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ON A CLASS OF SCALAR GENERALIZED VARIATIONAL INEQUALITIES

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On a class of scalar
generalized variational inequalities

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Abstract

We consider a class of variational inequalities defining the so-called hysteresis play operator. We propose a new approach to discontinuous BV-solutions based on measure theoretical arguments which enable us to infer the existence of solutions as a simple consequence of the classical theory. In this way we generalize a recent result where only continuous BV-solutions were studied. We also provide a representation formula which let us to deduce the continuity of the play operator from general theorems on hysteresis operators.

Keywords: Play operator; Evolution variational inequalities; Rate independence; Hysteresis; Elastoplasticity; Functions of bounded variation

AMS Subject Classification: 74N30; 74C05; 47H30; 34C55; 26A45

1 Introduction

The hysteresis play operator is an important mathematical tool in elasto-plasticity. In the monograph [1, Section 2] this operator is described by means of the following simple mechanical model. Let us consider a cylinder of length $2r$ which can move along the $y$-axis when driven by a piston. The position of the piston at the time $t$ is denoted by the coordinate $u(t)$, whereas $y(t)$ denotes the position of a point of the cylinder, for instance its center. Assume that the initial position $y(0)$ of the cylinder and the movement $u(t)$ of the piston are given. We want to describe the way the cylinder moves in a time interval $[0, T]$. Of course there must be $|u(0) - x(0)| \leq r$. Let us consider first the case when $u$ is monotone and continuous. We have that $y(t) = y(0)$ until $u(t)$ touches the top or the bottom of the cylinder, i.e. $|u(t) - y(0)| < r$. If $u$ increases and $u(t) \geq y(0) + r$, then $y(t) = u(t) + r$; whereas if $u$ is decreasing and $u(t) \leq y(0) - r$, then $y(t) = u(t) - r$. Shortly
we can write that \( y(t) = \max\{u(t) - r, y(0)\} = u(t) - \min\{r, u(t) - y(0)\} \) if \( u \) is increasing, and \( y(t) = \min\{u(t) + r, y(0)\} = u(t) - \max\{-r, u(t) - y(0)\} \) if \( u \) is decreasing. Even shorter is the formula \( y(t) = \max\{u(t) - r, \min\{u(t) + r, y(0)\}\} \). For continuous piecewise monotone \( u(t) \) one can reduce to the monotone case once one assumes a deterministic behaviour. Hence we have defined an operator \( P \) associating to \( u \) the function \( P(u) := y \). Often the suggestive terms \textit{input} and \textit{output} are used for the functions \( u \) and \( y \) respectively.

In order to give a mathematical description of this system for a more significant class of inputs, one can observe that if \( u \) is also absolutely continuous (\( u \in AC(0,T) \)), then \( y(t) \) solves the following variational inequality (cf. [2, Section III.2]):

\[
(u(t) - y(t) - z)y'(t) \geq 0 \quad \forall z \in [-r, r], \quad \text{for a.e. } t \in [0, T]. 
\]

Indeed it can be shown that when \( P \) is acting on the space \( C^p([0,T]) \) of continuous piecewise linear functions, it is uniformly continuous with respect to the topology induced by

\[
\|v\|_{AC(0,T)} = \|v\|_{L^1(0,T)} + \|v'\|_{L^1(0,T)}
\]

(cf e.g. [1, 2, 3]). Therefore it can be continuously extended to an operator \( P : AC(0,T) \rightarrow AC(0,T) \) by setting \( P(u) := y \), where \( y \) is the unique solution of the variational inequality (1.1), together with the conditions

\[
|u(t) - x(t)| \leq r \quad t \in [0, T],
\]

\[
u(0) - y(0) = z_0, \tag{1.3}
\]

for a prescribed \( z_0 \in [-r, r] \). Of course it is natural to wonder how to give an explicit analytic formulation of this model for inputs which are not necessarily absolutely continuous, i.e. how to extend the play operator to some operator \( \overline{P} \) defined on a larger space. In the paper [4] the following explicit variational integral formulation for continuous inputs with bounded variation is introduced: given \( u \in BV(0,T) \cap C(0,T) \) one has to look for \( y \in BV(0,T) \cap C(0,T) \) satisfying (1.2), (1.3) and

\[
\int_0^T (u(t) - y(t) - z(t)) \, dy(t) \geq 0 \quad \forall z \in C([0,T]), \quad z([0,T]) \subseteq [-r, r]. \tag{1.4}
\]

Here the integral has to be understood in the sense of Riemann-Stieltjes. In [4, Section I.3] it is proved that problem (1.2)–(1.4) admits a unique solution by means of a time discretization technique. Essentially (1.4) is replaced by an Euler implicit scheme which is solved by using variational techniques. Once one finds a priori estimates for the piecewise linear interpolation of the discrete solutions, the limit procedure is carried out by using suitable theorems for taking the limit under the Riemann-Stieltjes integral sign. Successively it is proved that for absolutely continuous inputs the two formulations are equivalent.

The case of discontinuous inputs is dealt with [5]. In that paper a definition of the play operator for input \( u \) with bounded variation is given. The integral in (1.4) has to be
understood in the Young sense, and the integrand map has to be replaced with its right continuous representative: hence if $u \in BV(0, T)$, then $y = \overline{P}(u) \in BV(0, T)$ is defined to be that function satisfying (1.2)–(1.3) together with the inequality

$$
\int_0^T (u(t+) - y(t+) - z(t)) \, dy(t) \geq 0 \quad \forall z \in BV(0, T), \quad z([0, T]) \subseteq [-r, r]. \quad (1.5)
$$

Also in this case the proof is essentially based on discretization procedure, but the procedure is more involved, due to the discontinuities of the input and output functions.

The aim of the present paper is to approach problem (1.5), (1.2)–(1.3) by using only elementary measure theory tools. The idea of the proof is suggested by a well-known feature of the play operator, namely rate independence, which can be expressed in this way: for any $u \in AC(0, T)$ and for every increasing surjective Lipschitz reparametrization $\varphi : [0, T] \to [0, T]$ one has

$$
P(u \circ \varphi) = P(u) \circ \varphi \quad (1.6)
$$

(rate independent operators, hysteresis, and other related problems are studied in [1, 2, 3, 4]). Whatever definition of generalized solution of problem (1.1)–(1.3) one may define, it is natural to expect the resulting operator solution $\overline{P}$ has still a rate independence property. Therefore if $u \in BV(0, T)$, we can consider the particular (discontinuous) reparametrization

$$
\varphi(t) := \ell_u(t) := \frac{T}{V(u, [0, T])} V(u, [0, t]), \quad t \in [0, T], \quad (1.7)
$$

where $V(u, [0, t])$ denotes the variation of $u$ on the interval $[0, t]$. Since one has the factorization

$$
u = \tilde{u} \circ \ell_u \quad (1.8)
$$

for a suitable Lipschitz map $\tilde{u} \in Lip(0, T)$ (see Proposition 3.1), it is natural to expect that

$$
P(u) = \overline{P}(\tilde{u} \circ \ell_u) = \overline{P}(\tilde{u}) \circ \ell_u. \quad (1.9)
$$

As $\tilde{u}$ is Lipschitz, the output $\overline{P}(\tilde{u})$ is known, and this suggests that formula (1.9) should be the generalized solution of the problem with $u \in BV(0, T)$. Hence we prove directly that $\overline{P}(\tilde{u}) \circ \ell_u$ is the solution of (1.5), thereby showing that the action of $\overline{P}$ on Lipschitz functions uniquely determines the action of $\overline{P}$ on $BV(0, T)$.

Let us remark that in [6] this procedure was carried out for functions $u \in BV(0, T) \cap C(0, T)$. If discontinuities occur, then the problem is more complicated and we need to exploit a suitable chain rule for the derivative of functions with bounded variation. Since the derivative $Df$ of a function $f \in BV(0, T)$ is a measure, we have to show that the integral in (1.5) can be replaced by the integral with respect to the measure $Dy$. In this way we can apply the chain rule and verify directly that (1.9) solves (1.5), using basic properties from measure theory, and exploiting the fact that solvability of (1.1)–(1.3) is
well known in the regular case. Then the elementary character of our proof allows to state that in a certain sense the solvability of the generalized problem is a direct consequence of the solvability of the classical problem.

As we said above, in [6] we dealt with the case when \( u \) is continuous and with bounded variation, and possibly with values in a Hilbert space \( \mathcal{H} \). This means that the product in (1.1) is replaced by the inner product, \( u \in BV(0, T; \mathcal{H}) \cap C(0, T; \mathcal{H}), \) and \( y \) is to be found in the same space. However when the dimension of \( \mathcal{H} \) is grater than 1, and \( u \) is not continuous, it can be checked that formula (1.9) does not provide the solution of (1.5). Therefore the procedure of the present paper, suitably adapted to the vectorial case, does not work.

Using representation formula (1.9) we are also able to apply general results on rate independent operators (see [7, 8]) and infer that \( \overline{P} \) is continuous when \( BV \) is endowed with its natural \emph{strict metric}

\[
d(u, v) := \|u - v\|_{L^1(0,T)} + |V(u) - V(v)|.
\]

Let us point out that also in [9] we studied the play operator by means of measures. In that paper the integral in (1.5) is replaced by the integral with respect to the measure \( Dy \), but the procedure is completely different: the analysis is restricted to right continuous inputs and we essentially follow the arguments of [5], i.e. we do not infer the existence of \( BV \) solutions as a simple consequence of the classical theory. Moreover in [9] the \( BV \)-continuity is not studied.

Let us conclude with a brief plan of the paper. In the following section we recall some basic facts about functions of bounded variation and signed measures. In Section 3 we state the main results of the paper and in Section 4 we review the existence results for the play operator with regular data. In Section 5 we give the detailed proofs of the existence theorem and finally in Section 6 we study the continuity of the play operator. In the Appendix we recall the definition of Young integral and we show its connections with the ordinary Lebesgue integral.

2 Preliminaries

Throughout the paper \( a, b \in \mathbb{R} \cup \{-\infty, \infty\}, a < b, \) and \( I := ]a, b[ \), the open interval with endpoints \( a, b \). We will deal with elements of the space \( \mathbb{R}^I \), i.e. with real functions defined on \( I \). By \( L^1 \) we denote the one-dimensional Lebesgue measure on \( I \) and we do not identify two functions agreeing \( L^1 \)-almost everywhere in \( I \). If \( f \in \mathbb{R}^I \), then \( \text{Cont}(f) \) represents the set of continuity points of \( f \) and \( \text{Discont}(f) := I \setminus \text{Cont}(f) \).

2.1 Step and regulated functions

If \( J \) is a subinterval of \( I \), the symbol \( St(J) \) will denote the set of \emph{step functions}, that is functions \( f : J \longrightarrow \mathbb{R} \) such that \( J \) can be partitioned into a finite number of (possibly
degenerate) intervals $J_1, \ldots, J_m$ and $f$ is constant on each $J_j$ for $j = 1, \ldots, m$. A function $f : J \to \mathbb{R}$ is called regulated on $J$ if at each point $t \in J$ there exist $f(t-) := \lim_{s \uparrow t} f(s)$ and $f(t+) := \lim_{s \downarrow t} f(s)$, and are finite, with the convention that $f(t-) := f(t)$ (resp. $f(t+) := f(t)$) if $t \in J$ and $t$ is the left (resp. the right) endpoint of $J$. We denote by $\text{Reg}(J)$ the set of regulated functions on $J$. It is well known that every $f \in \text{Reg}(J)$ is the uniform limit of a sequence $f_n \in \text{St}(J)$, hence the set $\{t \in J : f(t-) \neq f(t+)\}$ is at most countable, and if $J$ is compact then $f$ is bounded. Among the regulated functions we mention the class of monotone functions. In this regard we warn the reader about the terminology: by an increasing function on $J$, we mean a function $f : J \to \mathbb{R}$ such that $(f(t_1) - f(t_2))(t_1 - t_2) \geq 0$ for every $t_1, t_2 \in J$. Same convention is adopted for the term decreasing. Finally $f$ is monotone if it is increasing or if it is decreasing.

Let us recall that a subdivision of $J$ is a family of points $s = (t_j)_{j=0}^m$, $m \in \mathbb{N}$, with the property that $t_j \in J$ and $t_0 < t_1 < \cdots < t_m$. The set of all subdivisions of $J$ is indicated by $\mathcal{S}(J)$. If $f \in \mathbb{R}^J$, the pointwise variation of $f$ on $J$ is defined as

$$V_p(f, J) := \sup \left\{ \sum_{j=1}^m |f(t_j) - f(t_{j-1})| : m \in \mathbb{N}, (t_j)_{j=0}^m \in \mathcal{S}(J) \right\}$$

if $J \neq \emptyset$, otherwise we set $V_p(f, \emptyset) := 0$. It is well known that if $V_p(f, J) < \infty$ then $f$ is bounded, $f \in \text{Reg}(J)$, and $u(\inf J+)$ and $u(\sup J-)$ exist and are finite. If $J$ is open, the essential variation $V_e(u, J)$ of $u$ on $J$ is defined as

$$V_e(u, J) := \inf \{V_p(w, J) : u = w \ \mathcal{L}^1\text{-a.e. in } J \}.$$ 

If $v \in \mathbb{R}^J$ is such that $V_p(v, I) = V_e(u, J) < \infty$, then $v$ is called a good representative of the $\mathcal{L}^1$-class of $u$.

### 2.2 Signed measures

Now we recall some basic facts about finite signed measures on $I$. Proofs and details can be found, e.g., in [10, Chapter 3]. If $\mathcal{B}(I)$ denotes the family of Borel sets, a finite signed Borel measure on $I$ is a map $\nu : \mathcal{B}(I) \to \mathbb{R}$ which is countably additive, i.e. $\nu(\bigcup_{j=1}^\infty B_j) = \sum_{j=1}^\infty \nu(B_j)$ whenever $(B_j)$ is a family of mutually disjoint Borel sets. If the range of $\nu$ is $[0, \infty]$, the measure $\nu$ is also called a finite positive Borel measure, as opposed to positive measures which may attain the value $\infty$. From now on we omit the term “Borel”.

Many properties of a finite signed measure $\nu : \mathcal{B}(I) \to \mathbb{R}$ can be deduced from the theory of positive measures by means of the total variation of $\nu$, which is the smallest positive measure dominating the map $B \mapsto |\nu(B)|$ and will be denoted by $1\nu 1$. It turns out that $1\nu 1$ is finite. Setting $\nu_+ := (1\nu 1 + \nu)/2$ and $\nu_- := (1\nu 1 - \nu)/2$ we have defined two finite positive measures (the positive part and the negative part of $\nu$) such that $\nu = \nu_+ - \nu_-$ and $1\nu 1 = \nu_+ + \nu_-$. If $f$ is a $1\nu 1$-integrable function we define
Let us recall that whenever for every $f \in \operatorname{Reg}(I)$ is Borel measurable, hence every bounded regulated function belongs to $L^1(1_{I} \, 
u \mathbb{I})$. If $f \in L^1(1_{I} \, 
u \mathbb{I})$ then $\nu$ denotes the measure defined by $(\nu\mathbb{I})(B) := \int_B f \, d\nu$, $B \in \mathcal{B}(I)$.

Let us recall that every finite signed measure $\nu$ on $I$ can be uniquely decomposed in the sum $\nu = \nu_d + \nu_c$ where $\nu_c$ is such that $\nu_c(\{t\}) = 0$ for every $t \in I$ and $\nu_d = \sum_{j=1}^{\infty} c_j \delta_{t_j}$ for some real sequences $(c_j)$ and $(t_j)$ with $t_j \in I$ and $\sum_j |c_j| < \infty$, $\delta_{t_j}$ being the Dirac measure at $t_j$.

More generally we can consider Radon measures on $I$, i.e. countably additive $\mathbb{R}$-valued maps $\nu$ which are defined (and finite) on the family of bounded Borel subsets of $I$. Then there exists a smallest positive measure on $\mathcal{B}(I)$ dominating the map $B \mapsto |\nu(B)|$ for $B$ bounded Borel subset of $I$: it is called the total variation of the Radon measure $\nu$ and it is denoted by $1_{I} \, \nu \mathbb{I}$. If $1_{I} \, \nu \mathbb{I}$ is finite we can define the norm $\|\nu\|_I := 1_{I} \, \nu \mathbb{I} (I)$. The set of atoms of a Radon measure on $I$ is the set $A(\nu) := \{ t \in I : 1_{I} \, \nu \mathbb{I} (\{t\}) \neq 0 \}$. Radon measures arise for instance if we have a family of finite signed measures $\nu_j : \mathcal{B}(J) \longrightarrow \mathbb{R}$ indexed on the bounded open intervals $J \subseteq I$ with the property that $\nu_{j_1} = \nu_{j_2}$ on $\mathcal{B}(J_1)$ whenever $J_1 \subseteq J_2$: in this case a Radon measure is well defined by setting $\nu(B) = \nu_j(B)$ for $B \subseteq J \subseteq I$ with $B$ bounded Borel set, $J$ bounded open interval.

### 2.3 Functions of bounded variation

Let us recall that $I = ]a, b[,$ with $-\infty \leq a < b \leq \infty$.

**Definition 2.1.** We say that $u \in L^1(I)$ is of (essentially) bounded variation on $I$ if its distributional derivative $D u$ is a finite signed Borel measure on $I$, i.e. we have $-\int_I \varphi' u \, dL^1(I) = \int_I \varphi \, dDu$ for every $\varphi \in C_c^\infty(I)$. The space of all functions of bounded variation on $I$ is denoted by $BV(I)$. By $BV_{loc}(I)$ we indicate the space of functions which are of bounded variation on every bounded open subinterval of $I$. If $u \in BV_{loc}(I)$ then $D u$ is a Radon signed measure on $I$. We set $A(u) := A(D u)$.

We will mainly be interested in the particular case when a function $u \in BV_{loc}(I)$ is locally integrable on $I$, but its distributional derivative $D u$ is such that $|D u| (I) = \sup\{|D u| (K) : K \subseteq I, \ K \text{ compact} \} < \infty$. In the following result we summarize some well known properties of $BV$ functions whose proofs can be found e.g. in [11, Theorems 3.27, 3.28]).

**Proposition 2.1.** Assume $u \in L^1(I)$. Then $u \in BV(I)$ if and only if $V_e(u, I) < \infty$. In this case there exists a good representative $v \in \mathbb{R}^I$ of $u$ and we have $\|D u\|_I := |D u| (I) = V_e(u, I)$. Moreover there exists a unique constant $c \in \mathbb{R}$ such that the functions $u^r, u^l \in \mathbb{R}^I$ defined by

$$
\begin{align*}
u(t) := c + D u([a,t]), & \quad u^r(t) := c + D u([a,t]), \quad t \in I
\end{align*}
$$

(2.1)
are good representatives; \( u^r \) is right-continuous, \( u^l \) is left-continuous. Any other good representative \( v \) of \( u \) is characterized by the condition

\[
v(t) \in \{(1 - \lambda)u^l(t) + \lambda u^r(t) : \lambda \in [0, 1]\} \quad \forall t \in I; \tag{2.2}\]

moreover \( \text{Discont}(v) = A(u) = \{t \in I : |Du|(|\{t\}|) \neq 0\} \) is at most countable and the constant \( c \) appearing in (2.1) is equal to \( v(a+) \).

From (2.1) follows that \( Du([c, d]) = u^r(d) - u^r(c) \) whenever \( u \in BV(I) \) and \( a \leq c \leq d < b \). Let us recall the following formula (see, e.g., [10, Theorem 3.36, p. 107]), holding for \(-\infty < c < d < +\infty\):

\[
\int_{[c,d]} u \, dDv + \int_{[c,d]} v \, dDu = u(d-)v(d-) - u(c+)v(c+) \quad \forall u, v \in BV(I) \cap C(I). \tag{2.3}
\]

We recall that if \( J \subseteq I \) is a compact interval and if \( p \in [1, +\infty) \) then \( AC^p(J) \) denotes the space of absolutely continuous functions \( u : J \to \mathbb{R} \) such that \( u' \in L^p(J) \). It is well known that if \( u \in AC^p(J) \) then \( V_p(u, J) = \int_J |u'(t)| \, dt < \infty \). Clearly \( AC^p(J) \subseteq AC^q(J) \) for \( p > q \) and \( AC^\infty(J) = Lip(J) \), the space of functions having finite Lipschitz constant (see e.g. [11] for these facts). A very special subset of \( Lip(J) \) is \( C_p(J) \), the space of piecewise linear continuous functions, that is functions \( v \in C(J) \) such that there exists a subdivision \( (t_j)_{j=1}^m \) of \( J \) such that \( v \) is affine on each interval \([t_{j-1}, t_j], j = 1, \ldots, m, a = t_0, b = t_m \). Let us recall now two results whose proof can be found in the Appendix of [12].

**Proposition 2.2.** Assume that \( h \in BV(I) \) is a good representative and that \( N \subseteq I \) such that \( L^1(N) = 0 \). Then \( Dh(\{t \in \text{Cont}(h) : h(t) \in N\}) = 0 \).

**Proposition 2.3.** Assume that \( f \in Lip(\mathbb{R}) \) and \( h \in BV(I) \) is a good representative. Let \( f' \) denote any Lebesgue representative of the distributional derivative of \( f \). Define \( g_h : \mathbb{R} \to \mathbb{R} \) by

\[
g_h(t) := \begin{cases} f'(h(t)) & \text{if } t \in \text{Cont}(h) \\ \frac{f(h(t+)) - f(h(t-))}{h(t+) - h(t-)} & \text{if } t \in \text{Discont}(h) = A(h) \end{cases} \tag{2.4}
\]

Then \( f \circ h \in BV(I) \) and \( D(f \circ h) = g_h \, Dh \).

In order to approximate functions of bounded variation we need the following

**Lemma 2.1.** Assume \( u \in BV(I) \) is right-continuous and \( \text{Discont}(u) = \{t_n : n \in \mathbb{N}\} \). For every \( n \in \mathbb{N} \) define \( u_n : I \to \mathbb{R} \) by

\[
u_n(t) := u(a+) + (Du)_c([a, t]) + \sum_{1 \leq j \leq n, a < t_j \leq t} Du(\{t_j\}), \quad t \in I. \tag{2.5}\]
Then $u_n \in BV(I)$ is right-continuous, $\text{Discont}(u_n) = \{t_1, \ldots, t_n\}$, $u_n \to u$, $u_n' \to u'$ uniformly on $I$, as $n \to \infty$.

**Proof.** It is clear that $u_n$ is right-continuous. We have that $V_p(u, I) < \infty$, hence $\sum_{n=1}^{\infty} |u(t_j) - u(t_{j-})| < \infty$. Therefore for every $t \in I$ we have

$$|u'(t) - u_n'(t)| = \left| \sum_{0 < t_j < t} Du(\{t_j\}) - \sum_{1 \leq j \leq n} Du(\{t_j\}) \right| = \left| \sum_{j > n} \sum_{a < t_j < t} Du(\{t_j\}) \right|$$

$$\leq \sum_{j > n} |Du(\{t_j\})| \leq \sum_{j > n} |u(t_j) - u(t_{j-})| \to 0$$

as $n \to \infty$. It follows that $u_n' \to u'$ uniformly on $I$. Exactly in the same way we can check that $u_n \to u$ uniformly on $I$. $\square$

We finish with the following convergence property which is proved in [9, Proposition 3.2].

**Proposition 2.4.** Assume that $I$ is bounded, $u, u_n, v, v_n \in BV(I)$ and $u$ and $u_n$ are right-continuous for every $n \in \mathbb{N}$. If $\sup_n V_p(v_n, I) < \infty$ and $u_n \to u$ and $v_n \to v$ uniformly on $I$, then $\lim_n \int_I u_n dDv_n = \int_I u dDv$.

## 3 Statement of main results

In order to state the main results of the paper we need to introduce some technical tools.

### 3.1 A convention about extension of functions

In the sequel of the paper if a function $f : [c, d] \to \mathbb{R}$ is defined on a compact interval $[c, d]$, we will tacitly understand that $f$ is extended to $\mathbb{R}$ by setting

$$f(t) := \begin{cases} 
  f(c) & \text{if } t < c \\
  f(t) & \text{if } c \leq t \leq d \\
  f(d) & \text{if } t > d
\end{cases}, \quad t \in \mathbb{R}. \quad (3.1)$$

### 3.2 The arc-length

Let us fix a final time $T > 0$ and consider a function $u : [0, T] \to \mathbb{R}$ such that $u \in BV([0, T])$. Following the ideas of [13, Section 2.5.16], we introduce a sort of arc-length of $u$. First of all we extend $u$ to $\mathbb{R}$ as in Section 3.1, hence $u \in BV_{\text{loc}}(\mathbb{R})$, $Du$ is a finite measure on $\mathbb{R}$ and

$$\|Du\|_R = |Du||(\mathbb{R}) = |u'(0+) - u(0)| + |Du|(0, T] + |u(T) - u'(T-)|. \quad (3.2)$$
Observe also that \( u'(0-) = u'(0-) = u(0) \) and \( u'(T+) = u'(T+) = u(T) \). We define the normalized arc-length \( \ell_u : \mathbb{R} \to [0, T] \) of \( u \) by setting

\[
\ell_u(t) := \begin{cases}
\frac{T}{\|Du\|_\mathbb{R}} |Du|_{[-\infty, t]} & \text{if } \|Du\|_\mathbb{R} \neq 0, t \neq T \\
T & \text{if } \|Du\|_\mathbb{R} \neq 0, t = T, \quad t \in \mathbb{R}.
\end{cases}
\tag{3.3}
\]

The function \( \ell_u \) is increasing, is left-continuous at every point \( t \neq T \), \( \text{Discont}(\ell_u) = \text{Discont}(u') \), and \([0, T] \setminus \ell_u([0, T]) = \bigcup\{|\ell_u(t-), \ell_u(t+)\} : t \in \text{Discont}(u')\). We also have \( \ell_u(0) = 0, \ell_u(T) = T \). Now let \( w \in \mathbb{R}^\mathbb{R} \) be the good representative of \( u \) defined by

\[
w(T) := u(T), \quad w(t) := u'(t), \quad t \neq T.
\]

For every \( 0 \leq t_1 < t_2 < T \) we have \(|w(t_1) - w(t_2)| \leq V_p(w, [t_1, t_2]) = V_p(w, [-\infty, t_2]) - V_p(w, [-\infty, t_1]) = |Du|_{[\ell_u(0), t_2]} - |Du|_{[\ell_u(0), t_1]}|. \) Moreover \( |Du|_{[\ell_u(0), T]} = \|Du\|_\mathbb{R} \), hence we infer that

\[
|w(t_1) - w(t_2)| \leq \frac{\|Du\|_\mathbb{R}}{T}|\ell_u(t_1) - \ell_u(t_2)| \quad \forall t_1, t_2 \in \mathbb{R}.
\tag{3.4}
\]

This inequality implies that \( w(\ell_u^{-1}(\sigma)) \) is a singleton for every \( \sigma \in \ell_u(\mathbb{R}) \), therefore there is a unique \( U : \ell_u(\mathbb{R}) \to \mathbb{R} \) such that \( U \circ \ell_u = w \). From \( (3.4) \) it also follows that \( U \) is the unique Lipschitz function such that \( U \circ \ell_u = u \) \( \mathcal{L}^1 \)-a.e. in \( \mathbb{R} \). In order to extend \( U \) to \( \mathbb{R} \) we define \( \tilde{u} : \mathbb{R} \to \mathbb{R} \) by setting

\[
\tilde{u}(\sigma) := (1 - \lambda)w(t-) + \lambda w(t+) \quad \text{if } \sigma = (1 - \lambda)\ell_u(t-) + \lambda \ell_u(t+), \ t \in \mathbb{R}, \ \lambda \in [0, 1].
\]

Observe that \( \tilde{u}(0) = \tilde{u}(\ell_u(0)) = U(\ell_u(0)) = w(0) = u(0) \) and \( \tilde{u}(T) = \tilde{u}(\ell_u(T)) = U(\ell_u(T)) = w(T) = u(T) \), hence we have the first part of the following

**Proposition 3.1.** Let \( u : [0, T] \to \mathbb{R} \) be such that \( u \in BV([0, T]) \) and let \( \ell_u \) be given by \( (3.3) \). Then there exists a unique function \( \tilde{u} \in \text{Lip}(\mathbb{R}) \) such that, if \( u \) is extended to \( \mathbb{R} \) as in Section 3.1,

\[
u = \tilde{u} \circ \ell_u \quad \mathcal{L}^1 \text{-a.e. in } \mathbb{R},
\tag{3.5}
\]

\( \tilde{u} \) is affine on \([\ell_u(t-), \ell_u(t+)] \quad \forall t \in \text{Discont}(\ell_u). \tag{3.6}\]

We have that

\[
\tilde{u}(0) = u(0), \quad \tilde{u}(T) = u(T).
\]

Moreover \( \ell_u \in AC^p([0, T]) \) whenever \( u \in AC^p([0, T]) \), \( p \in [1, \infty] \). Finally if \( \phi : [0, T] \to [0, T] \) is increasing, surjective, and \( v := u \circ \phi \), then \( \ell_v = \ell_u \circ \phi \) and \( \tilde{v} = \tilde{u} \), or in other terms \( u \circ \phi = \tilde{u} \).
In order to finish the proof of this proposition let us observe that if \( u \in AC^p([0, T]) \) then for every \( t \in [0, T] \) we have \(|Du|(\cdot, t] = V_p(u', \cdot, t]) = \int_0^t |u'(s)| \, ds \), hence we get that \( \ell_u \in AC^p([0, T]) \). Now let us observe that the assumptions on \( \phi \) yield \(|Dv| (\cdot, t] = V_p(v', \cdot, t]) = V_p(u' \circ \phi, 0, t]) = V_p(u', 0, \phi(t]) = V_p(u, -\infty, \phi(t]) = |Du| (\cdot, \phi(t]) \), hence

\[
\ell_v(t) = \frac{T}{\|Dv\|_R} |Dv| (\cdot, t]) = \frac{T}{\|Du\|_R} |Du| (\cdot, \phi(t]) = (\ell_u \circ \phi)(t)
\]

for every \( t \not= T \). Thus we have \( \bar{u} \circ \ell_u = \bar{u} \circ \ell_u \circ \phi = u \circ \phi = v = \bar{v} \circ \ell_v \) and the last property of Proposition 3.1 follows from the uniqueness of \( \bar{v} \).

### 3.3 Main theorems

From now on \( r > 0 \) is fixed throughout the paper. Now we recall the existence result for the play operator with regular inputs which can be proved by using the classical theory on evolution variational inequalities (see Section 4, Proposition 4.1).

**Proposition 3.2.** Assume that \( p \in [1, \infty], z_0 \in [-r, r] \) and \( u \in AC^p([0, T]) \). Then there exists a unique \( y \in AC^p([0, T]) \) such that

\[
\begin{align*}
\quad u(t) - y(t) &\in [-r, r] \quad \forall t \in [0, T], \\
(u(t) - y(t) - z)y'(t) &\geq 0 \quad \forall z \in [-r, r], \quad \text{for } \mathcal{L}^1 \text{-a.e. } t \in [0, T], \\
u(0) - y(0) &= z_0.
\end{align*}
\]

The solution \( y \) is denoted by \( \mathcal{P}(u, z_0, [0, T]) \) or simply by \( \mathcal{P}(u, z_0) \).

The previous proposition allows to define an operator \( \mathcal{P} \) associating to \( u \) and \( z_0 \) the solution \( y \) of (3.11)–(3.13). This operator is usually called the play operator. Here is the statement of our main result.

**Theorem 3.1.** Assume that \( z_0 \in [-r, r], u : [0, T] \to \mathbb{R}, \) and \( u \in BV([0, T]) \). Let \( \ell_u \) and \( \bar{u} \) be the functions defined by Proposition 3.1. Let \( y : \mathbb{R} \to \mathbb{R} \) be the function defined by

\[
y := \mathcal{P}(\bar{u}, z_0) \circ \ell_u,
\]

where \( \mathcal{P}(\bar{u}, z_0) \) is defined in Proposition 3.2. Then \( y \in BV_{\text{loc}}(\mathbb{R}) \) and, extending \( u \) to \( \mathbb{R} \) as in Section 3.1, we have

\[
\begin{align*}
u(t) - y(t) &\in [-r, r] \quad \text{for } \mathcal{L}^1 \text{-a.e. } t \in [0, T], \\
\int_{[0,t]} (u^*(s) - y^*(s) - z(s)) \, dDy(s) &\geq 0 \\
\forall z \in \mathbb{R}^{[0,T]} \text{ Borel measurable, } z([0, T]) &\subseteq [-r, r], \quad \forall t \in [0, T], \\
u(0) - y(0) &= z_0.
\end{align*}
\]

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The next result shows that uniqueness is achieved if we identify functions which differ on sets of zero Lebesgue measure on $[0,T]$.

**Theorem 3.2.** Assume that $z_0 \in [-r,r]$, $u : [0,T] \rightarrow \mathbb{R}$ and $u \in BV([0,T])$. Let $\ell_u$ and $\bar{u}$ be the functions defined by Proposition 3.1 and let $y := P(\bar{u}, z_0) \circ \ell_u$. Then the set of solutions of (3.11)–(3.13) is

$$S_u := \left\{ w \in R^{[0,T]} : w \in BV([0,T]), \; w|_{[0,T]} = y|_{[0,T]}, \; w = y \; L^1 \text{-a.e. in } [0,T] \right\}. \quad (3.14)$$

We can define a solution operator $P$ if for every $u$ we select a unique element in $S_u$, for example the left continuous one in $[0,T]$, i.e.

$$P(u, z_0) := y = P(\bar{u}, z_0) \circ \ell_u. \quad (3.15)$$

The resulting operator is rate independent, that is $P(u \circ \varphi, z_0) = P(u, z_0) \circ \varphi$ for every $u : [0,T] \rightarrow \mathbb{R}$ with $u \in BV([0,T])$ and $\varphi : [0,T] \rightarrow [0,T]$ increasing and surjective. Finally if $p \in [1,\infty]$ and $u \in AC^p([0,T])$ then $P(u, z_0) = P(u, z_0)$, in other words $P$ extends $P$.

The continuity properties of the play operator $P$ will be studied in Section 6. Now let us comment the connection with the formulation studied in [5].

**Remark 3.1.**

(i) In [5] the following problem is considered: given $u \in Reg([0,T]) \cap BV([0,T])$, find $y \in Reg([0,T]) \cap BV([0,T])$ satisfying (3.13) and

$$u(t) - y(t) \in [-r,r], \; \forall t \in [0,T], \quad (3.16)$$

$$\int_{[0,t]} (u(s) - y(s) - z(s)) dy(s) \geq 0$$

$$\forall z \in Reg([0,T]), \; z([0,T]) \subseteq [-r,r], \; \forall t \in [0,T]. \quad (3.17)$$

Here the integral is meant in the sense of Young. In the Appendix we recall the definition of the Young integral and we show that the two integrals yield the same result. Therefore since the function (3.10) is a good representative, we obtain that it solves (3.16)–(3.17), (3.13). Our formulation allows to exploit only measure theory tools and to deduce the existence of solutions directly from the regular case. More precisely only the action of $P$ on $Lip([0,T])$ is needed.

(ii) Our procedure allows to infer that the set of solutions of (3.16)–(3.17), (3.13) is

$$\{ w \in S_u : w \in Reg([0,T]), |w(t) - u(t)| \leq r \; \forall t \in [0,T] \}. \quad (3.18)$$

where $S_u$ is defined in (3.14). Observe that also this set is not a singleton.
4 The play operator on regular inputs

In this section we recall some results about the play operator on regular functions. We fix $T_0, T_f \in \mathbb{R}$, $T_0 < T_f$, and we start with the

**Proposition 4.1.** Assume that $p \in [1, \infty]$, $z_0 \in [-r, r]$ and $u \in AC^p([T_0, T_f])$. Then there exists a unique $y \in AC^p([T_0, T_f])$ such that

$$u(t) - y(t) \in [-r, r] \quad \forall t \in [T_0, T_f],$$

$$(u(t) - y(t) - z)y'(t) \geq 0 \quad \forall z \in [-r, r], \text{ for } L^1\text{-a.e. } t \in [T_0, T_f],$$

$$u(T_0) - y(T_f) = z_0.$$

The solution $y$ is denoted by $P(u, z_0, [T_0, T_f])$ or simply by $P(u, z_0)$ if $T_0 = 0, T_f = T$.

**Proof.** It is a straightforward consequence of general results on evolution variational inequalities, indeed by [14, Ch. 3, Proposition 3.4 and Remark 3.7, p. 69] there exists a unique $x \in AC^p([T_0, T_f])$ such that $x(T_0) = z_0, x(t) \in [-r, r]$, and

$$(x(t) - z)(u'(t) - x'(t)) \geq 0 \quad \forall z \in [-r, r], \text{ for } L^1\text{-a.e. } t \in [T_0, T_f].$$

If we set $y := u - x$ we are done. \hfill \Box

It is well-known and it is easy to check that the play operator $P : AC^p([T_0, T_f]) \times [-r, r] \rightarrow AC^p([T_0, T_f])$ satisfies the following semigroup property: if $u \in AC^p([T_0, T_f])$ and $z_0 \in [-r, r]$ then we have that

$$P(u, z_0, [T_0, T_f])(t_1 + t_2) = P(u, P(u, z_0, [T_0, T_f])(t_2))(t_1)$$

$$\forall t_1, t_2, \quad t_1, t_2, t_1 + t_2 \in [T_0, T_f].$$

**Proposition 4.2.** Assume that $T_0 \leq c < d \leq T_f$ and that $u \in AC^p([T_0, T_f])$ is monotone on $[c, d]$. If $y := P(u, z_0, [T_0, T_f])$ then

$$y(t) = \max \{u(t) - r, \min \{u(t) + r, y(c)\}\} \quad \forall t \in [T_0, T_f].$$

In particular

$$(u(d) - y(d) - z)(y(d) - y(c)) \geq 0 \quad \forall z \in [-r, r],$$

and $P(\cdot, z_0, [T_0, T_f])$ is locally isotone, i.e. if $u$ is monotone on $[c, d]$, then $P(u, z_0, [T_0, T_f])$ is monotone on $[c, d]$ (see [7, Definition 2.9]).

**Proof.** Assume first that $u$ is monotone on $[T_0, T_f]$ ($c = T_0, d = T_f$). From (4.5) follows that $y(t) = \max \{u(t) - r, y(0)\}$ if $u$ increasing and $y(t) = \min \{u(t) + r, y(0)\}$ if $u$ is decreasing. Therefore $y \in \text{Lip}([T_0, T_f])$ and (4.1)–(4.3) are satisfied. The general case
\( T_0 \leq c < d \leq T_f \) is reduced to the previous one by using the semigroup property (4.4). From (4.5) we obtain that there exists \( t_0 \in [c, d] \) such that

\[
y(t) = \begin{cases} 
y(c) & \text{if } c \leq t \leq t_0 \\
y(t) + r & \text{if } t_0 < t \leq d \text{ and } u \text{ is decreasing} \\
y(t) - r & \text{if } t_0 < t \leq d \text{ and } u \text{ is increasing}
\end{cases}
\]  

This formula leads directly to the inequality (4.6).

From formula (4.5) and the semigroup property it can be deduced the following well-known continuity property of the play operator (see e.g. [3, Lemma 2.3.6, p. 46]).

**Proposition 4.3.** The restriction of the play operator to the space of piecewise linear continuous function,

\[
P(\cdot, z_0, [T_0, T_f]) : C_p([T_0, T_f]) \rightarrow C_p([T_0, T_f]),
\]

is continuous when \( C_p([T_0, T_f]) \) is endowed with the topology induced by the norm

\[
\|v\|_{AC^1([T_0, T_f])} = \|v\|_{L^1([T_0, T_f])} + \|v'\|_{L^1([T_0, T_f])}, \quad v \in AC^1([T_0, T_f]).
\]

### 5 Existence

This section is devoted to the proof of Theorems 3.1 and 3.2. We start with the following technical lemma.

**Lemma 5.1.** Assume that \(-\infty \leq a < c < d < b \leq \infty, I = ]a, b[, \) and let \( u \in BV(I) \) be right-continuous. Then

\[
2 \int_{]c, d[} u \, dD\, u = |u(d) - u(c)|^2 - |u(c)|^2 + \sum_{t \in \text{Discont}(u) \cap ]c, d[} |u(t) - u(t^-)|^2.
\]  

**Proof.** We can assume, without loss of generality, that \( \text{Discont}(u) = \{t_j, j \in \mathbb{N}\} \subseteq ]c, d[. \) Let \( u_n \) be defined as in (2.5), so that \( \text{Discont}(u_n) = \{t_1, \ldots, t_n\}. \) If we set \( s_n^n = t_n, s_0^n = c \) and \( s_{n+1} = d, \) then, thanks to (2.3) and the right continuity of \( u, \) we have

\[
\int_{]c, d[} u_n \, dD\, u_n = \sum_{j=1}^{n+1} \int_{s_{j-1}^n, s_j^n] u_n \, dD\, u_n + \sum_{j=1}^{n} u_n(s_j^n) D\, u_n(\{s_j^n\})
\]

\[
= \frac{1}{2} \sum_{j=1}^{n+1} (|u_n(s_j^n) - u_n(s_{j-1}^n)|^2) + \sum_{j=1}^{n} u_n(t_j)(u_n(t_j) - u_n(t_j^-))
\]

\[
= \frac{1}{2} (|u_n(d) - u_n(c)|^2 + \sum_{j=1}^{n} |u_n(t_j) - u_n(t_j^-)|^2)
\]

\[
= \frac{1}{2} (|u_n(d) - u_n(c)|^2 + \sum_{j=1}^{n} |u_n(t_j) - u_n(t_j^-)|^2)
\]

\[
= \frac{1}{2} (|u_n(d) - u_n(c)|^2 + \sum_{j=1}^{n} |u_n(t_j) - u_n(t_j^-)|^2)
\]

\[
= \frac{1}{2} (|u_n(d) - u_n(c)|^2 + \sum_{j=1}^{n} |u_n(t_j) - u_n(t_j^-)|^2)
\]
therefore, since \( \text{Discont}(u_n) \subseteq \text{Discont}(u_{n+1}) \subseteq \text{Discont}(u) \), thanks to Lemma 2.1 we have
\[
\lim_{n \to \infty} \int_{[c,d]} u_n \, dD u_n = \frac{1}{2} \left( |u(d) - u(c)|^2 + \frac{1}{2} \sum_{j=1}^{\infty} |u(t_j) - u(t_j-)|^2 \right)
\]
On the other hand by Proposition 2.4 we have that \( \int_{[c,d]} u_n \, dD u_n \to \int_{[c,d]} u \, dD u \), hence we deduce formula (5.1).

Using the additivity of the integral as a set function, we immediately obtain the following

**Corollary 5.1.** Assume that \(-\infty \leq a < c < d < b \leq \infty\), \( I = [a, b] \), and let \( u \in BV(I) \) be right-continuous. Then
\[
2 \int_{I} u \, dD u = |u(d)|^2 - |u(c^-)|^2 + \sum_{t \in A(u) \setminus [c,d]} |u(t) - u(t^-)|^2.
\]

Now we can prove Theorem 3.1. For simplicity we will often omit the initial datum \( z_0 \) and write \( P(v) \) rather than \( P(v, z_0) = P(v, z_0, [0, T]) \) for some function \( v \).

**Proof of Theorem 3.1.** Set \( y := P(\bar{u}) \circ \ell_u \). It is clear that (3.11) and (3.13) are satisfied. Observe that \( y'(t) = P(\bar{u})(\ell_u(t^+)) \) and \( \text{Discont}(y) = \text{Discont}(\ell_u) = A(u) \). Therefore, thanks to Proposition 2.3, for every Borel measurable \( z : [0, T] \to \mathbb{R} \) with \( z([0, T]) \subseteq [-r, r] \), we have
\[
\int_{[0,t]} (u^r(s) - y^r(s) - z(s)) \, dD y(s)
\]
\[
= \int_{[0,t]} \left( \bar{u}(\ell_u(s^+)) - P(\bar{u})(\ell_u(s^+)) - z(s) \right) \, dD(P(\bar{u}) \circ \ell_u)(s)
\]
\[
= \int_{[0,t] \setminus A(u)} \left( \bar{u}(\ell_u(s^+)) - P(\bar{u})(\ell_u(s^+)) - z(s) \right) \left( P(\bar{u})'(\ell_u(s^+)) - P(\bar{u})(\ell_u(s^+)) - P(\bar{u})(\ell_u(s^+)) \right) \, dD \ell_u(s)
\]
\[
+ \sum_{s \in A(u) \setminus [0,t]} \left( \bar{u}(\ell_u(s^+)) - P(\bar{u})(\ell_u(s^+)) - z(s) \right) \left( P(\bar{u})(\ell_u(s^+)) - P(\bar{u})(\ell_u(s^+)) \right).
\]

Now set
\[
F = \{ \sigma \in [0, T] : (\bar{u}(\sigma) - P(\bar{u})(\sigma) - z)(P(\bar{u}))'(\sigma) \geq 0 \quad \forall z \in [-r, r] \}.
\]

Thanks to Proposition 3.2, formula (3.8), we know that \( \mathcal{L}^1([0, T] \setminus F) = 0 \). Let us set
\[E := \{ s \in [0, T] : \ell_u(s) \in [0, T] \setminus F \}.\]
Since \( A(u) = \text{Discont}(\ell_u) \), in view of Proposition 2.2 we get that \( D \ell_u(E) = 0 \), therefore
\[
D \ell_u([0, T]) = D \ell_u([0, T] \setminus E) \leq D \ell_u\left( \{ s \in [0, T] : \ell_u(s) \in E \} \right)
\]
\[
= D \ell_u\left( \{ s \in [0, T] : [\bar{u}(\ell_u(s)) - P(\bar{u})(\ell_u(s)) - z(s)](P(\bar{u}))'(\ell_u(s)) \geq 0 \} \right).
\]
This implies that \([\tilde{u}(\ell_u(s)) - \mathbf{P}(\tilde{u})(\ell_u(s)) - z(s)](\mathbf{P}(\tilde{u}))'(\ell_u(s)) \geq 0\) for \(\mathbf{D}_{\ell_u}\)-a.e. \(s \in [0, t]\), therefore
\[
\int_{[0,t] \cap A(u)} (\tilde{u}(\ell_u(t)) - \mathbf{P}(\ell_u(t)) - z(t))(\mathbf{P}(\tilde{u}))'(\ell_u(t)) \, d\mathbf{D}_{\ell_u}(t) \geq 0.
\] (5.4)

Now let us recall that \(\tilde{u}\) is affine on the interval \([\ell_u(t-), \ell_u(t+)]\) for every \(t \in A(u)\), therefore thanks to (4.6) we get that
\[
(\tilde{u}(\ell_u(t+)) - \mathbf{P}(\tilde{u})(\ell_u(t+)) - z(t))(\mathbf{P}(\tilde{u})(\ell_u(t+)) - \mathbf{P}(\tilde{u})(\ell_u(t-))) \geq 0 \quad \forall t \in A(u). \] (5.5)

We can conclude collecting together (5.4), (5.5) and (5.3).

**Proof of Theorem 3.2.** Let us take \(z = u^* - (y_1^* + y_2^*)/2\), where \(y_1, y_2 \in \mathbb{R}^{[0,T]}\) are two solutions. By (5.2) and (3.13) we obtain
\[
0 \geq \int_{[0,t]} (y_1^* - y_2^*) \, d\mathbf{D}(y_1 - y_2) \geq (|y_1^*(t) - y_2^*(t)|)^2/2
\]
for every \(t \in [0, T]\) and we infer formula (3.14). Now we prove that \(\mathbf{P}\) is rate independent. Let \(\varphi : [0, T] \rightarrow [0, T]\) be increasing and onto and let \(v := u \circ \varphi\). Clearly \(v \in BV([0, T])\) and from the last statement of Proposition 3.1 we infer that \(\mathbf{P}(u \circ \varphi) = \mathbf{P}(\tilde{u} \circ \varphi) \circ \ell_v = \mathbf{P}(\tilde{u}) \circ \ell_u \circ \varphi = \mathbf{P}(u) \circ \varphi\). We finish by showing that \(\mathbf{P}\) extends \(\mathbf{P}\). The proof of this fact is analogous to the result of [6, Proposition 5.2]. Conditions (3.7) and (3.9) are easily checked. If \(y := \mathbf{P}(u)\) then \(y \in AC^p([0, T])\) and \(y'(t) = \ell'_u(t)(\mathbf{P}(\tilde{u}))'(\ell_u(t))\) for \(L^1\)-a.e. \(t\).

Hence we infer that for every \(z \in [-r, r]\) and for \(L^1\)-a.e. \(t \in [0, T]\)
\[
(\ell_u(t) - y(t) - z)y'(t)
= (\ell_u(t) - \mathbf{P}(\tilde{u})(\ell_u(t)) - z)(\ell'_u(t)(\mathbf{P}(\tilde{u}))'(\ell_u(t)))
= \ell'_u(t)(\tilde{u}(\ell_u(t)) - \mathbf{P}(\tilde{u})(\ell_u(t)) - z)((\mathbf{P}(\tilde{u}))'(\ell_u(t))).
\] (5.6)

Since \(\ell_u\) is absolutely continuous, we have that is \(L^1(\ell_u(N)) = 0\) whenever \(L^1(N) = 0\). Hence if we take
\[
A = \{ s \in [0, T] : (\tilde{u}(s) - \mathbf{P}(\tilde{u})(s) - z)(\mathbf{P}(\tilde{u}))'(s) \geq 0 \ \forall z \in [-r, r] \}
\]
and if \(N := [0, T] \setminus A\) we have \(L^1(N) = 0\) and therefore \(L^1(\ell_u(N)) = 0\), which together with (5.6) implies that the complement of \(\{ t \in [0, T] : (u(t) - y(t) - z)y'(t) \geq 0 \ \forall z \in [-r, r] \}\) is Lebesgue negligible. Thus (3.8) is proven. □

### 6 Continuity properties

In this section we study the continuity properties of the play operator. We fix \(T > 0, r > 0\) and \(z_0 \in [-r, r]\). We will use the notation \(\mathbf{P}\) for the operator \(\mathbf{P}(u, z_0) = \mathbf{P}(u, z_0, [0, T])\)
and $\mathcal{P}$ will denote the extension defined in (3.15). We have seen that $\mathcal{P}$ is acting in the set
$$D := \{ u : \mathbb{R}^{[0,T]} : u \in BV([0,T]) \}.$$
Extending any function of this space as in section 3.1, we endow it with the so-called strict semimetric:
$$d_s(u, v) := \| u - v \|_{L^1([0,T])} + \| \| D u \|_\mathbb{R} - \| D v \|_\mathbb{R} \|, \quad u, v \in D.$$  
This metric is very natural on $BV$ since if we approximate a function with bounded variation $v$ by convolution with a sequence $v_n$, we obtain that $d_s(v, v_n) \to 0$ as $n \to \infty$.

Now we are going to show that $\mathcal{P}$ is continuous with respect to the strict metric. Also in this case we show that this fact can be deduced directly from the continuity properties of $\mathcal{P}$ on $\text{Lip}([0,T])$, without referring to the integral formulation (3.12), but using only the representation formula (3.15). To this aim we apply the general results on rate independent operators contained on [7, 8]. However these results apply to functions defined in open intervals and do not take into account of jumps in the endpoint of $[0,T]$. Therefore in the proof of the following theorem, we have to “enlarge” the play operator to $BV([T_0,T_f])$, where $T_0 < 0 < T < T_f$.

**Theorem 6.1.** The operator $\mathcal{P}$ is continuous with respect to the strict convergence, i.e. if $u, u_n \in \mathbb{R}^{[0,T]}$ are such that $u, u_n \in BV([0,T])$ and $d_s(u_n, u) \to 0$, then $d_s(\mathcal{P}(u_n), \mathcal{P}(u)) \to 0$.

**Proof.** Let $T_0, T_f \in \mathbb{R}$ be such that $T_0 < 0 < T < T_f$, and define $R : AC^p([T_0,T_f]) \to AC^p([T_0,T_f])$ by $R(v) := \mathcal{P}(v, z_0, [T_0, T])$, $v \in AC^p([T_0,T_f])$. Instead we simply set $\mathcal{P}(u) := \mathcal{P}(u, z_0, [0,T])$ and we extend it to $\mathbb{R}$ as in Section 3.1. Let $u \in AC^p([0,T])$ and let us extend $u$ to $[T_0, T_f]$ as in Section 3.1. Since $R$ is rate independent, by [15, Lemma 4.1] we know that $R(u)$ is constant on $[T_0, 0]$ and on $[T, T_f]$ and it is equal respectively to $\mathcal{P}(u, z_0)(0)$ and $\mathcal{P}(u, z_0)(T)$. Therefore by the semigroup property (4.4) we infer that $R(u) = \mathcal{P}(u)$. Thanks to Proposition 4.2 we know that $R$ is locally isotone and by Proposition 4.3 we have that $R$ restricted to $C^p_d([T_0,T_f])$ is continuous with respect to the topology induced by the norm in (4.8). Therefore we can apply [8, Theorem 3.2] and infer that $R$ admits a unique continuous extension to $BV([T_0,T_f])$ and it is given by
$$\overline{R}(v) = R(\hat{v}) \circ s_v \quad \forall v \in BV([T_0,T_f]),$$  
where $s_v : [T_0, T_f] \to [T_0, T_f]$ is defined by
$$s_v(t) := \begin{cases} T_0 + \frac{T_f - T_0}{\| D u \|_\mathbb{R}} | D u | ([a, t]) & \text{if } \| D u \|_\mathbb{R} \neq 0, \\ T_0 & \text{if } \| D u \|_\mathbb{R} = 0, \end{cases} \quad t \in [T_0, T_f].$$  

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and \( \hat{u} \in Lip([T_0, T_f]) \) is the unique continuous function such that
\[
\hat{u} = \hat{u} \circ s_u \quad L^1\text{-a.e. in }]T_0, T_f[,
\]
(6.3)
\[
\hat{u} \text{ is affine on } [s_u(t-), s_u(t+)] \quad \forall t \in \text{Discont}(s_u) = A(u).
\]
(6.4)

It is clear that the theorem is proved if we show that \( \bar{R} = \bar{P} \). Observe that if \( 0 < t_1 < t_2 < T \) then
\[
|s_u(t_1) - s_u(t_2)| = \frac{T_f - T_0}{T} |\ell_u(t_1) - \ell_u(t_2)|
\]
therefore there exists a unique Lipschitz continuous function \( \varphi : \ell_u([T_0, T_f]) \rightarrow \mathbb{R} \) such that \( \varphi \circ \ell_u = s_u \). Now we extend \( \varphi \) to the whole interval \([T_0, T_f]\) by setting
\[
\varphi(\sigma) := (1 - \lambda)s_u(t) + \lambda s_u(t+) \quad \text{if} \ \sigma = (1 - \lambda)\ell_u(t-) + \lambda \ell_u(t+), \ t \in \mathbb{R}, \ \lambda \in [0, 1].
\]
Hence we have that \( \varphi : [T_0, T_f] \rightarrow [T_0, T_f] \) is an increasing Lipschitz continuous functions such that \( \varphi([T_0, T_f]) = [T_0, T_f] \) and \( s_u = \varphi \circ \ell_u \). Therefore we have that \( u = \hat{u} \circ s_u = \hat{u} \circ \varphi \circ \ell_u \) from which it follows that \( \bar{u} = \bar{\hat{u}} \circ \varphi \). Hence we infer that
\[
\bar{R}(u) = \bar{R}(\bar{\hat{u}}) \circ s_u = \bar{R}(\bar{\hat{u}}) \circ \varphi \circ \ell_u
\]
\[
= \bar{R}(\bar{\hat{u}} \circ \varphi) \circ \ell_u = \bar{R}(\bar{\hat{u}}) \circ \ell_u = \bar{P}(\bar{\hat{u}}) \circ \ell_u = \bar{P}(u).
\]
\(\square\)

7 Appendix

In this section we show that the Young integral with respect to a function \( g \) of bounded variation, is nothing but the ordinary Lebesgue integral with respect to the measure \( Dg \), the distributional derivative of \( g \). Let us now recall the definition of Young integral given in [5]. Assume \( -\infty < c < d < \infty \) and set \( J = [c, d] \). Let us consider \( f : [c, d] \rightarrow \mathbb{R} \) and \( g \in \text{Reg}(J) \). We extend \( g \) to \( \mathbb{R} \) as in section 3.1, in this way \( g(c-) := g(c) \) and \( g(d+) := g(d) \). Let us observe that if \( g \in BV([c, d]) \) then \( g \in BV_{\text{loc}}(\mathbb{R}) \) and \( Dg \) is a Radon measure on \( \mathbb{R} \), in particular \( Dg([c]) = g(c+) - g(c) \) and \( Dg([d]) = g(d) - g(d-) \). Let \( s = \{t_0, \ldots, t_m\} \in \mathcal{S}([c, d]) \) be a subdivision of \([c, d]\) such that \( t_0 = c \) and \( t_m = d \) and let \( c = (c_j)_{j=1}^m \) be a family of numbers that is consistent with \( s \), i.e. \( t_{j-1} < c_j < t_j \) for every \( j = 1, \ldots, m \). The Young integral sum is defined by
\[
S_Y(f, g, s, c) := \sum_{j=1}^m f(c_j)(g(t_j-) - g(t_{j-1}+)) + \sum_{j=0}^m f(t_j)(g(t_j+) - g(t_{j-1}-)).
\]
(7.1)

We say that \( f \) is Young integrable with respect to \( g \) if there exists \( L \in \mathbb{R} \) such that for every \( \varepsilon > 0 \) there exists \( s_\varepsilon \in \mathcal{S}([c, d]) \) which satisfies the inequality
\[
|L - S_Y(f, g, s, c)| < \varepsilon
\]
\[
\varepsilon
\]
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whenever \( s_\varepsilon \subseteq s \) and \( c \) is consistent with \( s \). The number \( L \) is uniquely determined and is called the \textit{Young integral of} \( f \) \textit{with respect to} \( g \) and is denoted by one of the symbols

\[
\int_a^b f \, dg, \quad \int_a^b f(t) \, dg(t).
\]

The proof of the following proposition is in [5, Lemma 3.2, p. 166].

**Lemma 7.1.** If \( f \in \mathbb{R}^J \) and \( g \in BV(J) \) then \( f \) is Young integrable with respect to \( g \) if and only if for every \( \varepsilon > 0 \) there exists \( s_\varepsilon \in \mathcal{G}([c,d]) \) such that for every \( s_j \subseteq s_\varepsilon \) and \( c_j \) consistent with \( s_j \), \( j = 1, 2 \), we have \( |S_Y(f, g, s_1, c_2) - S_Y(f, g, s_2, c_2)| < \varepsilon \).

Now we can compare the Young and Lebesgue integrals.

**Lemma 7.2.** If \( f \in \text{Reg}([c,d]) \) and \( g \in BV([c, d]) \cap \text{Reg}([c, d]) \) then \( f \in L^1(|DG|, [c, d]) \), \( f \) is Young integrable with respect to \( g \), and \( \int_c^d f \, dg = \int_{[c,d]} f \, dDg \).

**Proof.** First of all we have that \( f \in L^1(|DG|, [c, d]) \), because it is bounded. Let us neglect the trivial case when \( Dg \) is zero. Then there is a step map \( f_\varepsilon \) with respect to intervals such that \( \|f - f_\varepsilon\|_\infty < \varepsilon/(2\|Dg\|_R) \). We may assume that there are a subdivision \( (t_j)_{j=1}^m \) and numbers \( x_1, \ldots, x_m \) such that \( f_\varepsilon = \sum_{j=1}^m \chi_{[t_{j-1}, t_j]} x_j + \sum_{j=1}^m \chi_{[t_j, t_{j+1}]} f(t_j) \). Observe that \( \sup_{t_j} |f(t) - x_j| < \varepsilon/(2\|Dg\|_R) \) for every \( j \). Therefore if we take e.g. \( c_j := (t_{j-1} + t_j)/2 \), we have

\[
\begin{align*}
&\left| \sum_{j=1}^m f(c_j)(g(t_j) - g(t_{j-1})) + \sum_{j=1}^m f(t_j)(g(t_j) - g(t_{j-1})) - \int_{[c,d]} f \, dDg \right| \\
&\leq \left| \sum_{j=1}^m f(c_j)(g(t_j) - g(t_{j-1})) - \sum_{j=1}^m x_j(g(t_j) - g(t_{j-1})) \right| \\
&\quad + \left| \sum_{j=1}^m x_j(g(t_j) - g(t_{j-1})) + \sum_{j=1}^m f(t_j)(g(t_j) - g(t_{j-1})) - \int_{[c,d]} f \, dDg \right| \\
&= \left| \sum_{j=1}^m (x_j - f(c_j))(g(t_j) - g(t_{j-1})) \right| + \left| \int_{[c,d]} f_\varepsilon \, dDg - \int_{[c,d]} f \, dDg \right| \\
&\leq \sum_{j=1}^m |x_j - f(c_j)||g(t_j) - g(t_{j-1})| + \int_{[c,d]} |f_\varepsilon(t) - f(t)| \, d|DG| \, (t) \\
&\leq \sum_{j=1}^m \frac{\varepsilon}{2\|Dg\|_R} \, \text{d}|DG| \, ([t_{j-1}, t_j]) + \frac{\varepsilon}{2\|Dg\|_R} \, \text{d}|DG| \, ([c,d]) < \varepsilon.
\end{align*}
\]

Hence we have proved that \( f \) is Young integrable with respect to \( g \) and \( \int_c^d f \, dg = \int_{[c,d]} f \, dDg \). \( \square \)
References


