F^{(i)}_{j-1/2} \right],
\]

where the numerical fluxes at the \( i \)-th stage require the predictor steps:

\[
U^{(i)}_j = U^n_j - \lambda \sum_{i=1}^{\nu-1} a_i \left[ F^{(i)}_{j+1/2} - F^{(i)}_{j-1/2} \right].
\]

The evaluation of the numerical flux \( F \) at the \( i \)-th stage requires knowledge of the distribution function \( f^{(i)} \), which is not available. Note however that each explicit Runge-Kutta scheme can be written as a combination of explicit Euler steps. It is the particular combination of these Euler steps that yields the desired accuracy, see [19]. In our case, \( f^{(i)} \) cannot be computed explicitly using the macroscopic CFL, but we can obtain the desired accuracy evaluating \( f^{(i)} \) implicitly with Backward Euler, with a time step \( \Delta t^{(i)} = c_i \Delta t, c_i = \sum a_i \). Thus, once \( U^{(i)} \) is known, we evaluate the approximate Maxwellian \( M(U^{(i)}) \), and we compute \( f^{(i)} \) with (15), with the intermediate time step \( \Delta t^{(i)} \), and with \( \theta_k \equiv 1 \). Note that the computation of \( U^{(i)} \) requires only the numerical fluxes at the previous stages \( l, l = 1, \ldots, i - 1 \). Thus, again, the Maxwellian \( M(U^{(i)}) \) can be computed with no need to solve non linear equations.

The space discretization is again based on the splitting of the flux into positive and negative parts, except that the numerical flux is now applied to reconstructed values at the cell edges, namely:

\[
F_{j+1/2} = F^+(U^+_{j+1/2}, U^-_{j+1/2}) = F^+(U^-_{j+1/2}) + F^-(U^+_{j+1/2}).
\]

where \( U^+_{j+1/2} \) and \( U^-_{j+1/2} \) are the solution values extrapolated to the cell edges, with, for instance, a piecewise polynomial reconstruction, matching the accuracy of the Runge-Kutta scheme. For instance,
for second order accuracy, we use Heun as a second order ERK scheme, thus $\Delta t^{(1)} = 0$, $\Delta t^{(2)} = \Delta t$. In space, we use a piecewise polynomial reconstruction:

$$U_{j+1/2}^+ = U_j + \frac{1}{2}\sigma_j \quad U_{j+1/2}^- = U_j - \frac{1}{2}\sigma_j,$$

where $\sigma_j$ is a non oscillatory reconstructed slope, for instance:

$$\sigma_j = \text{MM}(U_{j+1} - U_j, U_j - U_{j-1}),$$

where MM denotes the MinMod function, see [25]. This piecewise linear reconstruction is second order accurate on smooth flows and away from extrema, where the limiter built in the MinMod function degrades accuracy to first order. This mechanism prevents the onset of spurious oscillations, which might develop with shock formation, see again [25]. In conservation laws, one would reconstruct only conserved variables, namely, density, momentum and total energy. Here however these variables are not enough to define the flux, and a new macroscopic variable needs to be reconstructed, in this case, the heat flux.

To update the distribution function, we distinguish between the stage values of $f$ which must be computed during the advection of macroscopic variables, and the actual update of $f$ after the new Maxwellian has been computed, at the end of the integration of the macroscopic conserved variables. Again, we consider the second order case.

The stage values are computed with the implicit Euler scheme. This scheme is highly diffusive and limiting may not be necessary. When this is the case, as for the smooth test used for the convergence history of the scheme, the slopes are obtained with central differences:

$$\sigma_j = \frac{f_{j+1} - f_{j-1}}{2}.$$

Then, using second order implicit upwinding, and recalling that $v_k^+ - v_k^- = |v_k|$ and $v_k^+ + v_k^- = v_k$, we find the linear system:

$$
\begin{align*}
(1 + \frac{3}{4}\lambda |v_k| + \frac{\Delta t}{f_j^{(2)}}) f^{(2)}_{k,j} + \lambda \left(v_k^- + \frac{1}{4}v_k\right) f^{(2)}_{k,j+1} - \lambda \left(v_k^+ + \frac{1}{4}v_k\right) f^{(2)}_{k,j-1} \\
+ \frac{\lambda}{4} v_k f^{(2)}_{k,j+2} - \frac{\lambda}{4} v_k f^{(2)}_{k,j+2} = f^n_{kj} + \frac{\Delta t}{f_j^{(2)}} M_{kj},
\end{align*}
$$

Thus $f^{(2)}$ is first order accurate in time and second order accurate in space. Note that the coefficient matrix is now pentadiagonal: as usual, when accuracy increases, the sparsity of the system decreases. This approach however may lead to spurious oscillations when discontinuities in space arise. To prevent this problem slopes must be limited; however, the use of slope limiting introduces nasty non linearities which require a non linear solver for the resulting system of equations, see [17]. To avoid this problem, we first estimate $f^{(2)}$ with $\tilde{f}^{(2)}$ computed with a first order accurate in space upwind scheme. Thus, $\tilde{f}^{(2)}$ is non oscillatory but it is only first order accurate. We use this estimate to compute the stencil used by the MinMod function applied to $\tilde{f}^{(2)}$. In this fashion we obtain a semi-linear recipe to compute the limited slopes. More precisely, let $f$ be the unknown function for which the limited slopes are needed, and let $\tilde{f}$ be the predicted value of $f$ which is used to compute the stencil. We construct the limited slopes as follows:

$$\sigma_j(f, \tilde{f}) = \begin{cases} 
0 & \text{if } s_j = (\tilde{f}_{j+1} - \tilde{f}_j)(\tilde{f}_j - \tilde{f}_{j-1}) \leq 0 \\
\tilde{f}_j - f_{j-1} & \text{if } s_j > 0 \text{ and } |\tilde{f}_{j+1} - \tilde{f}_j| > |\tilde{f}_j - \tilde{f}_{j-1}| \\
f_{j+1} - \tilde{f}_j & \text{if } s_j > 0 \text{ and } |\tilde{f}_{j+1} - \tilde{f}_j| < |\tilde{f}_j - \tilde{f}_{j-1}|.
\end{cases}$$

\[21\]
Note that \( \sigma(f, \tilde{f}) \) is linear in \( f \) and it is first order accurate away from extrema.

We introduce the space difference operator \( L_h(f, \sigma(f, \tilde{f})) \) which approximates \( h u \cdot \nabla_x f \):

\[
L_h(f, \sigma(f, \tilde{f}))_{k_j} = [v_k f_{k_j} - v_k^+ f_{k,j-1} + v_k^- f_{k,j+1} + \frac{1}{2} (v_k \sigma_{k_j} - v_k^+ \sigma_{k,j-1} - v_k^- \sigma_{k,j+1})].
\] (22)

We are now ready to compute the stage value \( f^{(2)} \) which is given by the linear system:

\[
(1 + \frac{\Delta t}{\tau^{(2)}}) f^{(2)} + \lambda L_h(f^{(2)}, \sigma(f^{(2)}, \tilde{f}^{(2)})) = f^n + \frac{\Delta t}{\tau^{(2)}} M_n^{(2)}.
\]

Finally, the update values \( f^{n+1} \) are computed with Crank-Nicolson scheme, which has a small amplitude error even at high CFL’s. Since this scheme has no dissipation, limiting the slopes will be necessary, and the limited slopes are computed via (21) using the already available \( f^{(2)} \) as predictor, namely \( \sigma = \sigma(f^{n+1}, f^{(2)}) \). Therefore,

\[
(1 + \frac{\Delta t}{2\tau^{n+1}}) f^{n+1} + \frac{\lambda}{2} L_h(f^{n+1}, \sigma(f^{n+1}, f^{(2)})) = (1 + \frac{\Delta t}{2\tau^n}) f^n - \frac{\lambda}{2} L_h(f^n, \sigma(f^n, f^n)) + \frac{\Delta t}{2} \left( \frac{1}{\tau^{n+1}} M_{n+1}^{(2)} + \frac{1}{\tau^n} M_n^{(2)} \right).
\] (23)

In this fashion, we still obtain a linear system of equations for the grid values of \( f^{n+1} \).

**Realining moment**

In regimes close to equilibrium, when \( \tau \ll 1 \), the microscopic equation for \( f \) reduces to a relaxation to the local Maxwellian. Thus, the macroscopic equations become a closed system of equations for the moments \( U \). When we are far away from equilibrium the moments \( U \) do depend on \( f \) and not only on the Maxwellian. However, in the schemes described so far we have

\[
U^{n+1} = U(f^n, U^n).
\]

To force dependence of \( U^{n+1} \) on \( f^{n+1} \), we correct the moments obtained by the macroscopic equations by computing the moments of the new distribution function. Namely:

\[
U^{n+1} = \left< f^{n+1}, \phi \right>.
\] (24)

This operation will be called *moment realignment* and it enforces a stronger coupling between \( f \) and its moments. Since this operation is necessary only away from equilibrium, we perform moments realignment following an adaptive strategy. We define the local Knudsen number as in [8]:

\[
Kn_{loc}(x, t) = \frac{\lambda}{L(x, t)}, \quad L(x, t) = \frac{\rho}{\rho_x}
\] (25)

and we set

\[
\mathcal{K}(t) = \max_x Kn_{loc}(x, t).
\] (26)

When \( \mathcal{K}(t) \) is larger than a given tolerance \( TOL \), moments realignment is performed. A local in space adaptive strategy would introduce spurious singularities in the moments and therefore it might produce small oscillations.
MiMe1 & BGK
\[ \text{Kn} = 10^{-1} \]

\begin{array}{cccccc}
N_x & \rho & u & T & \rho & u & T \\
40 & 1.868327E-03 & 2.242528E-03 & 1.719817E-03 & 2.639971E-03 & 2.211430E-03 & 4.730606E-03 \\
80 & 1.102374E-03 & 1.883790E-03 & 1.665598E-03 & 1.488081E-03 & 1.216610E-03 & 2.67432E-03 \\
\end{array}

\[ \text{Kn} = 10^{-2} \]

\begin{array}{cccccc}
N_x & \rho & u & T & \rho & u & T \\
40 & 1.780500E-03 & 1.935772E-03 & 2.314669E-03 & 2.628358E-03 & 2.281725E-03 & 5.493755E-03 \\
80 & 1.033931E-03 & 8.770891E-04 & 1.469653E-03 & 1.486475E-03 & 1.261112E-03 & 3.170525E-03 \\
\end{array}

\[ \text{Kn} = 10^{-3} \]

\begin{array}{cccccc}
N_x & \rho & u & T & \rho & u & T \\
40 & 1.638702E-03 & 1.749821E-03 & 3.270563E-03 & 2.593454E-03 & 2.507992E-03 & 6.620122E-03 \\
80 & 9.250416E-04 & 8.357159E-04 & 1.839296E-03 & 1.475344E-03 & 1.413790E-03 & 3.941407E-03 \\
\end{array}

Table 1: Test 1, absolute errors, MiMe1 vs BGK1

5 Numerical results

In this section we illustrate the characteristics of MiMe schemes using a few benchmark problems. We consider a smooth test problem, where the distribution function is locally Maxwellian, but with a macroscopic velocity depending on \(x\) (Test 1), see also [34]. Next we consider two Riemann problems. The first one (Test 2) was proposed in [13]. The second Riemann problem is Lax’ Riemann problem, with \(\gamma = 3\), which is a quite hard classical test problem in gas dynamics.

Test 1 We start with an initial distribution of the kind

\[ f(x, v, 0) = \frac{\rho}{\sqrt{2\pi RT}} \exp\left(-\frac{(v-u_0(x))^2}{2RT}\right), \quad x \in [-1, 1], \]

with constant density \(\rho = 1\) and temperature \(T = 1\) and with

\[ u_0(x) = \frac{1}{\sigma} \left(\exp\left(-\sigma x - 1\right)^2 - 2 \exp\left(-\sigma x + 3\right)^2\right) \]

with \(\sigma = 10\). Thus initially the distribution function \(f\) is smooth, with a localized perturbation in velocity, in a gas with a uniform density and temperature.

Test 2 We take as initial data a distribution which is discontinuous in space.

\[ f(x, v, 0) = \begin{cases} 
\rho_L (2\pi RT_L)^{-1/2} \exp\left(-\frac{(v-u_L)^2}{2RT_L}\right), & 0 \leq x \leq 0.5, \\
\rho_R (2\pi RT_R)^{-1/2} \exp\left(-\frac{(v-u_R)^2}{2RT_R}\right), & 0.5 < x \leq 1
\end{cases} \]

with \((\rho_L, u_L, T_L) = (2.25, 0, 1.125)\) and \((\rho_R, u_R, T_R) = (3/7, 0, 1/6)\). This test is derived from [13].
Table 2: Test 1, absolute errors, MiMe2 vs BGK2

<table>
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<tr>
<th>$\text{N}_x$</th>
<th>$\rho$</th>
<th>$u$</th>
<th>$T$</th>
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Test 3 Again, we take as initial data a distribution discontinuous in space of the form (27), on the interval $-3 < x < 3$ and with the discontinuity located in the middle. In this case, the parameters in (27) are given by: $(\rho_L, (pu)_L, E_L) = (0.4450, 0.311, 8.928), (\rho_R, (pu)_R, E_R) = (0.5, 0.14275), p = (E - \frac{1}{2}(pu)^2/\rho)(\gamma - 1)$ and $T = p/(\rho p)$. This test corresponds to Lax' problem, with $\gamma = 3$.
Gaussian-Hermite quadrature used here, see also [1].

Figure 1 shows the convergence history in the $L^\infty$ norm on test 1. It is clear that both schemes have the expected order as the grid is refined.

Figure 2 illustrates the influence of the moments realignment. We plot for both the first and the second order schemes the temperature profiles for test 2 for a large Knudsen number, $Kn = 10^{-1}$. For the first order scheme moments realignment enhances convergence and reduces artificial diffusion (see top of Figure 2). For the second order scheme, the lack of moments realignment results in spurious oscillations on finer grids where gradients are sharper and therefore rarefaction effects are stronger. For smaller Knudsen numbers the influence of moments realignment is less evident and therefore is not plotted. From now on, we consider only moments realignment with $TOL = 0.1$.

Figure 3 shows the temperature profiles on test 2 for the first and second order schemes. The enhancement of the solution obtained with the second order scheme is evident for all grids. Also note the sharpness of the shock profiles in the hydrodynamic limit for the second order scheme.

Figures 4 and 5 show the entropy decay in time for the first and the second order schemes in a smooth test (test 1) and on a Riemann problem (test 2). In the smooth case (Figure 4) the entropy decay is purely
numerical in the hydrodynamic limit (see right of Fig. 4). The decay is approximately linear in $h$ for the
first order scheme and clearly converges to 0 faster for the second order scheme. In the kinetic regime
the entropy dissipation converges from below to its asymptotic limit, again with a faster convergence in
the second order case. The extra entropy dissipation with respect to the asymptotic solutions is due to
the numerical dissipation of the scheme.

For the Riemann problem entropy dissipation is nonzero even in the hydrodynamic regime due to
entropy production across shocks. This appears clearly in Figure 5 in which the scale on the $y$ axis has
a wider range than in the smooth case. Still convergence occurs from below indicating that the schemes
are entropy stable.

Figure 6 contains the density profiles for the Lax shock tube problem in the hydrodynamic regime.
This test is a classic benchmark for computational gasdynamic and it is known to be quite a hard problem.
Both the first and the second order schemes reproduce the solution remarkably well even on the extremely
course $N_x = 40$ grid. The presence of oscillations whose amplitude decreases under grid refinement is
well known also for Euler solvers.

Figures 7 and 8 show the density, macroscopic velocity, temperature and entropy decay for $Kn = 10^{-1}$
and $Kn = 10^{-2}$, respectively, for the second order scheme. The coarse grid clearly gives very unaccurate
results, which however improve if moments realignment is applied with a stricter tolerance. The bad
behavior of the coarse grid is not surprising: a typical number of grid points for gas dynamics computations
of this test is above 100 nodes. The plots give indications on the evolution of Lax initial data for a rarefied
gas. It is interesting to see that the solution exhibits a rich structure which shows the need of limiters
even in the rarefied regimes.

Finally, Figure 9 shows the behavior of the condition number of the matrices defining the linear
systems (16) and (20) as a function of microscopic velocity values for a fixed $h$ ($N_x = 320$, left) and as
a function of $h$ for a fixed $v$ ($v = \max_\mathbf{k} |v_k|$, right), for several Knudsen numbers and for both the first
and the second order schemes. Here both schemes are fully implicit, with $\theta_k \equiv 1$. It is clear that, for
a fixed space grid, the condition number increases as $v$ increases, but still remaining small in all cases,
and actually approaching 1 in the hydrodynamic regime. On the other hand, for a fixed value of $v$, the

Figure 1: Test 1. Errors on the macroscopic quantities, $L^\infty$ norm (left: Kn = $10^{-1}$, right: Kn = $10^{-5}$),
MiMe1 and MiMe2. The thin lines represent the reference slopes for first and second order convergence.
conditioning of the matrix gets worse as the grid is refined, though the increase is by far sub-linear, and again the actual values remain small. Despite these data are obtained on problems with one degree of freedom in velocity, these results can be generalized to fully 3D problems simulating a real monoatomic gas. In fact the general case contains the sum of three velocity operators which have the same structure and therefore the same eigenvalues of the 1D problem. The coefficient matrix for the fully 3D case will have a banded structure requiring the use of iterative solvers. However these results show that the resulting matrix is well-conditioned and therefore the system is easily solvable e.g. with a few GMRES iterations.

6 Conclusions and perspectives

In this work, we have presented first and second order accurate schemes to integrate the BGK kinetic model. The schemes proposed are characterized by evolving the macroscopic variables explicitly and the distribution function implicitly (MiMe schemes). With the technique proposed, the evolution of the local Maxwellian is computed explicitly, and therefore the system of equations for the implicit values of the
distribution function becomes linear. Moreover, the whole construction is subject to a stability restriction linked to the macroscopic velocity and sound speed, which coincides with the classical CFL condition of Euler equations. For this reason, the schemes proposed are particularly indicated for hybrid schemes based on a domain decomposition in which the BGK model is used to solve the problem in rarefied regimes, while Navier-Stokes or Euler are used for small Knudsen numbers \[15, 16\]: MiMe schemes allow to use the same time step in the whole computational domain, thus minimizing spurious interface effects.

Another aspect we wish to underline is the treatment of implicit limiting in the second order MiMe scheme. It is well known that second and higher order schemes must be non linear, even for the linear advection equation, to prevent the onset of spurious oscillations in the presence of discontinuities. This fact has always hindered the application of implicit time integration for conservation laws, since non linear limiting would result in a non linear system of equations, see for instance \[17\]. We believe that the technique adopted here, based on lower order prediction to determine non oscillatory stencils, might be of interest also to develop implicit schemes for conservation laws with stiff propagation speeds, as in low Mach number gas dynamics. A work in this direction is in preparation.

Further investigation will involve two dimensional problems and a more realistic 3D space for the microscopic velocities, taking into account the reduction of variables found in \[12\], and the extension of

Figure 3: Test 2. Temperature profiles. Top: MiMe1, bottom: MiMe2. Left: \(Kn = 10^{-2}\), right: \(Kn = 10^{-5}\).
Figure 4: Entropy decay, test 1 (smooth test). Top: MiMe1, bottom: MiMe2. Left: Kn = 10^{-1}, right: Kn = 10^{-5}. 
Figure 5: Entropy decay, test 2 (Riemann problem). Top: MiMe1, bottom: MiMe2. Left: $Kn = 10^{-1}$, right: $Kn = 10^{-5}$.

Figure 6: Test 3. Density profiles, $Kn = 10^{-5}$. Left: MiMe1, right: MiMe2.
Figure 7: Test 3, Kn = 10\(^{-1}\), MiMe2. Left to right and top to bottom: \(\rho\), \(u\), \(T\), entropy decay.
Figure 8: Test 3, $Kn = 10^{-2}$, MiMe2. Left to right and top to bottom: $\rho$, $u$, $T$, entropy decay.

Figure 9: Condition number of the matrix defining the linear systems (16) and (20) versus $v$ for $N_x = 320$ (left) and versus $h$ for $v = \max_k |v_k|$ (right)
the scheme to the ES-BGK model.

References


