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Abstract
Photon transport is studied in an interstellar cloud occupying a time-dependent region $V(t) \subset \mathbb{R}^3$. First, under the assumption that the surface $\partial V(t)$ is a known function of $t \in [0, t_{\text{max}}]$, existence and uniqueness of the solution of the corresponding direct problem are proven. Then, an inverse problem is examined, under the assumptions that the time behaviour of $\partial V(t)$ is unknown and the photon distribution function is measured at a location far from the cloud. Some numerical experiments are then provided.

Keywords: photon transport, interstellar clouds, inverse problems.

AMS subject classification: 85A25, 82C70, 34K29, 65M32

1 Introduction
In this paper, we study photon transport in an interstellar cloud, [1, 2, 3]. Interstellar clouds are astronomical objects [4] that occupy large regions of the interstellar space: the diameter of an average cloud may range from $10^{-1}$ to 10 parsec, i.e. from $10^3$ to $10^5$ times the diameter of our solar system (which at present is moving through a “local” cloud and may exit from it during the next $10^4$ years, [5]).

The clouds under consideration are composed of a low density mixture of gases and dust grains, mainly hydrogen molecules with some 1%-2% (of the total mass) of small size carbonaceous grains and silicates, [6]. Typical particle densities may be of the order of $10^4$ particles/cm$^2$, i.e. $10^{-15}$ times the density of earth atmosphere at sea level and $10^4$ times the density in the intergalactic vacuum. Interstellar clouds may also contain “clumps”, i.e. relatively small regions where the particle density is approximately $10^4$ times the average density in the cloud.

With reference to Figure 1 we consider an interstellar cloud which occupies the time-dependent region $V(t) \subset \mathbb{R}^3$ bounded by the surface $\partial V(t)$. The state variable of the system is the distribution function

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*Aldo has left us on June 20th 2009. The paper was finished before he passed away. The last editing was made by the coauthors.*
of photons inside the cloud. We first consider the direct evolution problem and prove existence and uniqueness of the solution, under the assumption that the surface $\partial V(t)$ is suitably given as a function of $t \in [0, t_{\text{max}}]$. In the second part of the paper, we propose a simplified version of the problem, by assuming that the cloud remains homogeneous during the observation time. Using such an assumption, we introduce a perturbation on the law of cloud motion and we provide an error estimate on the photon distribution function with respect to the unperturbed case. This part is then devoted to an inverse problem, of some interest in astrophysics, under the conceivable assumption that the cloud has a spherical symmetry. In fact, we shall assume that the photon distribution function is measured at a location $\hat{x}$, far from the cloud ("far field" measurement), at times $0 = t_0 < t_1 < t_2 < \cdots < t_n = t_{\text{max}}$. From the knowledge of these $(n + 1)$ data, we shall indicate how the time behaviour of $\partial V(t)$ can be inferred $\forall t \in (0, t_{\text{max}}]$. Such a behaviour may give useful indications on the evolution of the cloud also at times $t > t_{\text{max}}$.

Some numerical experiments have been performed in the spherical symmetry case, both on the direct and on the inverse problem. Here some results obtained on the direct problem are reported, showing the behaviour of the photons distribution function on the boundary of the cloud. Starting from the knowledge of data obtained by the direct simulations, some computations on the inverse problem are performed.

2 The mathematical model

Let the interstellar cloud under consideration occupy the region $V(t) \subset \mathbb{R}^3$, bounded by the closed "regular" surface $\partial V(t)$. We shall agree that the following holds (see Figure 1).

1. Given any $x \in V(t)$ and any unit vector $u \in S$ (where $S$ is the surface of the unit sphere), let $R(x, u, t) > 0$ be such that $x - R(x, u, t)u \in \partial V(t)$; then, for each $t \in [0, t_{\text{max}}]$, $R(x, u, t)$ is assumed to be a continuous function of $(x, u) \in V(t) \times S$ and such that $\partial R/\partial t$ exists and is strictly negative.

2. At any $t \in [0, t_{\text{max}}]$, $V_{\text{min}} \subset V(t) \subset V_{\text{max}}$ (and $V(t + h) \subset V(t)$, $\forall t, t + h \in [0, t_{\text{max}}]$, $h > 0$ because of item (1)).

3. UV-photon sources (stars) are contained in the time-independent region $V_q \subset V_{\text{min}}$ and are modelled by a "source term" of the form $q(x, u, t)\chi_q(x)$, where $\chi_q(x)$ is the characteristic function of $V_q$ and $q(x, u, t)$ will be given in a suitable way.

4. The scattering and the total macroscopic cross section (which take into account interactions photons-cloud) have the form $\sigma_s(x, t)\chi_{V(t)}(x)$, $\sigma(x, t)\chi_{V(t)}(x)$, respectively, where $\chi_{V(t)}(x)$ is the characteristic function of $V(t)$ and $\sigma_s \geq 0$, $\sigma \geq 0$ belong to $L^\infty(V_{\text{max}})$ at each given $t \in [0, t_{\text{max}}]$, and are such that $\sigma \leq \sigma_s$.

Remark 1 According to assumption (1), the region $V(t)$ (i.e., the cloud) shrinks during the time interval $[0, t_{\text{max}}]$. Such a behaviour may be the first step towards the creation of a protostar [2, 3]. We also remark that the case in which a cloud expands or has a periodic behaviour may be dealt with in a similar way. □

We are now in a position to write the photon transport equation in the region $V_{\text{max}} \supset V(t)$. If $N(x, u, t)$ is the distribution function of photons that, at time $t \in [0, t_{\text{max}}]$, are at $x \in V_{\text{max}}$ and have
velocity $cu$, being $c$ the speed of light, then we have

$$
\partial_t N(x,u,t) = -cu \cdot \nabla_x N(x,u,t) - c\sigma(x,t)\chi_{\nu(x)}(x)N(x,u,t) \\
+ \frac{c}{4\pi} \sigma_s(x,t)\chi_{\nu(x)}(x) \int_S N(x,u',t)du' + cq(x,u,t)\chi_\nu(x),
$$

(1)

with $(x,u,t) \in V_{\text{max}} \times S \times (0,t_{\text{max}})$, and

$$
N(x,u,0) = N_0(x,u)\chi_\alpha(x), \quad (x,u) \in V_{\text{max}} \times S
$$

(2)

$$
N(y,u,t) = 0 \quad \text{if} \quad y \in \partial V_{\text{max}} \quad \text{and} \quad u \cdot \nu(y) < 0.
$$

(3)

In the initial condition (2), $N_0$ is given and $\chi_\alpha(x) = \chi_{\nu(x)}(x)$, whereas, in the non-re-entry boundary condition (3), $\nu(y)$ is the outward directed normal to $\partial V_{\text{max}}$ at $y = x - R_{\text{max}}(x,u)u \in \partial V_{\text{max}}$, see Figure 1. Furthermore, the integral term on the r.h.s. of (1) has a rather simple form because scattering is assumed to be isotropic.

Figure 1: The region $V(t)$, occupied by the cloud, and the time-independent region $V_q$ containing the $UV$-photon sources. Note that $y = x - R_{\text{max}}(x,u)u \in \partial V_{\text{max}}$. 

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By a standard procedure, [7, 8, 9], system (1)-(3) can be transformed into the following integral equation
\[ N(x, u, t) = N_0(x - ctu, u)x(x - ctu)g_{ct}(x, u, t) + \int_0^{ct} dr \gamma_r(x, u, t) \{ q(x - ru, u, t - r/c)x(x - ru) \right. \]
\[ + \frac{1}{4\pi} \sigma_s(x - ru, t - r/c)x_{V_{(t-r/c)}}(x - ru) \int_S N(x - ru, u', t - r/c)du' \}, \]
\[ (4) \]
with \((x, u, t) \in V_{max} \times S \times [0, t_{max}],\) where
\[ \gamma_r(x, u, t) = \exp \left[ - \int_0^t \sigma(x - r' u, t - r/c)x_{V_{(t-r/c)}}(x - r' u)dr' \right]. \] (5)
Note that the introduction of the characteristic function \(\chi\) allows us to write the upper bounds of the various integrals on the r.h.s of (4) in a simple way (instead of using for instance \(R^*(x, u, t) = \min \{ ct, R(x, u, t) \}).

3 Existence and uniqueness

Let \(X\) denote the space \(L^\infty(V_{max} \times S \times [0, t_{max}])\), equipped with the usual norm \(\|f\| = \text{ess sup} |f(x, u, t)|, (x, u, t) \in V_{max} \times S \times [0, t_{max}])\), and consider the integral equation (4). Setting
\[ Q(x, u, t) := N_0(x - ctu, u)x(x - ctu)g_{ct}(x, u, t) \]
\[ + \int_0^{ct} dr \gamma_r(x, u, t)q(x - ru, t - r/c)x(x - ru), \] (6)
\[ (Hf)(x, u, t) := \frac{1}{4\pi} \int_0^{ct} dr \gamma_r(x, u, t)\sigma_s(x - ru, t - r/c)x_{V_{(t-r/c)}}(x - ru) \]
\[ \times \int_S f(x - ru, u', t - r/c)du', \quad f \in D(H) = X, \] (7)
then (4) can be written as follows
\[ N = Q + HN \] (8)
and it will be studied in the Banach space \(X\).

We have from definitions (5) and (7)
\[ |Hf(x, u, t)| \leq \|f\| \int_0^{ct} dr \frac{\sigma_s(x - ru, t - r/c)}{\sigma(x - ru, t - r/c)} \left| \frac{d}{dr} \gamma_r(x, u, t) \right| \]
\[ \leq -\alpha \|f\| \int_0^{ct} \frac{d}{dr} \gamma_r(x, u, t)dr \leq \alpha \|f\| \] (9)
where
\[ \alpha = \text{ess sup}_{x,t} \frac{\sigma_s(x, t)}{\sigma(x, t)} \]
We set \( \sigma_s(x, t) = \sigma_{s0} n(x, t) \), \( \sigma(x, t) = \sigma_n n(x, t) \), where \( n(x, t) \) is the number density at \( x \in V(t) \) of the particles of the host medium (the cloud). Further, \( \sigma_{s0} \) and \( \sigma_0 \) are the microscopic cross sections, which depend only on the physical properties of the host particles (and on the photon energy). Hence, \( \sigma_s(x, t)/\sigma(x, t) = \sigma_{s0}/\sigma_0 < 1 \) because \( \sigma_0 = \sigma_{s0} + \sigma_{o} \), with \( 0 < \sigma_{o} = \) the microscopic capture cross section.

Since \( \alpha < 1 \), the unique solution in \( X \) of equation (8) has the form

\[
N = (I - H)^{-1} Q
\]

with \( \|N\| \leq (1 - \alpha)^{-1}\|Q\| \).

Note that the above results still hold even if \( \partial_t R \) is not strictly negative, see assumption (1) of Section 2.

**Remark 2** System (1)-(3) could also be studied by using the theory of semigroups, see [10, 11]. Indeed, if we set

\[
(Tg)(x, u) = -cu \cdot \nabla_x g(x, u), \quad D(T) = \{ g : g \in X_1, \, cu \cdot \nabla_x g \in X_1, \, g \text{ satisfies the b.c. (3)} \},
\]

\[
(B_1(t)g)(x, u) = -\sigma(x, t)\chi_{V(t)}(x)g(x, u), \quad D(B_1) = X_1,
\]

\[
(B_2(t)g)(x, u) = \frac{c}{4\pi} \sigma(x, t)\chi_{V(t)}(x) \int_S g(x, u')du', \quad D(B_2) = X_1,
\]

where, for instance, \( X_1 = L^1(V_{\text{max}} \times S) \), then the abstract version of (1)-(3) is the following:

\[
\frac{d}{dt} N(t) = TN(t) + B_1(t)N(t) + B_2(t)N(t) + cq(t), \quad t > 0
\]

in problem (14), \( N(t) = N(\cdot, \cdot, t) \) and \( q(t) = q(\cdot, \cdot, t) \) are now maps from \([0, t_{\text{max}}]\) into \( X_1 \) and \( N_0 \) is a given element of \( X_1 \). Since it can be shown that \( T \) is the infinitesimal generator of a contractive \( C_0 \)-semigroup on \( X_1 \) [9, 10, 11], the unique strong solution of system (14) can be found by using some results on bounded time-dependent perturbations (which, in our case, are rather simple because they are given by time-independent operators multiplied by bounded real functions of \( x \) and \( t \) [12, 13, 14]). However, the first step is to transform (14) into an abstract integral equation, which is formally identical to (4). \( \square \)

### 4 Small changes of the shrinkage law

For simplicity, we shall assume that the cloud remains homogeneous at any \( t \in [0, t_{\text{max}}] \). This implies that the cloud number density depends only on time: \( n = n(t), \forall x \in V(t), \, t \in [0, t_{\text{max}}] \). As a consequence, we have that

\[
\sigma_s(x, t) = \sigma_s(t) = \sigma_{s0} n(t) = \sigma_{s0} n_{\text{tot}}/m(t)
\]

\[
\sigma(x, t) = \sigma(t) = \sigma_0 n(t) = \sigma_0 n_{\text{tot}}/m(t),
\]

where \( n_{\text{tot}} \) is the total number of particles in the cloud (i.e. of the host medium) and

\[
m(t) = \int_S du \int_0^{R(x, u, t)} r^2 dr = \frac{1}{3} \int_S R^3(x, u, t)du
\]
is the volume of the region $V(t)$, which can be proved to be independent of $\mathbf{x} \in V(t)$. Further, (15) gives at $t = 0$

$$\sigma_s^{(0)} = \sigma_s(0) = \sigma_{s0} n_{tot}/m(0), \quad \sigma_r^{(0)} = \sigma_r(0) = \sigma_{r0} n_{tot}/m(0)$$

and so (15) becomes

$$\sigma_s(t) = \sigma_s^{(0)} m(0)/m(t) = \sigma_s(0) b(t), \quad \sigma_r(t) = \sigma_r^{(0)} m(0)/m(t) = \sigma_r(0) b(t)$$

(17)

with $b(t) = m(0)/m(t)$. Correspondingly, we have from (4)

$$N(\mathbf{x}, \mathbf{u}, t) = N_0(\mathbf{x} - ct \mathbf{u}, \mathbf{u}) \chi_0(\mathbf{x} - ct \mathbf{u})$$

$$+ \frac{c}{4\pi} b(\mathbf{r}) \chi_0(\mathbf{x} - ct \mathbf{r}) \int_S N(\mathbf{x} - ct \mathbf{r}, \mathbf{u}', \mathbf{r}) d\mathbf{u}'$$

(18)

where

$$\gamma^*_x(\mathbf{x}, \mathbf{u}, t) = \exp \left[ -c \sigma_s^{(0)} \int_0^t b(\mathbf{r}) \chi_0(\mathbf{x} - c(t - \mathbf{r}) \mathbf{u}) d\mathbf{r} \right].$$

(19)

**Remark 3** We note that a “homogeneous” cloud has a poor physical meaning because the cloud particle density must approach zero close to the surface $\partial V(t)$. However, one could think of a homogeneous cloud “equivalent”, in some sense, to a given real cloud as far as some specific measurements are concerned [1, 2]. The simplest non homogeneous case leading to an explicit expression of $b(t)$ is perhaps the following. Assume that the shrinkage (or, more in general, the time behaviour) of $V(t)$ is homothetic with respect to some suitable center $C \in \Gamma_{\min}$ (this is reasonable from a physical viewpoint [8]). Then the polar equation of $\partial V(t)$ may be written, at each $t \in [0, t_{\max}]$, as follows: $\rho = R(\mathbf{C}, \mathbf{u}, t)$, i.e. $\rho = \phi(t) R(\mathbf{C}, \mathbf{u}, 0)$, $\mathbf{u} \in S$, where $\phi(t)$ is given, with $\phi(0) = 1$. Assume also that, at time $t \in [0, t_{\max}]$, the number density of its particles is a given function $n(r, t)$ with $0 \leq r \leq \phi(t) R(\mathbf{C}, \mathbf{u}, 0)$. Then, the cloud particles, which were in the volume element $r^2 dr d\mathbf{u}$ (between $r$ and $r + dr$) at time $t = 0$, can be found at time $t > 0$ in the volume element between $r\phi(t)$ and $(r + dr)\phi(t)$ whose measure is $(r\phi)^2 dr d\mathbf{u} \phi d\theta = r^2 \phi^3(t) dr d\mathbf{u}$. Thus, particle conservation implies that:

- $n(r, 0) r^2 dr d\mathbf{u} = n(r\phi(t), t) r^2 \phi^3(t) dr d\mathbf{u}$, and so
- $n(r\phi(t), t) = n(r, 0) / \phi^3(t)$ for $r \in [0, R(\mathbf{C}, \mathbf{u}, 0)]$, or else
- $n(r', t) = n(r' / \phi(t), 0) / \phi^3(t)$ for $r' \in [0, \phi(t) R(\mathbf{C}, \mathbf{u}, 0)]$.

It follows that:

- $\sigma(r', t) = \sigma_0 n(r', t) = \sigma_0 n(r' / \phi(t), 0) / \phi^3(t) = \sigma(r' / \phi(t), 0) / \phi^3(t)$,
- $\sigma_s(r', t) = \sigma_{s0} n(r', t) = \sigma_s(r' / \phi(t), 0) / \phi^3(t)$,
where we recall that \( \sigma_0, \sigma_{s0} \) are the microscopic cross sections of the cloud. Note also that (16) now becomes
\[
m(t) = \frac{1}{3} \int_S R^3(C, u, 0)\phi^3 \, du = m(0)\phi^3(t) \tag{20}
\]
and so \( b(t) = m(0)/m(t) = 1/\phi^3(t) \). □

Assume that, at each \( t \in [0, t_{\text{max}}] \), the new shrinkage law is defined by
\[
R_*(x, u, t) = R(x, u, t) + \eta(x, u, t), \quad (x, u) \in V(t) \times S,
\tag{21}
\]
with
\[(i) \ \eta(x, u, 0) = 0, \ \eta(x, u, t) \geq 0; \]
\[(ii) \ \eta(x, u, t) \leq \epsilon < \delta_q, \text{ where } \delta_q \text{ is the diameter of } V_q; \]
\[(iii) \ \partial_t R_*(x, u, t) = \partial_t R(x, u, t) + \partial_t \eta(x, u, t) < 0, \forall (x, u, t) \in V(t) \times S \times [0, t_{\text{max}}], \text{ where we recall that } \partial_t R \text{ is strictly negative, see assumption (1)}.\]

We remark that (i) implies that the new region \( V_1(t) \), defined by (21), “shrinks less” than \( V(t) \), whereas (ii) shows that \( V_1(t) \) is always “very close” to \( V(t) \). Further, assumption (iii) simplifies calculations, but it could be released.

If \( m_1(t) \) is the volume of the new region \( V_1(t) \), definition (21) gives
\[
m_1(t) = \frac{1}{3} \int_S R_*^3(x, u, t) \, du = m(t) + \mu(t) \tag{22}
\]
where
\[
\mu(t) = \int_S \eta \left[ R^2 - R_0 q + \frac{1}{3} \eta^2 \right] \, du \leq \epsilon \int_S \left[ R^2 + R_0 q + \frac{1}{3} \eta^2 \right] \, du \leq h \epsilon,
\tag{23}
\]
where \( h = 4\pi[2\delta_{\text{max}} + \epsilon \delta_{\text{max}} + \frac{1}{3} \epsilon^2] \), with \( \delta_{\text{max}} \) = diameter of \( V_{\text{max}} \).

Hence,
\[
b_1(t) = \frac{m_1(0)}{m_1(t)} = \frac{m(0)}{m(t) + \mu(t)} = b(t) + \beta(t),
\tag{24}
\]
with
\[
\beta(t) = -\frac{\mu(t)}{m(t) + \mu(t)} b(t), \quad |\beta(t)| \leq \frac{h \epsilon}{m(t_{\text{max}})} b(t) \ll b(t). \tag{25}
\]

Equation (18), with \( N \) substituted by \( N_1 \), \( b \) by \( b_1 \) and \( V \) by \( V_1 \) (thus, \( N_1 \) is the photon distribution function with the new shrinkage law), has the form
\[
N_1(x, u, t) = N_0(x - c t u, u)\chi_u(x - c t u)\gamma_{\eta,1}(x, u, t)
+ c \int_0^t d\tau \gamma_{\tau,1}(x, u, t) \left\{ \text{sinc}(x - c(t - \tau)u, u, \tau)\chi_u(x - c(t - \tau)u) \\
+ c \frac{\tau(0)}{4\pi} b_1(\tau)\chi_{v_1(\tau)}(x - c(t - \tau)u) \int_S N_1(x - c(t - \tau)u, u', \tau) \, du' \right\},
\tag{26}
\]
where (see (19))
\[ \gamma_{\tau,1}(x, u, t) = \exp \left[ -c\sigma(0) \int_{t}^{t} b_1(\theta) \chi_{V_1(\theta)}(x - c(t - \theta)u) \, d\theta \right]. \]  

(27)

If we consider for instance the first term on the r.h.s. of (26), we have from (27) with \( \tau = 0 \)
\[ \gamma_{0,1}(x, u, t) = \exp \left\{ -c\sigma(0) \int_{0}^{t} \left[ b(\theta) + \beta(\theta) \right] \chi_{V_1(\theta)}(x - c(t - \theta)u) \right\} \]

(28)

where we used (24) and the fact that \( V_1 = V \cup (V_1 - V) \) and so \( \chi_{V_1} = \chi_V + \chi_{V_1 - V} \). Hence,
\[ \gamma_{0,1}^*(x, u, t) = \gamma_0^*(x, u, t) \exp[\psi(x, u, t)] \]

where
\[ \psi(x, u, t) = -c\sigma(0) \int_{0}^{t} b(\theta) \chi_{V_1(\theta)}(x - c(t - \theta)u) \, d\theta + \int_{0}^{t} \beta(\theta) \chi_{V_1(\theta)}(x - c(t - \theta)u) \, d\theta. \]

(29)

We have from (29)
\[ |\psi(x, u, t)| \leq c\sigma(0) \left\{ \frac{m(0)}{m(t)} \int_{0}^{t} \chi_{V_1(\theta) - V(\theta)}(x - c(t - \theta)u) \, d\theta \right. \]
\[ + \left. \frac{he}{m(t_{\text{max}})} \int_{0}^{t} b(\theta) \chi_{V_1(\theta)}(x - c(t - \theta)u) \, d\theta \right\} \]
\[ \leq c\sigma(0) \left\{ \frac{m(0)}{m(t_{\text{max}})} \int_{0}^{t} \chi_{V_1(\theta) - V(\theta)}(x - c(t - \theta)u) \, d\theta \right. \]
\[ + \left. \frac{he}{m(t_{\text{max}})} \int_{0}^{t} \chi_{V_1(\theta)}(x - c(t - \theta)u) \, d\theta \right\}. \]

(30)

On the other hand,
\[ \frac{he}{m(t_{\text{max}})} \int_{0}^{t} \chi_{V_1(\theta)}(x - c(t - \theta)u) \, d\theta \leq \frac{he}{m(t_{\text{max}})} \frac{t_{\text{max}}}{t_{\text{max}}} \leq \frac{ht_{\text{max}}}{m(t_{\text{max}})} \]

because \( 0 \leq \chi_{V_1(\theta)} \leq 1 \). Furthermore,
\[ \int_{0}^{t} \chi_{V_1(\theta) - V(\theta)}(x - c(t - \theta)u) \, d\theta \leq \frac{2\epsilon}{c} \]

since the length of any segment contained in \( V_1 - V \) is smaller than \( 2\epsilon \), see assumption (ii).

We conclude that
\[ |\psi(x, u, t)| \leq k\epsilon, \quad \exp(-ke) \leq \exp(\psi(x, u, t)) \leq \exp(ke) \]

(31)

where
\[ k = c\sigma(0) \frac{m(0)}{m(t_{\text{max}})} \left[ \frac{2}{c} + \frac{ht_{\text{max}}}{m(t_{\text{max}})} \right] \epsilon. \]

As a consequence,
\[ |1 - \exp(\psi(x, u, t))| \leq k\epsilon \exp(ke) \]
and so
\[ |\gamma_{0,1}^*(x, u, t) - \gamma_0^*(x, u, t)| = \gamma_0^*(x, u, t)|1 - \exp[\psi(x, u, t)]| \]
\[ \leq \epsilon[k \exp(k \epsilon)]\gamma_0^*(x, u, t). \]  
(32)

Since a similar result holds for the other two terms on the r.h.s. of (26), it is not difficult to show that \( N_1 - N \) satisfies the equation
\[ N_1 - N = M + H(N_1 - N), \]  
(33)
where the “source term” \( M \) has norm of order of \( \epsilon \). Then,
\[ N_1 - N = (I - H)^{-1}M \]
has also norm of order \( \epsilon \) because
\[ \|N_1 - N\| \leq \frac{1}{1 - \alpha} \|M\|, \]
see (10). This shows that \( N \) is “continuous” with respect to the shrinkage law and that the error in substituting \( N \) with \( N_1 \) is of the order of \( \epsilon \).

We also remark that a similar procedure can be used to prove that \( N \) is “continuous” with respect to small variations of the microscopic cross sections \( \sigma_0 \) and \( \sigma_0 \).

5 An application in spherical symmetry

In order to deal with the inverse problem, for sake of simplicity, let us now consider the spherical symmetry case, where the source \( q \) does not depend anymore on the unit vector \( u \). We refer to Figure 2 to set some notations. Let \( P \) denote position of the test photon and let
\[ OA = R = \text{const.}, \quad 0 \leq OP = r \leq R, \quad OP = r' = r'(r; \mu), \quad PP' = \rho. \]
with
\[ P'O = r'(r; \rho, \mu) = \sqrt{r^2 + \rho^2 - 2\rho r}, \quad PA = \rho_{\max}(r, \mu) = \mu r + \sqrt{R^2 - (1 - \mu^2)r^2} \]
\[ \mu' = \mu'(r; \mu) = \cos \phi' = \frac{r\mu - \rho}{r'}, \quad \mu'_{\max} = \frac{r\mu - \rho_{\max}}{r'_{\max}} = \frac{\sqrt{R^2 + (1 - \mu^2)r^2}}{R}. \]

The transport equation (1) in spherical symmetry reads [7]
\[ \partial_t N(r, \mu, t) = -c\mu \frac{\partial N}{\partial r} (r, \mu, t) - \frac{1}{r} \mu^2 \frac{\partial N}{\partial \mu} - c\sigma(r, t)N(r, \mu, t) \]
\[ + \frac{c}{2} \sigma_s(r, t) \int_{-1}^{1} N(r, \mu, t) d\bar{\mu} + cq(r, t), \quad 0 \leq r < R, \quad -1 \leq \mu \leq 1, \]  
(34)
Figure 2: Notations in spherical symmetry

with initial data

\[ N(r, \mu, 0) = N_0(r, \mu) \]

and non-reentry boundary conditions

\[ N(\overline{R}, \mu, t) = 0 \quad \text{if} \quad \mu < 0. \]

The integral form of equation (34) is given by

\[
N(r, \mu, t) = \\
\exp \left[ - \int_0^{ct} \sigma(r'(\tilde{\rho}; r, \mu), t - \tilde{\rho}/c)d\tilde{\rho} \right] N_0(r'(ct; r, \mu), \mu'(ct; r, \mu))U(\rho_{\text{max}} - ct) \\
+ \int_0^{\rho^*} d\rho \exp \left[ - \int_0^{\rho} \sigma(r'(\tilde{\rho}; r, \mu), t - \tilde{\rho}/c)d\tilde{\rho} \right] \\
\times \left\{ q(r'(\rho; r, \mu), t - \rho/c) + \frac{1}{2} \sigma_s(r'(\rho; r, \mu), t - \rho/c) \int_{-1}^{1} N(r'(\rho; r, \mu), \tilde{\mu}, t - \rho/c)d\tilde{\mu} \right\},
\]

where

\[
\rho^* = \rho^*(r, \mu, t) = \min\{ct, \rho_{\text{max}}(r, \mu)\}
\]

with

\[
\rho_{\text{max}}(r, \mu) = r\mu + \sqrt{R^2 - (1 - \mu^2)r^2},
\]

and \( U \) is the Heaviside function. In Figure 3 we represent three possible directions of the photon motion across the region. In particular, we have denoted with subscripts 1, 2, 3, respectively, the cases in which
the straight line $\overline{AP}$ crosses the cloud, is outside the cloud itself and crosses both the cloud and the source.

Let us suppose that the cloud lies in the (time-dependent) region $0 \leq r \leq R(t)$, $t \in [0, t_{\text{max}}]$, with

$$R(t) = R(0) - vt = \overline{R} - vt \Rightarrow R(0) - vt_{\text{max}} \leq R(t) \leq R(0),$$

where $v > 0$ is the shrinking velocity which hereinafter will be the unknown of the inverse problem we are going to formulate. Moreover, we have put $\overline{R} = R(0)$.

Furthermore, let us assume that $0 < R_q \leq R(t) = \overline{R} - vt$ $\forall t \in [0, t_{\text{max}}] \Rightarrow 0 < R_q \leq \overline{R} - vt_{\text{max}}$

where $R_q$ is the (time-independent) region occupied by the source $q$ which is assumed homogeneous inside $R_q$, thus implying $q(r, t) = q(t)U(R_q - r)$, see Figure 3.

Finally, let us assume that the cloud is homogeneous $\forall t \in [0, t_{\text{max}}]$. Consequently, the cloud particles density $n$ (host medium) depends only on time. Hence, preservation of particles number inside the cloud implies

$$n(0)\frac{4}{3}\pi R^3 = n(t)\frac{4}{3}\pi R^3(t) \Rightarrow n(t) = n(0)\frac{R^3}{R^3(t)} = n(0)b(t)$$

with

$$b(t) = \frac{R^3}{R^3(t)} = \frac{R^3}{(\overline{R} - vt)^3}$$

The scattering $\sigma_s$ and the total $\sigma$ cross sections become

$$\sigma_s(r, t) = \sigma_s^{(0)}b(t)U(R(t) - r), \quad \sigma(r, t) = \sigma^{(0)}b(t)U(R(t) - r),$$

Figure 3: The cloud in spherical symmetry
where \( \sigma_s^{(0)} \) and \( \sigma^{(0)} \) are the macroscopic cross sections at \( t = 0 \).

**Remark 4** Formally, the cloud still occupies the region \( 0 \leq r \leq R \) whereas for \( R(t) < r < \overline{R} \) the region should be empty. This is accomplished by the cross sections, through the Heaviside function. \( \square \)

By using equations (39), equation (37) becomes

\[
N(r, \mu, t) = \\
\exp \left[ -\sigma^{(0)} \int_0^t b(t - \hat{\rho}/c)U(R(t - \hat{\rho}/c) - r'(\hat{\rho}; r, \mu))d\hat{\rho} \right] \\
\times N_0 (r'(ct; r, \mu), \mu' (ct; r, \mu))U(\rho_{max} - ct) \\
+ \int_0^{\rho'} d\rho \exp \left[ -\sigma^{(0)} \int_0^t b(t - \hat{\rho}/c)U(R(t - \hat{\rho}/c) - r'(\hat{\rho}; r, \mu))d\hat{\rho} \right] \\
\times \left\{ q(t - \rho/c)U(R_q - r'(\rho; r, \mu)) + \frac{1}{2}\sigma_s^{(0)}b(t - \rho/c)U(R(t - \rho/c) - r'(\rho; r, \mu)) \\
\times \int_{-1}^1 N(r'(\rho; r, \mu), \tilde{\mu}, t - \rho/c) d\tilde{\mu} \right\} .
\]

The Heaviside functions in (40) rule the three different cases shown in Figure 3.

Putting as photon density
\[
\psi(r, t) = \int_{-1}^1 N(r, \tilde{\mu}, t) d\tilde{\mu}
\]
equation (40) gives

\[
\psi(r, t) = \int_{-1}^1 d\mu \exp \left[ -\sigma^{(0)} \int_0^t b(t - \hat{\rho}/c)U(R(t - \hat{\rho}/c) - r'(\hat{\rho}; r, \mu))d\hat{\rho} \right] \\
\times N_0 (r'(ct; r, \mu), \mu' (ct; r, \mu))U(\rho_{max} - ct) \\
+ \int_{-1}^1 d\mu \int_0^{\rho'(r, \mu, ct)} d\rho \exp \left[ -\sigma^{(0)} \int_0^t b(t - \hat{\rho}/c)U(R(t - \hat{\rho}/c) - r'(\hat{\rho}; r, \mu))d\hat{\rho} \right] \\
\times \left\{ q(t - \rho/c)U(R_q - r'(\rho; r, \mu)) \\
+ \frac{1}{2}\sigma_s^{(0)}b(t - \rho/c)U(R(t - \rho/c) - r'(\rho; r, \mu))\psi(r'(\rho; r, \mu), t - \rho/c) \right\} .
\]

5.1 The inverse problem

The inverse problem we are proposing consists in the determination of the cloud shrinking velocity \( v \) (recall that \( R(t) = R(0) - vt \)) once the radiation \( N(R(t), 1, t) \), coming out from the cloud itself, is known (far-field measurement).

With reference to Figure 4 we have:

\[
\overline{OA} = \overline{R} = R(0), \quad \overline{FO} = R(t) = R(0) - vt, \quad \overline{PA} = R(t) + R(0) = \rho_{max}(R(t), 1)
\]

12
since $\mu = \cos \varphi = 1$. Moreover, in the present situation, it results $r'(\rho; r, \mu) = r'(\rho; R(t), 1) = R(t) - \rho$ if $\rho < R(t)$ and $r'(\rho; R(t), 1) = \rho - R(t)$ if $\rho > R(t)$; consequently $r'(\rho; R(t), 1) = |R(t) - \rho|, \forall \rho \in [0, \rho_{\text{max}}(R(t), 1)]$, with $\rho_{\text{max}}(R(t), 1) = R(0) + R(t) = PA$.

Furthermore we have

$$r'(ct; R(t), 1) = |R(t) - ct|, \quad \mu'(ct; R(t), 1) = \frac{R(t) - ct}{|R(t) - ct|},$$

$$U(R(t - \bar{\rho}/c) - r'(\bar{\rho}; R(t), 1)) = U(R(t - \bar{\rho}/c) - |R(t) - \bar{\rho}|),$$

$$U(R(t - \rho/c) - r'(\rho; R(t), 1)) = U(R(t - \rho/c) - |R(t) - \rho|),$$

$$U(R_q - r'(\rho; R(t), 1)) = U(R_q - |R(t) - \rho|).$$

Finally, equation (40) becomes:

$$N(R(t), 1, t) = \exp \left[ -\sigma(0) \int_0^{ct} b(t - \bar{\rho}/c) U(R(t - \bar{\rho}/c) - |R(t) - \bar{\rho}|) d\bar{\rho} \right]$$

$$\times N_0 \left( \frac{|R(t) - ct|}{R(t) - ct} \right) U(\rho_{\text{max}} - ct)$$

$$+ \int_0^{r'(R(t), 1, t)} d\rho \exp \left[ -\sigma(0) \int_0^\rho b(t - \rho/c) U(R(t - \rho/c) - |R(t) - \rho|) d\bar{\rho} \right]$$

$$\times \left\{ q(t - \rho/c) U(R_q - |R(t) - \rho|) \right.$$  

$$+ \frac{1}{2} \sigma(0) b(t - \rho/c) U(R(t - \rho/c) - |R(t) - \rho|) \psi(|R(t) - \rho|, t - \rho/c) \right\},$$

\[ 43 \]
where $\rho^*(R(t), 1, t) = \min\{ct, \rho_{\max}(R(t), 1)\}$, with $\rho_{\max}(R(t), 1) = R(t) + R(0)$, consequently $\rho^*(R(t), 1, t) = \min\{ct, R(t) + R(0)\}$.

The inverse problem consists then in finding $v$ which satisfies equation (43) once its l.h.s., i.e. $N(R(t), 1, t)$, is a known datum.

In view of the proposed applications we make the following simplifying assumption

$$q(t) = q_0 \text{ (constant).}$$

### 5.2 Some numerical results

Focusing on the spherical symmetry case, we have performed some numerical experiments as follows. First, the direct problem has been treated. Let us consider equations (42) and (43), with $r \in [0, R]$ and $t \in [0, t_{\text{max}}]$. We have introduced uniform grids in space and time on such intervals, with $n_r$ and $n_t$ nodes, respectively. All the Heaviside functions in the integrals appearing in equations (42) and (43) have been converted in proper integration extrema for such integrals. Further on, numerical integration has been performed with Gaussian quadrature rule with $n_\mu$ and $n_\rho$ nodes for integrals in $\mu$ and $\rho$, respectively. Values of $v$ ranging from $10^{-3}c$ to $10^{-1}c$ have been considered. In all computations the initial radius of the cloud has been set equal to 1 parsec, with the radius of the source equal to 0.5 parsec. Furthermore, we have set $t_{\text{max}} = 1.8$, $\sigma_v^{(0)} = \frac{1}{3} \sigma(0)$, $q_0 = 100$.

Some of the results obtained with $n_r = n_t = 50$ and $n_\mu = n_\rho = 12$ are reported in Figures 5-6, in which we plot the computed values for $N(R(t), 1, t)$ obtained from equation (43), corresponding to different values of $v$, $\sigma^{(0)}$, and $N_0$. More in details, Figure 5 refers to $N_0 = 0$, whereas Figure 6 refers to $N_0 = 100$. In all these pictures, results obtained for several values of $v$ are compared in the same plot.

In Figure 7 we report details of pictures of Figure 6, zooming on the left side of the pictures, corresponding to the initial part of the phenomenon, where $N$ is almost constant, and using a logarithmic scale on the $y$ axis.

As can be noted, the behaviour of $N(R(t), 1, t)$ is the following. If we start from $N_0 = 0$, in the first part of the process the photons measured at the boundary of the cloud are essentially 0. Starting with $N_0 = 100$, the photons measured at the boundary of the cloud slightly decrease due to scattering and absorption. This decrease is more evident if higher values of the cross sections are considered (see right picture in Figure 7). When photons produced from the source reach the boundary of the cloud, the number of photons measured on the boundary rapidly increases up to a maximum value; then, again due to scattering and absorption, the number of photons decreases. The decrease is more evident in the case $N_0 = 100$.

Concerning the behaviour corresponding to different values of the shrinking velocity, we clearly notice that for low values of $v$, as the cloud shrinks slowly, the photons produced from the source reach the boundary later than in a case with higher $v$, as can be seen from all the pictures of Figures 5-6. Moreover, for higher values of $v$, the decrease of photons on the boundary in the final part of the process is much more evident. This effect is due to the fact that as the volume of the cloud decreases rapidly, the number of interactions increases.

In Figure 8 we show a result on the photon density $\psi$, obtained from equation (42), focusing on $v = 0.1c$ and $\sigma^{(0)} = 3 \cdot 10^{-3}$. From the Figure we can clearly see that, for all $t$, photon density is higher in a region corresponding to the source (we recall that in our experiments the radius of the source is half the radius of the cloud).
Next, we turn to the inverse problem. Let us consider equation (43) and assume that the values $N_j = N(R(t_j), 1, t_j), j = 1, 2, ..., n_t$ are known measurements. Note that this data refer to the point $\zeta$ on the surface of the cloud whereas the so-called far-field measurement should be referred to point $\hat{\xi}$ (see Figure 1). Therefore, we assume

$$N(\zeta, \hat{\xi}, t_j) = N(\hat{\xi}, \hat{\xi}, \hat{t}_j)$$

with $\hat{t}_j = t_j - |\hat{\xi} - \zeta|/c$. In order to solve numerically the inverse problem, we use a strategy based on an iterative approach. For a fixed time $\hat{t}$, given two initial values $v_1 < v_2$, we apply a bisection procedure to find an approximate $\hat{v}$ solving equation (43) with respect to $v$. We start with $\hat{t} = t_2$ and $[v_1, v_2] = [0, 0.9c]$. Since in our model $v$ is assumed to be independent of $t$, when $\hat{v}$ has been found for a given $\hat{t}$, the same value should be found for all the $t_j$. Hence, in order to speed up the process, in all the subsequent steps we start with $[v_1, v_2] = [\hat{v} - \Delta, \hat{v} + \Delta]$ for a given $\Delta$. Moreover, to perform some experiments on the inverse problem, we have used as measured data the values $N_j$ obtained by a direct computation. With this approach we were able to successfully solve the inverse problem for several values of $v$, with maximum relative errors as reported in Table 1.

<table>
<thead>
<tr>
<th>$v$</th>
<th>max relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.972e-2</td>
<td>3.5804e-05</td>
</tr>
<tr>
<td>2.387e-3</td>
<td>1.2494e-03</td>
</tr>
<tr>
<td>6.281e-4</td>
<td>4.7325e-03</td>
</tr>
</tbody>
</table>

Table 1: Maximum relative errors on the runs with the inverse problems

6 Concluding remarks

The knowledge of the photon distribution function is important because interactions between photons and the particles of the cloud (mainly hydrogen molecules and a small percentage of dust grains) play a
crucial role in the chemistry of the cloud itself. On the other hand, the form of the photon distribution function depends on the cross sections and on the shape of the region \( V(t) \) (i.e., \( \partial V(t) \)).

This justifies the mathematical machinery of Sections 2 and 3, and the study of the inverse problem of Section 5 to determine the motion of \( \partial V(t) \) (which also can give some indications of the evolution of \( \partial V(t) \) at times \( t > t_{\text{max}} \)). Note that, as shown in Section 4, small errors in the identification of \( \partial V(t) \) produce small errors on the corresponding distribution function.
Figure 7: Details of Figure 6 (logarithmic scale on $y$ axis).
Figure 8: $\psi(r, t)$ for $v = 0.1c, \sigma^{(0)} = 3 \cdot 10^{-3}, N_0 = 0$. 
References


