Politecnico di Torino

Department of Mathematics

PhD Day

Booklet of Abstracts

December 15, 2006

PRESENTATION

Gli studenti giunti al termine del secondo anno di Dottorato hanno di fatto già messo a punto un programma di ricerca e consolidato risultati, sovente pubblicati o in corso di pubblicazione. Alcuni di loro stanno svolgendo stages di ricerca a livello internazionale, altri svolgeranno lo stage durante il terzo anno. È quindi un buon momento per incontrarsi, comunicare al Dipartimento ed alla Scuola di Dottorato i risultati raggiunti, consigliarsi sul futuro, e quindi anche concludere il "PhD Day" in allegria.

Il programma che segue è limitato alla presentazione solo di coloro che attualmente sono a Torino; gli altri (Giacomo Como e Federica Garin), ai quali invio un caro saluto, presenteranno i loro risultati nel corso di una seconda giornata, non appena tutti e due saranno di nuovo a Torino. Tuttavia, gli abstracts che seguono il programma includono le ricerche di tutti i dottorandi attualmente al secondo anno. In coda al programma è riportato il titolo delle loro comunicazioni.

Si sa che quello attuale è un momento difficile per tutti coloro che hanno dedicato una parte significativa della loro vita alla ricerca. Il momento è particolarmente difficile per i giovani, tuttavia per un giorno desideriamo abbandonare il pessimismo e pensare con convinzione che la ricerca ed i risultati competitivi a livello internazionale abbiano riconoscimento prima in questo Ateneo e poi anche nel paese.

Nicola Bellomo

PROGRAMME

14.30: Astanin Sergey – *Model of tumour growth: simulation of cord formation and its microenvironment* –

15.00: Bianca Carlo – *Periodic orbits and chaos of the nonequilibirum Ehrenfest wind-tree model* –

15.30: Brazzoli Ilaria – *Methods of the discrete mathematical kinetic theory for active particles with application to complex systems in applied sciences* –

16.00: Fransos Davide – *Deterministic and stochastic numerical models for wind engineering* –

16.30 – 17.00 – *Break* –

17.00 Imparato Daniele Enrico – *Relative entropy, non parametric exponential models and their relationship to the minimal distance martingale measure and portfolio optimization in incomplete markets* –

17.30: Lanza Valentina – *Limit cycles and bifurcation analysis in nonlinear oscillatory networks and applications to spiral waves propagation* –

18.00: Tosin Andrea – *Simulation and optimization models. Vehicular traffic problems on road networks* –

18.30 *Chiusura Lavori*

Nota: La Presentazione di Giacomo Como e Federica Garin sarà programmata nel 2007 in data da definirsi.

Como Giacomo – *On coding theory for non-binary high performances codes* –

Garin Federica – *Asymptotic analysis of concatenated coding schemes for communication using higher order modulation* –

MODEL OF TUMOUR GROWTH: SIMULATION OF CORD FORMATION AND ITS MICROENVIRONMENT

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Keywords: tumour growth, tumour microenvironment, tissue mechanics, nonlinear diffusion, finite differences, interface capturing methods.

Abstract

Cancer disease manifests itself as abnormally excessive cell proliferation. This is the result of dysregulation of normal constrains on cellular proliferation [1].

This fact has serious implications on the morphology of growth. While most of the cells in the human body are within few cell diameters from a blood vessel, intensive proliferation of tumour cells creates cell populations distant from blood vessels and deprived from nutrient and oxygen supply [9].

In some tumours this gives rise to formation of cylindrical structures around blood vessels. These structures are composed of tumour cells and are often surrounded by region of necrosis. These structures are referred to as *tumour cords* [12, 7, 10].

Tumour cords are known to have stationary radial structure with the most intensive proliferation near the blood vessel. This proliferation pushes tumour cells further away from the vessel, until they reach hypoxic or nutrient poor zone, where cells eventually die. The processes of growth and death tend to be in equilibrium as the cord reaches a certain radius. This radial structure has been described in a series of mathematical models [5, 4, 3].

The focus of our model is at describing propagation of the cord along the vessel and formation of its radial structure. Thus the model is essentially multidimensional. For qualitative modelling axial symmetry of the cord is assumed.

We consider tumour cord as the growth of tumour tissue around the blood vessel and assume that it is surrounded by the normal tissue of the host. We consider these two tissues to be immiscible and exhibiting similar mechanical properties. However, the last assumption is arguable, and might be reconsidered in the future.

Tissues are described as porous media. This approach has been already successfully applied to tumour tissue modelling in [6]. Both tissues in our model may be compressed or stretched and constitute continuous medium.

We assume that growth and death occur only in tumour tissue, and thus suppose that tumour is a major consumer of nutrients and oxygen.

There is flow of nutrient and oxygen from the vessel through the tissue. While normal regulation of cell proliferation in tumour may be broken, its growth is still certainly limited by availability of nutrients and oxygen and by maximum achievable cell packing density. The death results from insufficient supply of nutrients and oxygen.

The growth is likely to be limited by either supply of oxygen or supply of one of the nutrients. For the sake of definiteness we assume oxygen-limited growth, but the model stays practically the same for a highly diffusive nutrient.

According to [1], four cell functions tend to be incorrectly regulated in neoplasm: normal constraints on cellular proliferation, differentiation programme, stability of genetic organization, and apoptosis. As far as only excessive proliferation of tumour tissue is concerned, the model takes into account only inefficacy of proliferation constraints. This leaves out of scope of the model such phenomena as evolution of tumour behaviour with time (mutable genotype) and changes of tissue properties due to incomplete differentiation programme.

Tumour growth is a very complex process, and the model being developed grasps only the minimal range of phenomena required to describe anisotropic growth of tumour cords. However the model has greater potential. We consider the model as a framework for further simulation of more complex processes, like delivery of drugs inside the tumour, tumour interactions with extracellular matrix, and changes of tumour behaviour due to changing microenvironment conditions.

For example, according to [9], various factors influence drug penetration in tumours, including physicochemical properties of drugs. Distribution of water-soluble macromolecules depends strongly on the level of interstitial fluid pressurxpdf e and properties of extracellular matrix. Both factors are model variables, and are already part of the model framework. This opens a natural way of integration between growth model and drug distribution models.

However, even at this stage the model allows to describe tumour cord propagation along the vessel as well as its morphology, formation of hypoxic regions and implications of growth on the tissue density and nutrient distribution.

We assume that tissue may be described as a saturated porous medium consisting of three phases: cells, extracellular liquid, and rigid extracellular matrix. We enforce mixture saturation condition and employ quasi-static approximation of mechanical stress balance in the tissue.

A high degree of flexibility of the model lies in the way cell–cell contacts are described. In most connective tissues intercellular adhesion forces may be important, though in some tissues this is not the case. Some tissues are easily compressible, the others are not. We assume that the mode of cell-cell interactions is a tissue property. The stress tensor of the cellular fraction depends on the function describing intercellular forces for a given tissue. This produces nonlinear diffusion terms in equations of cellular fraction movement, but offers more modelling opportunities.

There is a significant simplification of the model in an assumption that friction between cells and intercellular liquid is negligible. This implies constant pressure for the liquid phase. Thus the model is reduced to just one independent variable, related to tissue packing density.

We consider that oxygen is supplied by diffusion from the vessel and consumed by tumour cells at a constant rate. As oxygen diffusion is a much faster process than tumour growth, we use Poisson equation to describe oxygen distribution in the tissue.

Cell proliferation and cell death under the conditions of non-uniform nutrient distribution play an important role in forming the stationary structure of tumour cord [3]. This is the most crucial and the most delicate part of the model. At this moment we assume that cells proliferate when the concentration of oxygen is above certain threshold value, and cells die when it is below another threshold. This is an extremely rough description of the underlying physiological processes, as we do not take into account cell cycle effects and effects of other factors except of oxygen level.

One of the distinctive features of the model is explicitly defined interface between tumour mass and surrounding tissue. This reflects an assumption that these tissues are immiscible. Imposing boundary conditions on this interface is not a trivial question, because this interface is moving, and there is flow of liquid through the interface.

Mechanical considerations suggest continuity of stress on this moving boundary. The velocity of interface movement is the velocity of the cellular fraction of tumour cells. If the cord is surrounded by inviscid liquid mass, this will result in a free boundary problem with a Dirichlet condition on a free boundary. If the cord is surrounded by connective tissue, and mechanical properties of the two tissues are different, packing density jump may be observed across the interface. Under an assumption of identical mechanical response functions of both tissues, the condition of continuity of stress becomes the condition of continuity of packing density across the interface.

Preliminary simulations have been done for the case of growth surrounded by liquid [2]. In this case a deformable quadrilateral grid has been used to approximate variables inside the tumour domain. Method of indefinite coefficients [8] allowed to construct conservative finite difference approximation.

A more general approach has been used in later simulations for the case of the cord surrounded by connective tissue. In this case a regular quadrilateral grid was used and level-set method [11] employed to capture the position of the interface.

Capturing interface position raises the question of choosing an appropriate interface capturing method. While level-set method is the easiest for implementation, it does not conserve the mass. Hybrid interface capturing methods might be considered for future implementations. Phase field methods might be useful if immiscibility condition is dropped.

The first simulations demonstrated steady expansion of the cord along the vessel and formation of its radial structure with necrotic shell. These simulations serve as a proof of idea implementation and are a good starting point for developing more advanced models of tumour cords.

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PERIODIC ORBITS AND CHAOS OF THE NONEQUILIBRIUM EHRENFEST WIND-TREE MODEL

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Keywords: Chaotic hypothesis, Enherefest wind tree-model, Gaussian thermostat

Abstract

Recent advances in nanotechnology make possible the manufacturing of microporous membranes with pores of several nanometres, with the desired geometry and porosity. These membranes have important medical applications, such as the protection of xenograft tissues from the immune system, and the controlled delivery of drugs. Several dynamical theories of Irreversible Thermodynamics[5] have been proposed[2, 3, 6, 8, 9, 10, 12, 13] , which could be adapted to the study of transport in such microporous media. These theories lead to experimentally verifiable results, like various fluctuation relations[6, 7, 8, 9, 10, 12], and motivate the study of stochastic related processes. This study is of interest in both fundamental and applied research. Apart from the medical applications mentioned above, a succesful model of transport through microporous membranes will favor a deeper understanding of the current chaotic hypotheses (CH) .

The development of a such theory requires specific mathematical models. One of these is the periodic Ehrenfest wind-tree model which is a one-particle system whose dynamics is rather complex. It consists of one moving particle of mass m, which is elastically scattered by a set of rhomboidal fixed obstacles arranged on a triangular lattice. Between any two collision, the particle moves freely along straight lines. The periodic structure of the lattice allows us to follow the motion of the particle, looking at the periodic image of its trajectory in just one hexagonal cell.

In [14] these authors considered a nonequilibrium version of the model, with an external field ϵ and a Gaussian thermostat which constrains the kinetic energy of the particle to its initial value K. The asymptotic behavior of the system for nonvanishing(but small) fields is trivial, namely all trajectories eventually approach a periodic or quasi periodic orbit. However, before reaching the asymptotic regime, the system exhibits a diffusive behaviour, up a number of collisions n_c, which is part of a long transient(the time duration of it is several millions collisions for fields of order $10^{-3} - 10^{-2}$) during which the dynamics looks non trivial, as if the system were chaotic. They found that for smaller ϵ the iterates of the map during the transient fill more and more uniformly the square.

Accordingly, upon switching on the field, two of the four Lyapunov exponents remain zero (the exponents corresponding respectively to the conserved kinetic energy, λ_1, and to the direction of the flow, λ_2) while the other two are numerically seen to approach a negative or vanishing value:

$$0 = \lambda_1 = \lambda_2 \geq \lambda_3 > \lambda_4.$$

In practice, for small fields like those mostly considered here, the exponent λ_3 is often so small that numerically we can hardly distinguish it from zero, while λ_4 takes definitely negative values. The mere existence of a trivial asymptotic motion for $\epsilon \neq 0$ does not however exclude quite

complicated behaviour. Indeed, as dissipation is weak for small ϵ a long transient is requied to reach the periodic orbit. During the transient the motion of the particle looks erratic, and covers almost uniformly a large fraction of the phase space. Furthermore, the behaviour of the system on this time scale appears to be almost stationary, and can be described in a statistical way.

Then they conclude that, for the generic initial conditions, the particle spends a considerably long time exploring pratically all the available phase space, until it eventually reaches the periodic orbit. It follows that, because a positive current is observed in the stationary state as well as in the transient, the sum of the Lyapunov exponents is always negative. Therefore, one exponent at least is negative, while the largest exponent is either marginally negative or very close to zero. Finally their conjecture was that at small field the real stationary state is a periodic orbit (or almost), but there is a transient state that lasts n_c collisions. The ensuing divergence of n_c with small fields is thus fully consistent with the conclusion that genuine diffusive behaviour is found for the wind-tree model without an applied field. Therefore this seems to contradict the claim that observing diffusive behaviour in a given physical system is a good indicator that the corresponding molecular dynamics is chaotic. The conclusion they draw is that, in spite of the absence of any source of chaos, the complex dynamics of our system in the transient states is almost indistinguishable from genuine chaotic dynamics.

Since the asymptotic state stydied in [14] was always found to be either a periodic orbit or quasi periodic it is important to find some results of existence of periodic orbits. We investigated periodic orbits of period two three and four and we proved their existence for a generic geometry and initial conditions as a function of various parameters. For this orbits is possible to compute analitically the Lyapunov exponents.

We performed numerical simulations of the model starting with random initial conditions and considering the eletric field in the range $0.02 - 1$. We computed the lyapunov exponents by using the Benettin's algorithm. We observed more cases with positive lyapunov exponents than with negative exponents. In contrast to the expectations of [14], we have that the asymptotic behaviour of the system is not trivial most of the time. In fact for such eletric fields the motion of the particle looks erratic and covers a large fraction of the phase space. The distribution is still complex and when the stationary state has been reachead it looks not trivial. We have found a periodic orbit of period four which is embedded in the attractor and has positive lyapunov exponent, strongly suggesting the presence of chaos.

The most interesting behaviour is when the eletric field is in the range $0.002 - 0.1$. As before we performed numerical simulations of the model starting with random initial conditions and by using the Benettin's algorithm. As before we observe more cases with positive lyapunov exponents than with negative exponents.

The asymptotic behaviour of the system is double. In fact the state is either chaotic or a periodic (or quasi periodic) orbit. In the latter case we oberve a striking decrease of the lyapunov exponent towards negative values, as time grows. The exponent looks to have converged for a large number of still collisions but after a longer time it starts to decrease. For each field in the above range we performed numerical simulations. The situation is almost the same as the previous one: some exponents become negative after a long time and the stationary state is a periodic or quasi-periodic orbit while other exponents remain positive.

Other models such as the mechanical pump of Benettin and Rondonis[1] and the polygonal channels of Jepps and Rondonis[11] will be considered in the next stage of our investigation.

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METHODS OF THE DISCRETE MATHEMATICAL KINETIC THEORY FOR ACTIVE PARTICLES WITH APPLICATION TO COMPLEX SYSTEMS IN APPLIED SCIENCES

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Keywords: Kinetic theory, active particles, discrete microscopic variable, immune competition.

Abstract

An interesting research field of applied mathematics is the modelling of large systems of interacting entities whose microscopic state includes, in addition to mechanical variables, also additional variables related to their somehow organized and even "intelligent" behavior. These entities interact among themselves - individual based interactions - thus exchanging informations which modify their individual state.

Mathematical methods can be developed to describe by equations the collective behavior starting from a description of microscopic interactions. Methods of the mathematical kinetic theory can be properly developed towards the above target. It is a quite natural approach considering that classical models of the kinetic theory, e.g. the Boltzmann and Vlasov equations, lead to models which describe the collective behavior of classical particles which cannot be individually identified in a large system, while individual interactions are modelled within the framework of classical mechanics.

On the other hand, these specific systems are characterized by the additional difficulty that interactions do not follow rules of classical mechanics. Moreover the identification of the microscopic state itself needs a deep analysis of physical systems characterized by an high level of complexity.

The application of methods of the mathematical kinetic theory to the modelling and simulation of complex systems in applied sciences is documented in several papers related, for instance, to traffic flow modelling [2], social systems [9], [20-21], evolution of personal feelings [13-14], modelling multicellular systems with application to the immune competition [4], [7], [18].

Additional bibliography can be recovered in papers [1] and [8], devoted to the methodological aspects on the derivation of kinetic equations, as well as in [3] or in [23] which concerns the mathematical description of living fluids and tissues.

Some pioneers papers have proposed suitable generalizations and developments of the above mathematical approach to model large complex systems in different field of applied sciences. Among others, Prigogine and Herman proposed a mathematical theory of traffic flow by Boltzmann type equations [22]. The same approach has been applied to model the social behavior of colonies of insects by Jager and Segel [17], or cell populations with special attention to the immune competition [6].

A mathematical discrete kinetic theory was developed in [12] towards the modelling and analysis of systems in the following two fields:
- immune competition [10]
- degradation phenomena in conservation sciences [11].

Paper [12] is motivated by the mathematical approach proposed in [9], where a class of evolution equations was proposed to deal with large systems of interacting individuals such that the microscopic state is a **discrete variable**. Remark that the hint to analyze this type of systems is motivated by modelling requirements rather than by the aim of reducing computational complexity.

The contents refer to the modelling of large complex systems by mathematical methods of the kinetic theory. These systems are mathematically characterized by the fact that the number of variables to describe the evolution of the system is greater than the number of evolution equations. As a matter of fact, microscopic interactions between the entities - called **active particles** - make them change their microscopic state, which is not only characterized by mechanical properties, but also by a social or biological discrete variable called **activity**. Thus we have to define the activity of the entities, which cannot generally be described by the standard evolution equations.

Paper [12] deals with the above mentioned topics with the aim of providing a variety of mathematical tools to be used towards modelling in applied sciences. Therefore the contents refer to methodological aspects, while a specific application is dealt with in [10] with reference to the immune competition between tumor and immune cells. The application of methods of the mathematical kinetic theory to this competition was initiated in [6] and developed by various authors as documented in the review paper [5], as well as in various recent applications [7-8], [15,16], [19].

In [10] a qualitative and computational analysis of the model proposed is developed with the aim of visualizing the predictive ability of the model with special attention to the asymptotic behavior of the solution. More in details I present some simulations which have the aim to focus the following aspects:

- The model, under suitable selection of the parameters which appear in the evolutions equations, is able to show, after an initial situation of growth, either the progressive destruction of tumor cells, due to the action of active immune cells, or the blow up of tumor cells.
- the activation of immune cells may modify the output of the competition.

A suitable development of the methods of the kinetic theory for active particles is proposed in [11] applied to open systems subject to external actions. I modelled degradation phenomena for works of art under the action applied by external agents. The model consists in an evolution equation for the probability distribution of the degradation stage. The interpretation of empirical data provides the identification of the parameters of the model and a quantitative prediction of degradation events.

At present, the research program is addressed to a deeper analysis of open systems with special attention to biological sciences.

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ON CODING THEORY FOR NON BINARY HIGH PERFORMANCES CODES

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Keywords: coding theorems, capacity, error exponent, nonbinary codes, group codes, low-density parity-check codes.

Abstract

Information theory was started as a mathematical discipline by some pioneering works of Claude Shannon in the late '40s of last century [14]. Since then, its primary motivation -both as main field of application and source of problems- has been communication engineering, which deals with the issue of designing systems for reliable transmission of data. During the second half of last century, this has been one of the most rapidly developing research areas with many achievements of great relevance both from the applicative as well as from the theoretical viewpoint : for a background we refer to textbooks [9, 5].

A fundamental issue characterizing information theory since its beginning is the use of the probabilistic method for proving coding theorems. Rather then considering single coding schemes, ensemble of codes are usually introduced (i.e. sets of coding schemes equipped with a probability measure). Once estimated an ensemble's performances with the aid of combinatorial and probabilistic techniques, probabilistic statements can be formulated about its behaviour. A first step is considering the average performance of an ensemble: this is the technique originally proposed by Shannon [14], and actually many capacity achievement results can be shown in this way. However it turns out that in many contexts there is a lack of concentration of the ensemble performances around their expected value. In these cases average based analyses are either too conservative in evaluating error exponents or do not even allow to show exponential behaviours characterizing typical instances of the ensemble. In many contexts where more refined information theoretic parameters than the capacity are of interest, more sophisticated probabilistic techniques, such as second moment methods, expurgation arguments, concentration inequalities, martingales and large deviations, can often give fundamental insight. Another aspect of classical information theory is that many fundamental results are proved only for binary alphabets using the simple algebraic structure of the binary field. While this assumption is often justified by simplicity and versatility reasons, there are many applications where bandwidth efficiency requirements motivate extensions of the theory to non binary alphabets and the use of more complicate algebraic structures.

Among the research lines which have been developing during last years, the analysis and design of high performance coding schemes is certainly one of the most relevant. Starting with the introduction of turbo codes in 1993 [3] and the rediscovery of Gallager's low-density parity-check (LDPC) codes [8] in 1995 [11], this field has attracted an increasing attention from many researchers not only in communication theory but also in different disciplines such as statistical physics. The applicative success of both turbo and LDPC coding schemes -witnessed by their implementation in the most recent standards of communication- relies on the excellent tradeoff between performances and complexity guaranteed by the use of a suboptimal iterative decoding algorithm. This algorithm exploits the sparsity of the graph representation of these codes and has been recognized as an instance of a more general algorithm known as belief propagation and used in many areas of statistics and combinatorial optimization [20]. Among the mathematical

techniques that allowed to obtain the most recent and important theoretical achievements in this field are the combinatorics of permutations, random graphs theory and concentration theory.

Up to now, my research work has focused mainly on the development of mathematical tools for the analysis of non-binary high-performances coding schemes. The core idea consists of considering extensions of binary LDPC codes to the framework of group codes matched to geometrically uniform modulations ([7],[10]). The research explored mainly the following guidelines.

A first issue is the extension of the theory of linear codes over memoryless symmetric channels to group codes. The fact that linear codes over finite fields allow to achieve Shannon capacity and error exponent of a symmetric memoryless channel is a well known result of classical information theory [9]. We addressed the question whether the same holds true in the more general context of group codes: a result in this sense was conjectured by Loeliger in [10]. We solved this problem for a generical Abelian group G [23]: it turns out that classical information theoretic theorems generalize in a nontrivial way. A new concept of capacity is introduced, which we called G-capacity: it conveys information about Shannon capacities of subchannels associated to subgroups of G, and it is shown to be exactly the information theoretical limit achievable by G-codes. Examples are presented showing that in some cases G-capacity and Shannon capacity coincide while in other cases the former is strictly less than the latter. The theory of group actions and the analysis of the evolution of probability measures over finite groups have been the main mathematical tools used. I'm currently working in extending to the Abelian group case other classical results concerning linear binary codes such as the relationship between the Gilbert-Varshamov bound and the typical minimum distance and error exponent [1] of ensembles. The main problem left for future research is an extension of the theory to non Abelian groups.

A second issue consists in an analysis of the maximum-likelihood decoding performances of LDPC codes over cyclic groups matched to nonbinary modulations. Up to now, the problem has been addressed in the literature either with an empirical approach [16] or essentially restricting the analysis to the finite field case [2, 6]. The analysis we developed is performed by evaluating average distance spectra of ensembles of LDPC codes and deriving bounds on the performances based on them, following the research line of [2, 15]. The main results obtained show that, for a given factor graph degree pair, regular ensembles of LDPC codes over an arbitrary cyclic group allow to achieve a nonzero capacity with average error rate decreasing to zero as a power of the blocklength. Expurgation arguments lead to show an exponential decaying for the error rate of a typical code sequence for high enough signal to noise ratio. Moreover, as the factor graph density is increased, capacity and error exponent of LDPC ensembles are shown to approach those of the ensemble of all cyclic group codes. I'm currently analyzing how different choices of the nonzero entries of the LDPC matrix affect the performances of the codes. There is strong empirical evidence of the relevance of this design parameter and tools from Fourier analysis on finite groups seem promising in giving insight in the problem. I'm also working in extending the ideas of typical case analysis as opposed to average based one to the study of distance distribution and error exponent of different ensembles of binary and non binary codes [24].

The problem of control with information constraints is receiving increasing attention from the research community. Motivated by the relationship between the problem of stabilizing unstable plants and that of channel coding with feedback [17], many traditional concepts of information theory are being reconsidered, and particularly the role of feedback and the notions of capacity, delay and error exponent [18, 12, 13]. In this context I'm interested in the development of a mathematical framework for the analysis and design of high performance, non binary sequential coding schemes for non ergodic sources, to be used in estimation and control problems for dynamical systems. To this extent, I would like to generalize my previous analysis from the memoryless setting to channels with memory and feedback.

I'm currently spending an eight months period as a visiting assistant in research at Yale University. My research activity here focused on the problem of the error exponent for channels with memory and feedback. Fundamental works addressing the problem date back to the '60s and 70's [4, 21] and are being recently reconsidered [18, 19]. Preliminary results obtained for noiseless feedback Markov channels have shown as the theory of binary hypothesis testing and large deviations theory of Markov chains can give some fundamental insight in the problem.

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DETERMINISTIC AND STOCHASTIC NUMERICAL MODELS FOR WIND ENGINEERING

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Keywords: stochastic PDEs, generalized polynomial chaos, sparse grids, geometric uncertainty, wind engineering, flutter derivatives.

Abstract

Uncertainty in the knowledge of physical quantities, such as material properties, geometry and others, is a matter of increasing importance for engineering and physics. In fact, these uncertainties can deeply affect the modeled phenomena. For instance, this is the case of the aerodynamic or aeroelastic behavior of the not streamlined bodies of interest in Wind Engineering.
The first part of the thesis is devoted to the development of mathematical tools for the numerical solution of the complex models describing such uncertainty. In the second part, new deterministic models are developed, in order to characterize aeroelastic systems by means of computational tools. Finally, in the third part, uncertainty is inserted in the latter models in the form of random parameters describing the geometry of the considered aeroelastic system. The numerical solution is obtained through the methods introduced in the first part of the thesis.

Part I - Numerical methods for the solution of SPDEs in random domains

Stochastic partial differential equations (SPDEs) provide richer mathematical models than standard (deterministic) PDEs, in that they also account for possible uncertainties in the phenomena under modelization. Usually, uncertainties are incorporated into the model by assuming that one or more components which define the PDE problem (the coefficients, the initial or boundary data, the domain) are suitable random variables. The price for dealing with a richer model is obviously an (often significant) increase in the complexity of the problem. This is primarily manifested by the introduction of a (possibly large) number of new independent variables (the *stochastic* variables), in addition to the standard *deterministic* variables such as space and time.
The numerical solution of a SPDE is therefore an extremely challenging task, which is attracting a growing interest in literature (see, e.g, the contributions in [15]).
The approach followed in the thesis has received considerable attention in the last decade, and relies on the global expansion of the random variables defining the PDE into a basis of "elementary" random variables, and the corresponding representation of the solution in terms of such variables. The most popular examples of such expansions are the Karhunen-Loève expansions (see, e.g., [12]) and the Polynomial Chaos (PC) expansions [16], which exploit the classical technique of weighted orthogonal polynomials. The stochastic problem is then transformed into a deterministic one in higher dimension.
The first part of the thesis focuses on the numerical solution of SPDEs of elliptic and parabolic type, posed in a domain whose boundary is described by random variables. Possible solution approaches are based on the Fictitious Domain method [5] and on stochastic mappings [17]. Following the latter, a PC expansion expresses the stochastic parametrization of the boundary. Then,

it is possible to extend the parametrization inside the domain, i.e., to map the random domain onto a fixed one: the PDE of interest is converted into a new one, posed in a non-random domain but with random coefficients and data (recent results for such equations can be found in [1]). Discretization is accomplished by h-type finite elements in the deterministic variables and orthogonal polynomial projection in the stochastic variables. In the case of a small number of PC variables, the discrete variational formulation features a Galerkin projection onto complete tensor-product spaces of stochastic variables and tensor-product numerical integration in these variables. The resulting scheme is equivalent to a collocation scheme, i.e., to a non-intrusive treatment of stochasticity. This yields a very efficient method, e.g. in comparison with Monte Carlo methods. When the number of PC variables becomes larger than a certain value, the need for "sparse" quadratures arises. The latter case is considered and will be addressed in the work [4].

Part II - Determination of the transfer functions for a linear aeroelastic system by means of Computational Wind Engineering

The study of flutter instability of long, flexible structures is usually simplified by considering the two-dimensional airflow around a representative rigid section along the main span of the structure. The motion of the section and the aeroelastic force, induced by the motion itself, can be seen as input and output of a dynamical system, which for streamlined bodies can be supposed linear. Even if Wind Engineering usually deals with bluff bodies (for instance bridge cross sections), for which the assumption of linearity no longer holds, Scanlan and Tomko [14] firstly formulated a linearized model, introducing empirical coefficients, the so-called *flutter derivatives*, to characterize the aeroelastic transfer function. They are generally obtained through wind-tunnel test on oscillatory section models. The computational determination of flutter derivatives is a challenging task for Computational Wind Engineering. The most popular method to achieve this goal has been firstly proposed by Larsen and Walther [10], and adopts sinusoidal imposed vibrations of the section in perfect analogy with experimental tests. Hence, the transfer function in the frequency domain is obtained frequency by frequency. This method, when applied to the computational approach, is very expensive, because a transient simulation is required for each frequency. Following the idea firstly introduced in [11], a new quasi-indicial method is developed in the thesis. A smoothed-ramp motion of the section is proposed, on one hand in order to avoid the computational problems involved by the infinite acceleration provided by a step-wise input of the boundary condition. On the other hand, the smoothed ramp has a frequency content large enough to evaluate the transfer function at each frequency of interest by means of a unique computational simulation. The Arbitrary Lagrangian-Eulerian formulation of the Navier-Stokes equations is employed to simulate the effects of such motion on the flowfield. The numerical procedures are optimized and the reliability and efficiency of the method is shown ([8, 7, 6]).
The application of the method to the classical thin flat plate allows to investigate the physical characteristics that mostly vary between Aeronautic and Civil Engineering applications. In particular, the Reynolds number effects have been evaluated ([2, 3]).

Part III - Geometric uncertainty quantification in Wind Engineering

In Wind Engineering, as well in Aerospace Engineering [9], random discrepancies between the ideal geometries conceived in the design phase and their actual realization tested in wind tunnels may lead to significant variations in the resulting flow field. In particular, the case study of the Sunshine Skyway Bridge deck is addressed in the latest part of the thesis. Large differences between the mean forces acting on the design deck section (with sharpe edges) and the wind tunnel model (with rounded edges, due to the realization technology) have been observed, both by experimental tests and computational simulations [13].
The aim of the work is to introduce suitable random variables in order to describe these geometric uncertainties. The resulting SPDE will be treated by means of the methods introduced in the first part of the thesis, along with the identification techniques introduced in the second part, in order to evaluate the effects of randomness on the static and aeroelastic forces acting on the section.

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ASYMPTOTIC ANALYSIS OF CONCATENATED CODING SCHEMES FOR COMMUNICATION USING HIGHER ORDER MODULATION

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Keywords: turbo codes, convolutional codes, discrete-time dynamical systems, trellis-coded modulation, symmetries, ensembles of permutations.

Abstract

Channel coding theory is the study of how to add redundancy to a source message in order to guarantee a correct received message even when the communication channel is noisy. In his pioneering work [19], Shannon gave a mathematical formalization of the problem of digital communication, introducing probabilistic models of sources and channels and studying both source coding (data compression) and channel coding (error correction). His surprising channel coding theorem guarantees that error probability can be made arbitrarily small by introducing bounded redundancy, at the price of augmenting code complexity. Shannon's proof was based on probabilistic arguments, and showed that randomly constructed codes are good with high probability. However, random codes have in general an unfeasible complexity, particularly in the decoding process. Thus, for more than forty years, classical algebraic coding theory (see e.g. [16]) focused on the construction of codes with very strong structural properties, in order to allow an easy decoding, but the performances of such codes were far away from the theoretical limit given by Shannon capacity.

Two modern classes of codes give an excellent performance/complexity tradeoff: turbo codes (introduced in 1993 [4]) and LDPC codes (introduced in 1963 [11] but re-discovered [15] and widely studied only recently). The amazing performances of those two families of codes are due to an iterative decoding algorithm which reduces the decoding complexity. The performances have been showed by extensive simulations via Monte-Carlo methods, but there isn't yet a fully satisfactory theoretic explanation, even if some analysis has given partial understanding. On one side, the theoretical work can focus on the intrinsic properties of the codes, analyzing minimum distance and error probability under maximum likelihood decoding (a study initiated in [2]); on the other side it can analyze the iterative decoder and its convergence properties (starting from [20]; see [17] and the reference therein). Both approaches can give more insight in the performances and provide criteria for code design, and for both topics there are many open questions to address.

We focus on the first approach and we consider the family of turbo codes, including later generalized 'turbo-like' codes, in particular serially-concatenated turbo codes introduced in [3]. A serial turbo encoder is the composition of an outer encoder $\phi_o : \mathbb{Z}_2^{kN} \to \mathbb{Z}_2^{rN}$, an interleaver, i.e. a permutation $\pi \in S_{rN}$ which rearranges the outer code bits in a different order, and an inner encoder $\phi_i : \mathbb{Z}_2^{rN} \to \mathbb{Z}_2^{nN}$. Both the outer and the inner encoder are obtained by restriction of a convolutional encoder to finite sequences. The first analysis of such schemes [3] separates the design of the constituent convolutional encoders from the interleaver. Their technique, named 'uniform interleaver', is an ensemble analysis quite common in information theory: instead of studying one single code, they analyze a family, introducing an *ad hoc* probability distribution on it. In this case the family consist in encoders made by the same fixed constituent encoders ϕ_o and

ϕ_i while the interleaver Π is a r.v. uniformly distributed over S_{rN}. The performance parameter to be studied is the error probability under the assumption of maximum likelihood decoding, and in particular its asymptotic behavior when N tends to infinity. The result in [3] was an approximate upper bound for the average error probability of the ensemble, having a polynomial decay C/N^{μ}, with an exponent $\mu = \lfloor (d_f^o + 1)/2 \rfloor$, where d_f^o is the free distance (i.e. minimum Hamming weight) of the outer code. The upper bound was then proved more rigorously in [12]. Our contribution was to add a lower bound which is equal to the upper bound up to a constant factor.

Moreover, we have looked for more precise probabilistic results instead of only the average. We have found that the average is affected by a small fraction of very bad codes, and the typical codes sequence has a much better behavior, having a sub-exponential decay. More precisely, assuming that $\{\Pi_N\}$ is a sequence of independent r.v.s each uniformly distributed over S_{rN}, and defining the r.v. $X_N = \log(-\log(P(e)|\Pi_N))/\log N$, X_N converges to β in probability, while almost every sequence densely covers all the interval $[\alpha, \beta]$, for two parameters $0 < \alpha < \beta < 1$ which are increasing functions of d_f^o (see [5]). The proofs are based on an analysis of the minimum distance by techniques introduced in [1, 13], involving system-theoretic properties of the constituent convolutional encoders for bounding their weight enumerating coefficients, counting arguments, graph-theoretic arguments for a deterministic bound on the minimum distance; other mathematical tools are Borel-Cantelli Lemma and a conditioning technique known as 'expurgation'.

Another generalization of the results in [3] was to look at non-binary alphabets. The engineering motivation is transmission which uses more than two waveforms, in order to achieve a better spectral efficiency and to save power and bandwidth. Some of the most used modulations and channel models have symmetries which can be exploited in the code design by choosing an alphabet whose algebraic structure respects these symmetries: see [10, 14] for a theory of Geometrically Uniform constellations and their generating groups. In particular, we extended the average-based asymptotic analysis of error probability to serial ensembles over Abelian groups, finding (under proper assumptions on the constituent encoders) a polynomial decay $P(e|\Pi_N) \asymp N^{-\mu}$ for $N \to \infty$, where μ is described as the solution of an integer optimization problem, involving error events of the constituent encoders, and satisfies the bounds $\lfloor (d_f^o + 1)/2 \rfloor \leq \mu \leq d_f^o$ (see [9]).

Another significant variation is looking at smaller ensembles, where permutations are restricted to some specified subset of S_{rN}; this is motivated by simulation results [6] showing that some classes of permutations exhibit a better convergence abscissa for the iterative decoding algorithm.

To prove our main result for non-binary codes, we use system-theoretic properties of the constituent convolutional encoders for bounding their weight enumerating coefficients,a second-order method for the lower bound and bounds on the size of the orbit of some codewords under the action of the permutation group. For this reason, our proof generalizes to smaller ensembles where the subset is a subgroup of permutations, considering proper distances and weights that are invariants under the action of the subgroup (see [7]). As future work, we would like to address the case when the subset is a coset of a subgroup: our conjecture is that this could improve both the value of μ and the iterative decoding convergence.

There are further generalizations that we would like to address. First of all, our results about typical codes concern only the binary case and our proofs heavily rely on this assumption, which simplifies the dynamics of the convolutional encoders and allows for tighter bounds of their weight enumerating coefficients: we will look for new bounding techniques giving more general results.

Another interesting generalization is the serial concatenation of more than two encoders: partial results about distances in [1] suggest that typical code sequences could have exponentially decreasing error probability.

Finally, some recent results [18] extend to turbo codes the concept of stopping sets, a tool for analyzing the iterative decoder (at least on a particular channel, the BEC) which was previously used only for LDPC. The definition of stopping sets can then be generalized for all turbo-like codes, and we think that we can apply to stopping sets enumerating functions some bounding techniques related to those we are currently using for weight enumerating functions.

A further important generalization would be the extension of some results to codes over non-Abelian groups; however this would involve completely new techniques, as many of the properties of convolutional encoders that we use are proved for rings and then extended to Abelian groups via the Kronecker Decomposition Theorem (see e.g. [8]).

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RELATIVE ENTROPY, NON PARAMETRIC EXPONENTIAL MODELS AND THEIR RELATIONSHIP TO THE MINIMAL DISTANCE MARTINGALE MEASURE AND PORTFOLIO OPTIMIZATION IN INCOMPLETE MARKETS

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Keywords: Exponential Orlicz spaces, Exponential model, relative entropy, Girsanov densities, (BDG)-inequalities.

Abstract

The aim of the present research is to investigate in detail the theory of non parametric exponential models and connections with minimization problems of the relative entropy. Applications are considered to both Martingale Theory and Mathematical Finance.

Let $(\Omega, \mathcal{F}, \mu, (\mathcal{F}_t))$, with $t \in [0, T]$, $T < \infty$, be a filtered probability space satisfying the usual conditions. Pistone and Sempi (1995) and Pistone and Rogantin (1999) construct a differential geometrical manifold structure locally modeled on the topology of centered random variables of the exponential Orlicz space $L_0^{\exp}(\mu)$ - sub-0 denotes centered r.v.'s - for the set of μ-almost surely strictly positive densities; this work is a functional framework for Information Geometry, see e.g. Amari and Nagaoka (2000). Within such a framework, they define the *maximal exponential model at p* as the set of densities which can be connected to p by an open exponential arc. In our work we consider extended exponential models, studying in particular their topological closure. Recent literature, in fact, has been considering these concepts; see e.g. Czizár and Matůs (2005).

Burkholder, Davis and Gundy's (BDG) inequalities relate the norm of a martingale with that of its quadratic variation in L^p spaces and, more in general, in moderate Orlicz spaces. By using the previous geometrical structure, in our research we find a result which is similar to (BDG) classical inequalities, but in the framework of exponential (non moderate) Orlicz spaces -see Imparato and Pistone (2005)-. Recently, in fact, this analytical framework has been given renewed relevance - see e.g. Alsmeyer (2006) - and may have applications to Mathematical Finance. For instance, in Rheinländer (2005) semimartingales such that their quadratic variation belongs to the exponential Orlicz space are considered.

In Mathematical Finance, if a market is incomplete optimal portfolios can be obtained by maximizing a given expected utility. By considering the dual problem, this corresponds to a minimization problem among equivalent martingale measures. Such a set can be infinite, so that one has to choose the optimal one with respect to a suitable criterion. In general, minimal distance martingale measures are considered.

In Frittelli (2000) the problem of finding the equivalent martingale measure which minimizes the relative entropy is considered. Under the hypotheses that this set is non empty, he discusses existence and uniqueness of such a solution and connections to the maximization of the expected exponential utility.

Our aim is to set up this classical problem within the previous analytical framework.

As a first application to Martingale Theory, we prove that locally Girsanov density processes can be described by an exponential model. Regularity properties are proved within the Wiener

space. Some of these results were partially presented at IGAIA2 conference - see Imparato (2005a).

We recall that the relative entropy between two densities p and q is defined as $D(q||p) := \mathbb{E}_p[q/p \log(q/p)]$ and takes infinite value when the expectation diverges. Such an integral is always non negative and is not if and only if $q = p$. In most literature, theorems regarding the relative entropy are stated with the assumption of finiteness of such a functional. Therefore, it makes sense to find conditions for the relative entropy to be finite. In Cena and Pistone (2006) regularity of the relative entropy and connections with exponential models are discussed. They observe that the finiteness of this functional can be reinterpreted within the topology of the Orlicz space which is the conjugate of the exponential one. In particular, it is proved that any density which belongs to the maximal exponential model at p has finite relative entropy with respect to p.

In the present work we further investigate such links. More precisely, our aim is to describe a new approach to the problem of minimizing the relative entropy among given densities; see e.g Czisár (1975) as classical reference. In that paper, minimization problems are discussed in which the set of densities is convex; furthermore, the existence of a solution for a class of constrained problems is proved.

The analytical framework we use in order to state regularity properties of the relative entropy enables us to extend those classical results to a wider class of constrained problems. These considerations may have straightforward applications to the problem of finding the minimal relative entropy equivalent martingale measure. A first discussion on this topic was presented at the National Conference on Stochastic Methods in Mathematical Finance - see Imparato (2006).

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LIMIT CYCLES AND BIFURCATION ANALYSIS IN NONLINEAR OSCILLATORY NETWORKS AND APPLICATIONS TO SPIRAL WAVE PROPAGATION

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Keywords: nonlinear oscillatory networks, CNN, describing function technique, spiral waves.

Abstract

There are several methods that are exploited to predict the existence of limit cycles in nonlinear systems and to study their stability, both in time and frequency domain. The so-called Poincaré map method, for instance, is a time-domain method and it widely used, but, when applied to high dimension problems, a number of difficulties appear. In these cases, spectral methods, such as harmonic balance and the describing function technique [14], are more powerful. Their basic idea is to represent a periodic solution by a Fourier series and to try to find a period and a set of Fourier coefficients which satisfy the system equations. In this way, instead of solving a set of nonlinear ordinary differential equations, one has to consider a system of nonlinear algebraic equations, whose solutions can be achieved more easily. In particular, the describing function technique [14] consists in approximating the periodic solution up to the first harmonic. Although it could seem a too strong approximation, in most cases it yields remarkably successful results.

In the first part of my work, I have shown [12, 13] that limit cycle Floquet's multipliers and the related bifurcation phenomena can be estimated through a spectral approach, entirely based on the describing function technique, both in general and for systems that admit a Lur'e representation. To illustrate the proposed method, we analyze different key examples proposed in literature [1] and we compare our results with the available ones. In order to confirm the accuracy of the predictions, the Floquet's multipliers have also been computed (and the related bifurcation processes have been identified), by detecting the limit cycle through the HB technique. We have shown that the most significant bifurcation curves are easily detected and that in most cases our approach yields more accurate prediction than the method proposed in [1]. In addition it allows one to estimate the whole sets of Floquet's multipliers and hence it provides a complete characterization of the stability properties of each limit cycle.

The second part of my work is devoted to the study and the analysis of nonlinear oscillatory networks. From a mathematical point of view an oscillatory network is formed by a large system of coupled nonlinear ordinary equations, that may exhibit a rich spatio-temporal dynamics, including several attractors and bifurcation phenomena. Such networks can be adequately modeled as Cellular Neural/Nonlinear Networks (CNNs), a new paradigm that was introduced several years ago in the electrical engineering community [2, 3]. A CNN is a high-dimensional dynamic nonlinear circuit, given by a local interconnection of simple circuit units called cells or neurons. The model for a standard CNN [3] is constituted by a bidimensional array of $M \times N$ cells in which every cell is a nonlinear dynamic system of order 1. The dynamics of every cell n_{ij} depends on the state of the cells that interact with it, that is the neighboring cells that belong to the influence sphere of n_{ij}, eventually on an external input (in case of non autonomous CNNs), and on a constant term, called threshold or bias. Notice that the coupling law between the cells can be nonlinear and

can involve functional operators, as convolution operator, nonlinear delay operator and Volterra series. The evolution in time of a CNN is specified when are known the initial conditions and the input for every cell, and the boundary conditions. In fact, for the cells that are on the boundary, boundary conditions are needed. In general, they can be of Dirichlet, Neumann or periodic type.

For weakly connected networks, under the assumptions that every cell in the uncoupled system has an hyperbolic stable limit cycle attractor, Malkin's theorem [9] permits to describe the dynamics of the oscillatory system in terms of the phase deviation from the natural oscillation. Following this approach, each limit cycle (either stable or unstable) of the weakly connected network corresponds to an equilibrium point of the phase deviation equation. In order to apply Malkin's theorem to weakly connected network and to compute the phase deviation equation the knowledge of the limit cycle trajectories in absence of coupling is required. The method of [7] is based on the idea that this cycle can be approximately computed by exploiting the describing function technique.

An interesting approach to the analysis of oscillatory arrays, proposed by [16], is widely used in microwave theory to study a set of antenna-loaded, single-device oscillators. In this paper a Kurokawa analysis [11] is used to transform the frequency domain network description into a set of equations describing the oscillator amplitude and phase dynamics. In fact, when the oscillators are strongly coupled, the representation only through the phase is not sufficient, because the amplitude dynamics (that were ignored in the previous analysis) become important. In [16] also the eventual dependence of the coupling on the frequency is taken into account. In this approach, however, they do not study an actual nonlinear oscillatory network because in the frequency domain they linearize the nonlinear circuit elements.

In our work, we have exploited the technique proposed in [16] to a real nonlinear oscillatory system, wherein every cell is resistively coupled with the neighboring ones. In order to characterize each oscillator by using amplitude and phase variables, a method based on a generalized version of the describing function technique is proposed [5]. Without hypothesis on the coupling strength, just asking the amplitudes and phases to be slowly-varying function of time, this method allows us to derive a set of nonlinear ordinary differential equations governing the amplitude and phase dynamics. These assumptions on the system solution can be viewed as a perturbation of the steady-state solution obtained via the describing function technique; then, by analyzing the evolution of amplitudes and phases we can investigate the stability properties of the limit cycles of the whole system. We have proved that in case of weakly connected network, the same results as [7] are obtained, namely we can argue that in this case all the amplitudes of the oscillators remain close to the amplitudes of the uncoupled oscillators and we are able to derive the same differential equation for the sole phase variables as in [7]. Then, we have studied the stability for a chain of oscillators in which each cell interacts only with its adjacent ones. Moreover, we have considered Dirichlet boundary conditions. We have computed the equilibrium configurations for the amplitudes and phases: for the phase only two values of phase shifts are admissible, that is 0 and π, while the amplitudes of equilibrium are those provided by the describing function technique applied to the whole chain, as in [6]. Their stability properties have been studied, of course, computing the Jacobian matrix. As case study, we have applied our approach to a chain of four diffusively coupled Chua's circuits and we have predicted the bifurcations of limit cycles depending on the coupling strength. The results [5] are quantitatively and qualitatively in good agreement with those in literature [6]. Thus, in this work, we have proposed a more general approach to study an oscillatory network, without assumptions on the coupling and without simplifications of the system to be studied.

Pattern formation and nonlinear waves propagation have been observed and reported from almost every scientific disciplines, from engineering to biology and chemistry. In particular, autowaves and spiral waves have been noticed in many excitable media, including the cardiac muscle, retinae, cultures of bacteria, chemical waves and chemical oscillators such as the Belousov-Zhabotinsky reaction. Most of these systems have been successfully modeled by continuous models via partial differential equations (see, for instance, for cardiac arrhythmias, [8]), but they can be well described by discrete models too. Following Kuramoto's approach [10], in fact, what looks like a single periodic process in a macroscopic level often turns out to be a collective oscillation resulting from the mutual synchronization among an huge number of the constituent oscillators. In particular, it is well demonstrated how cellular nonlinear networks can be used for modeling and generating pattern and nonlinear wave phenomena [2, 15].

Our future purpose is to apply the approach in [5] to the oscillatory models proposed in literature to describe the propagation of spiral waves in the cardiac muscle. Exploiting this technique, in fact, we would be able to describe the spatial wave dynamics in terms of temporal amplitudes and phases dynamics of each oscillator forming the network.

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SIMULATION AND OPTIMIZATION MODELS
VEHICULAR TRAFFIC PROBLEMS ON ROAD NETWORKS

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Keywords: Traffic flow modeling, discrete kinetic theory, differential equations, qualitative analysis of mathematical problems.

Abstract

Vehicular traffic is attracting a growing scientific interest because of its connections with other important problems, like e.g. environmental pollution and cities congestion; rational planning and management of vehicles fluxes are key topics in modern societies under both economical and social points of view. In spite of their importance, however, these questions cannot be effectively handled by simple experimental approaches. Observation and data recording certainly provide useful information on the trend of the traffic in uniform flow conditions, but they are completely unable to catch the real unsteady dynamics; in other words, they are scarcely predictive tools. This makes vehicular traffic not only an engineering matter, but also a challenging mathematical problem.

The mathematical modeling of vehicular traffic requires, first of all, the choice of the scale of representation. The relevant literature offers many examples of models at any scale, from the microscopic to the macroscopic through the kinetic one (see e.g., Bellomo *et al.* [1] for an extended review on the subject). Each of them implies some technical approximations, and suffers therefore from related drawbacks, either analytical or computational. The microscopic scale is like a magnifying glass focused on each single vehicle, and by consequence it usually requires a large number of ordinary differential equations to be solved, which makes this approach not competitive for computational purposes. Furthermore, from the analytical point of view it is often difficult to investigate the relevant global features of the system. On the other hand, both kinetic and macroscopic scales aim at describing the big picture of the phenomenon without looking specifically at each single vehicle, hence they are computationally more efficient: few partial differential equations, that can be solved numerically in a reasonable time, are normally involved. Nevertheless, the modeling itself is now less accurate than in the previous case, due to a sort of continuum hypothesis that underlies both approaches (the number of vehicles should be large enough so that it makes sense to introduce the concepts of distribution function and macroscopic density), clearly not physically satisfied by vehicles along a road. Therefore it must be accepted in the abstract, which may somehow give the annoying feeling of invalidating the foundations of the mathematical theory itself. However, besides this critical point, the kinetic approach offers several advantages over the macroscopic one. Firstly, despite the overall description of the system, it allows a microscopic modeling of the interactions among the vehicles, without the need for devising closure relations (the so-called *fundamental diagrams*) to get a self-consistent system of equations. Macroscopic models recover this missing information from experimental observations under steady flow conditions; instead, in the kinetic approach fundamental diagrams are not assumed *a priori*, but they are mathematically studied as equilibrium solutions of the model itself and possibly

compared with the available experimental data. Secondly, all interesting macroscopic quantities can be readily recovered *a posteriori* from the kinetic results via suitable average procedures, so that both levels of description are finally accessible.

The kinetic approach to the modeling of traffic flow firstly proposed by Prigogine and Herman in their seminal work [10] relies strongly on the main ideas of the classical kinetic theory of gases. Vehicles are assimilated to particles whose microscopic state is identified via position and velocity; a one-particle distribution function is then introduced over such microscopic state, for which an evolution equation in time and space is deduced via suitable balance laws within any volume element of the state space, under the main assumptions of binary localized interactions and vehicular chaos. The resulting mathematical structure consists in an integro-differential equation inspired by the celebrated Boltzmann equation, which incorporates an advection part and a gain-loss term.

A second thread is due to Klar, Wegener and their co-workers (see e.g., Günther *et al.* [7], Klar and Wegener [9]). They refer to an Enskog-like framework, giving up the idea of localized interactions and introducing instead the concept of *thresholds of interaction*, namely minimal finite distances among the vehicles which trigger accelerations, brakings, overtakings and so on.

The reserach presented here is in turn concerned with the mathematical modeling of vehicular traffic at the kinetic scale. More in detail, following some general ideas proposed by Bertotti and Delitala [3] on the discrete kinetic and stochastic game theory, it is focused on a discrete kinetic framework, in which the velocity of the vehicles is not regarded as a continuous variable but can take a finite number of values only. Discrete kinetic models have historically been conceived in connection with the Boltzmann equation, primarily as mathematical tools to reduce the analytical complexity of the latter (see e.g., Bellomo and Gatignol [2], Gatignol [6]). In the present context, however, the discretization of the velocity plays a specific role in modeling the system rather than being only a mathematical simplification, because it allows to relax the continuum hypothesis for the velocity variable and to include, at least partially, the strongly granular nature of the flow of vehicles in the mathematical theory of traffic. The discrete velocity framework also gives rise to an interesting structure of the interaction terms of the kinetic equations. The encounters among the vehicles are described in an essentially probabilistic way, introducing the probability that a velocity transition occurs after an interaction between a vehicle and another vehicle located in front of it. The set of all these transition probabilities constitutes the so-called *table of games*, a third order tensor which is furthermore assumed to depend on the solution of the problem itself via the macroscopic density of vehicles. This reflects the influence of the global traffic conditions on the behavior of the vehicles: the probabilities of acceleration-and-overtaking or of braking-and-queuing change dramatically according to the crowding of the road. It is worth remarking that stochasticity is an essential ingredient in order to capture the real essence of the interactions among the vehicles. Velocity transitions are described without invoking any classical concept of force, since vehicles do not interact mechanically: they simply "see" each other and adjust their velocity according to the behavioral rules coded in the table of games. Finally, interactions are binary like in the classical kinetic theory, but not localized: an interaction length is introduced, which defines a visibility zone in front of each vehicle. Moreover, interactions occur at higher or lower rates on the basis of the local free space available along the road. The final outcome on the speed of a vehicle is then determined by a weighted average of the interactions that it experiences with all other vehicles comprised in its visibility zone.

In both the spatially homogeneous and the spatially inhomogeneous case the model has shown ability to describe specific traffic flow phenomena well documented in the specialized literature (see e.g., Kerner [8]).

For the spatially homogeneous problem, in which the one-particle distribution function is assumed to be independed of the spatial variable, existence, uniqueness, and *a priori* boundedness of a physically relevant solution have been proved; moreover, the dependence of the equilibrium configurations on a phenomenological parameter linked to the quality of the road has been investigated, at least in some relatively simple but representative cases. Numerical simulations provide fundamental diagrams and equilibrium velocity distributions highly close to the corresponding available experimental data. More in detail, fundamental diagrams exhibit a strong variation in their behaviors when passing from low to high density, which demonstrates the ability of the model to capture the *phase transition* between free and congested flow experimentally found in uniform flow conditions. It is worth stressing that this result is a consequence of the model, not a modeling assumption. In other words, phase transition has not been postulated *a priori*, as it happens in other works present in the literature (see e.g., Colombo [4]); rather, it is correctly reproduced by the model as a consequence of more general mathematical and physical assumptions.

The spatially inhomogeneous problem, which describes the spatial and temporal evolution of traffic subjected to suitable initial and boundary conditions, has been analyzed by addressing some representative phenomena like, for instance, the formation and backward propagation of a queue both in a road with constant capacity and in a bottleneck. The results yield a vehicles density profile which in the second case closely follows that of the bottleneck, and which in both cases never overcomes the maximum value fixed by the road capacity. If the dislocation of the interactions is undoubtedly responsible for backward density waves in spite of the unidirectional motion of the vehicles, the specific form of the interaction rate introduced in the interaction terms of the kinetic equations favors instead the boundedness of the density. Indeed, the rate of the interactions among the vehicles is conceived so as to strongly increase when the macroscopic density locally approaches the upper limit threshold, so that a larger number of velocity transitions toward lower velocity classes is induced in such a case, thus preventing the superposition of different density waves which may locally sum to more than the threshold.

The research described here has originated the paper [5] by M. Delitala and A. T., which has been accepted for publication in *Mathematical Models and Methods in Applied Sciences.*

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