Ten Lectures on Mathematical Modelling of Complex Living Systems - Part I - Foundations

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"To those who do not know mathematics, it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature. If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in". (Richard Feynman)
Lecture 1. - Common Features of Complex Living Systems

Identification the common relevant characteristic of living systems viewed as complex systems and of the guidelines toward a system approach to their modeling.

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Bibliography


- N. Bellomo, C. Bianca, and M. Delitala, Complexity analysis and mathematical tools towards the modelling of living systems, Physics of Life Reviews, 6 (2009) 144–175.
The view point of a biologist


- Biological systems are very different from the physical or chemical systems analyzed by statistical mechanics or hydrodynamics. Statistical mechanics typically deals with systems containing many copies of a few interacting components, whereas cells contain from millions to a few copies of each of thousands of different components, each with very specific interactions.

- Although living systems obey the laws of physics and chemistry, the notion of function or purpose differentiate biology from other natural sciences. Organisms exist to reproduce, whereas, outside religious belief rocks and stars have no purpose. Selection for function has produced the living cell, with a unique set of properties which distinguish it from inanimate systems of interacting molecules. Cells exist far from thermal equilibrium by harvesting energy from their environment.
Lecture 1. - Common Features of Complex Living Systems

The viewpoint of Philosophers and Physicists


**E. Schrödinger, P. Dirac - 1933**, *What is Life?, I living systems have the ability to extract entropy to keep their own at low levels.*

**R. May**, *Science 2003 In the physical sciences, mathematical theory and experimental investigation have always marched together. Mathematics has been less intrusive in the life sciences, possibly because they have been until recently descriptive, lacking the invariance principles and fundamental natural constants of physics.*
The study of complex systems, namely systems of many individuals interacting in a non-linear manner, has received in recent years a remarkable increase of interest among applied mathematicians, physicists as well as researchers in various other fields as economy or social sciences.

Their collective overall behavior is determined by the dynamics of their interactions. On the other hand, a traditional modeling of individual dynamics does not lead in a straightforward way to a mathematical description of collective emerging behaviors.

In particular it is very difficult to understand and model these systems based on the sole description of the dynamics and interactions of a few individual entities localized in space and time.
Ten Common Features and Sources of Complexity

• **1. Ability to express a strategy:** Living entities are capable to develop specific *strategies* and *organization abilities* that depend on the state of the surrounding environment. These can be expressed without the application of any external organizing principle. In general, they typically operate *out-of-equilibrium*. For example, a constant struggle with the environment is developed to remain in a particular out-of-equilibrium state, namely stay alive.

• **2. Heterogeneity:** The ability to express a strategy is not the same for all entities: *Heterogeneity* characterizes a great part of living systems, namely, the characteristics of interacting entities can even differ from an entity to another belonging to the same structure. In biology, this is due to different phenotype expressions generated by the same genotype.
Ten Common Features and Sources of Complexity

3. **Interactions**: Interactions are nonlinear (nonlinearly additive) and involve immediate neighbors, but in some cases also distant particles. Indeed, living systems have the ability to communicate and may possibly choose different observation paths.

4. **Topology of interactions**: In some cases, the topological distribution of a fixed number of neighbors can play a prominent role in the development of the strategy and interactions. Namely, interactions do not involve all entities that are within an interaction domain.

5. **Stochastic games**: Interactions modify their state according to the strategy they develop. Living entities *play a game at each interaction* with an output that is technically related to their strategy often related to surviving and adaptation ability. This dynamics is also related to the fact that living systems receive a feedback from their environments, which enables them to learn from their experiences.
5. Ten Common Features and Sources of Complexity

• **6. Large deviations:** Interactions involving entities and those with the outer environment can lead to *large effects*, which in turn lead to even larger effects due to a natural trend far from equilibrium.

• **7. Large number of components:** Complexity in living systems is *induced by a large number of variables*, which are needed to describe their overall state. Therefore, the number of equations needed for the modeling approach may be too large to be practically treated. For instance, biological systems are different from the physical systems analyzed by statistical mechanics, which typically deals with systems containing many copies of a few interacting components, whereas cells contain from millions to a few copies of each of thousands of different components, each with specific interactions.

• **8. Learning ability:** Living systems have the ability to learn from past experience. Therefore their strategic ability and the characteristics of interactions among living entities evolve in time.
Ten Common Features and Sources of Complexity

9. **Multiscale aspects:** The study of complex living systems always needs a *multiscale approach*. Specifically, the dynamics of a cell at the molecular (genetic) level determines the cellular behaviors. As a matter of fact, the structure of macroscopic tissues depends on such a dynamics. Moreover, living systems are generally constituted by a *large number of components*. Finally, one only observation and representation scale is not sufficient to describe the dynamics of living systems.

10. **Darwinian selection and time as a key variable:** All living systems are evolutionary. For instance birth processes can generate individuals more fitted to the environment, who in turn generate new individuals again more fitted to the outer environment. Neglecting this aspect means that the time scale of observation and modeling of the system itself is not long enough to observe evolutionary events. Such a time scale can be very short for cellular systems and very long for vertebrates.
Three key questions and a dilemma

The above preliminary analysis enables us to derive three *key questions* to applied mathematicians, that can possibly contribute to define methodological guidelines to modeling in view of a mathematical theory.

1. *Can mathematics contribute to reduce the complexity of the overall system by splitting it into suitable subsystems?*

2. *Can mathematics offer suitable tools to describe the five features proposed above as common to all complex biological systems?*

3. *How far is the state-of-the-art from the development of a biological-mathematical theory of living systems and how can an appropriate understanding of multiscale issues contribute to this ambitious objective?*
Three key questions and a dilemma

One more question can be added, which is somehow related to the third one.

4. Should mathematics attempt to reproduce experiments by equations whose parameters are identified on the basis of empirical data, or develop new structures, hopefully a new theory, able to capture the complexity of the biological phenomena and finally to base experiments on theoretical foundations?

This last question witnesses the presence of a dilemma, which occasionally is the object of intellectual conflicts within the scientific community. However, we are inclined to assert the second perspective, since we firmly believe that it can also give a contribution to further substantial developments of mathematical sciences.
**Scaling Problems** *The first step in the modeling approach is the assessment of the observation and representation scales.*

- The *microscopic scale* corresponds to modeling the evolution of the variable suitable to describe the physical state of each single object viewed as a whole.

- An alternative to the above approach can be developed if the system is constituted by a large number of elements and it is possible to obtain suitable locally in space averages of their state in an elementary space volume ideally tending to zero. In this case the *macroscopic scale* can be used referring to locally averaged quantities, called *macroscopic variables*.

- Generally, these two scales are sufficient to model, by different methods, the dynamics of systems of the inert matter. On the other hand, living systems need the addition of models derived at lower scales. Therefore, the *representation of a living system needs the use of several scales.*
Scaling Problems

• In several cases the averaging process to obtain macroscopic quantities cannot be performed. For instance when the density of the interacting elements is not large enough.

• Moreover averaging processes kill some of the characteristics of complexity, namely the heterogeneous behavior of the components of the system. Further, macroscopic equations need closures related to the material behavior of the system. These are valid in equilibrium, or small deviations from equilibrium, while living systems ”live” far from equilibrium.

• Moreover averaging processes kill some of the characteristics of complexity, namely the heterogeneous behavior of the components of the system. Further, macroscopic equations need closures related to the material behavior of the system. These are valid in equilibrium, or small deviations from equilibrium, while living systems ”live” far from equilibrium.
As an alternative the statistical (kinetic) representation can be used. The state of the whole system is described by a suitable probability distribution over the microscopic state of the interacting elements, that is the dependent variable in kinetic models.

Macroscopic observable quantities are computed by weighted moments of the distribution function. In the kinetic representation the dynamics, namely time and space evolution, of the distribution function is determined by interactions at the microscopic level.

Models designed at the microscopic scale are generally stated in terms of ordinary differential equations. The dependent variable is the overall state of each sub-system composing the overall system.
Different classes of equations correspond to the above scaling.

- Models at the *macroscopic scale* are generally stated in terms of *partial differential equations*. The dependent variable is the local average of the state, at the microscopic level, of each sub-system composing the overall system.

- Models, which use the *statistical (kinetic) representation* are stated in terms of *integro-differential equations*. The *dynamics of the distribution function* is determined by interactions at the microscopic level.
Lecture 1. Common Features of Complex Living Systems

- Critical Analysis

The modeling approach to complex systems should take into account the afore-said common complexity features and can be developed through four steps as follows:

**Step 1:** Derivation of a mathematical structure suitable to describe the evolution of a complex system in a way that the structure retains the common complexity features. It needs to be regarded as a potential ability to be characterized for each specific case study.

**Step 2:** The second step specifically refers to a well defined real system. It consists in selecting the specific complexity characteristics of the system under consideration and in adjusting the afore-said structure to such structure to the specific case.

**Step 3:** The third step consists in characterizing the variables and the parameters that characterize the model referring to the real system that is object of the modeling approach. This basically means (as we shall see in the next lectures) modeling the interaction at the individual (microscopic scale).
Step 4: Validation of the model by comparison of the dynamics predicted by it with empirical data. More precisely, the model should show quantitative agreement with data that are quantitatively known and qualitative agreement with emerging behaviors that are observed in the real behavior.

Remark 1.1. Some models (too many) include artificially empirical data. For instance trend to steady state conditions. This is not correct, prediction of the model should be a consequence of interactions at the microscopic scale.

Remark 1.2. Generally quantitative empirical data are available only for steady state (may be equilibrium) conditions, while emerging behaviors are observed in dynamical conditions. These behaviors are observed and repeated only at a qualitative level as they are very sensitive to environmental conditions.

Remark 1.3. Some models may possibly depict emerging behaviors that have not yet been observed. In this case, specific experiments can be organized to verify the predictions of the model.
Lecture 2. - Reducing Complexity and Representation

Lecture 1. - Common Features of Complex Living Systems

Lecture 2. - Reducing Complexity and Representations Guidelines to represent complex systems looking ahead at the objective of reducing complexity.

Lecture 3. - The Kinetic Theory of Active Particles with Linear Interactions

Lecture 4. - Nonlinear Interactions and Learning
Motivations to Reduce Analytic and Computational Complexity

Motivations to use the macroscopic scale instead of the microscopic one are also related to the practical objective of reducing analytic and computational complexity. For instance, when systems involve a large number of interacting elements, the number of equations of the model may be too large to be computationally tractable. Moreover, only macroscopic quantities are often of practical interest.

Classically, macroscopic models are obtained by conservation or equilibrium equations closed by phenomenological constitutive models. However, the closure is generally obtained by relations valid at equilibrium, while macroscopic models should be derived from the underlying microscopic description delivered by kinetic models. Derivation is obtained by asymptotic methods letting the distance among entities to zero.
Preliminary reasonings towards modeling

Some preliminary reasonings on the modeling approach should be focused on the selection of the appropriate representation scale and on the related variable that are supposed to describe, in the mathematical model, the state of the system under consideration. This selection should however face the problem of reducing complexity.

1. The modeling of systems of the inert matter is developed within the framework of deterministic causality principles. Namely for a given cause, the effect is deterministically identified. This principle needs to be weakened, or even suppressed, in the case of the living matter.

2. The description of the physical state of living systems needs the introduction of additional variables suitable to describe the strategy that each element of the system develops. Therefore, the use of of models at the macroscopic scale ends up with killing the presence of such variable, which is, as documented in Lecture 1, heterogeneously distributed.
• Preliminary reasonings towards modeling

3. If system is constituted by a large number of interacting living entities, one may call them *active particles*, their microscopic state should include, in addition to geometrical and mechanical variables, also an additional variable, called *activity*. This variable should be appropriate to model the ability of each entity to express a specific strategy.

4. The ability to express a strategy is heterogeneously distributed over the active particles. Therefore, a modeling approach can be based on the description of the overall state of the system by a *probability distribution function* over such microscopic state.

**Remark 2.1.** *The activity variable is selected, as it will be discussed in Lecture 3, as a scalar variable.*
Preliminary reasonings towards modeling

5. *Methods of the mathematical kinetic theory* can be developed to model the evolution in time and space of the distribution function. For instance using a suitable balance in each elementary volume of the space of the microscopic states, where the inflow and outflow of particles in the said volume is determined by interactions among particles.

6. *Living systems*, particularly in biology (e.g. multicellular systems), are constituted different types of active particles, so that several different functions are expressed. This is a relevant source of complexity. Therefore, it is useful reducing it *by decomposing the system into suitable functional subsystems* according to well defined rules that will be stated in the sequel.
Preliminary reasonings towards modeling

7. A *functional subsystem* is a collection of active particles, which have the ability to express collectively the same *activity* regarded as a scalar variable. The whole system is constituted by several interacting functional subsystems. The link between a functional subsystem and its activity depends also on the specific phenomena under consideration.

8. The *decomposition into functional subsystems* is a flexible approach to be adapted at each particular investigation. Specifically, the identification of each functional sub-system is related to the activity they express as well as to the specificity of the investigation that is developed. Namely, given the same system, different decompositions can be developed corresponding to different specific investigation.
Preliminary reasonings towards modeling

9. Active particles of the same functional subsystem may, however, differ for specific characteristics. For instance in a multicellular system, cells with a different genotype may however collectively express, with other cell, the same function. Different decomposition may correspond, according to the afore-said statements, to different composition of each functional subsystem.

10. Considering that the subsystems composing a system may be linked by networks, the modeling approach needs treating both interactions within the same functional subsystem in a node of the network and interactions involving functional subsystems of different nodes. In some cases nodes of the network may identify different sub systems. Dealing also with interactions among subsystems may need, in some cases, use of different scales.
Systems of Active Particle

- The system is constituted by a large number of interacting entities, called *active particles* organized into $n$ interacting populations labeled by the indexes $i = 1, \ldots, n$.

- The variable charged to describe the state of each particles is called *microscopic state*, which is denoted by the variable $w$ that includes the geometrical and mechanical description as well as of the *activity variable*. In the simplest case: $w = \{x, v, u\}$, where $x \in D_x$ is *position*, $v \in D_v$ is *mechanical state*, e.g. linear velocity, and $u \in D_u$ is the *activity*.

- Each population corresponds to a *functional subsystem*, namely active particles in each functional subsystem develop collectively a common strategy.
Statistical Representation

- The description of the overall state of the system is defined by the **generalized one-particle distribution function**

\[
    f_i = f_i(t, x, v, u), \quad i = 1, \ldots, p,
\]

such that \( f_i(t, x, v, u) \, dx \, dv \, du \) denotes the number of active particles whose state, at time \( t \), is in the interval \([w, w + dw]\). where \( w = x, v, u \) is an element of the **space of the microscopic states**.

**Remark 2.2.** If the number of active particles is constant in time, then the distribution function can be normalized with respect to such a number and consequently is a probability density.
Marginal Densities

- Marginal densities are computed as follows:

\[
f^m_i(t, x, v) = \int_{D_u} f_i(t, x, v, u) \, du,
\]

\[
f^a_i(t, x, u) = \int_{D_v} f_i(t, x, v, u) \, dv.
\]

and

\[
f^b_i(t, u) = \int_{D_x \times D_v} f_i(t, x, v, u) \, dx \, dv.
\]
Moments and Macroscopic Quantities

- The **local density** of the $i^{th}$ functional subsystem is:

$$\nu[f_i](t, x) = \int_{D_v \times D_u} f_i(t, x, v, u) \, dv \, du = \int_{D_v} f_i^m(t, x, v) \, dv.$$ 

- Integration over the volume $D_x$ containing the particles gives the **total size** of the $i^{th}$ subsystem:

$$N_i[f_i](t) = \int_{D_x} \nu[f_i](t, x) \, dx,$$

which can depend on time due to the role of proliferative or destructive interactions, as well as to the flux of particles through the boundaries of the volume.
Moments and Macroscopic Quantities

- First order moments provide either linear mechanical macroscopic quantities, or linear activity macroscopic quantities.

- The flux of particles, at the time $t$ in the position $x$, is given by

$$Q[f_i](t, x) = \int_{D_v \times D_u} \mathbf{v} f_i(t, x, \mathbf{v}, u) \, d\mathbf{v} \, du = \int_{D_v} f_i^m(t, x, \mathbf{v}) \, d\mathbf{v}.$$ 

- The mass velocity of particles, at the time $t$ in the position $x$, is given by

$$U[f_i](t, x) = \frac{1}{\nu_i[f_i](t, x)} \int_{D_v \times D_u} \mathbf{v} f_i(t, x, \mathbf{v}, u) \, d\mathbf{v} \, du.$$
Moments and Macroscopic Quantities

- The activity terms are computed as follows:

\[ a[f_i](t, x) = \int_{D \times D_u} u f_i(t, x, v, u) \, dv \, du , \]

while the local activation density is given by:

\[ a^d[f_i](t, x) = \frac{a_j[f_i](t, x)}{\nu_i[f_i](t, x)} = \frac{1}{\nu_i[f_i](t, x)} \int_{D \times D_u} u f_i(t, x, v, u) \, dv \, du. \]

- Integration over space provides global quantities:

\[ A[f_i](t, x) = \int_\Omega a[f_i](t, x) \, dx , \]

\[ A^d[f_i](t, x) = \int_\Omega a^d[f_i](t, x) \, dx , \]
Moments and Macroscopic Quantities

- **Local quadratic activity** can be computed by second order moments:

\[ e[f_i](t, x) = \int_{D_v \times D_u} u^2 f_i(t, x, v, u) \, dv \, du , \]

while the **local quadratic density** is given by:

\[ \varepsilon[f_i](t, x) = \frac{e[f_i](t, x)}{\nu[f_i](t, x)} = \frac{1}{\nu[f_i](t, x)} \int_{D_v \times D_u} u^2 f_i(t, x, v, u) \, dv \, du . \]

**Remark 2.3.** The analogy with quadratic quantities and energy needs to be specified with reference the real system under consideration.
Remark 2.4. Particular systems in life sciences are such that the microscopic state is identified only by the activity variable. Namely, space and velocity variables are not significant to describe the microscopic state of the active particles. In this case, the overall state of the system is described by the distribution function $f_i(t, u)$ over the activity variable only.

Remark 2.5. The microscopic state may identified only by the space and activity variables. Namely, the velocity variable is not significant to describe the microscopic state, while it is important knowing the localization of the active particles. Therefore, the representation is given by $f_i(t, x, u)$ and the definition of vanishing variable is used.

Remark 2.6. The space variable can be used, in some cases to identify the regionalization of active particles, where the activity variable is the same in each region, but the communication rules, namely the dynamics at the microscopic scale, differ from region to region. In this case the localization identifies different functional subsystems.
Systems with Discrete Microscopic States

- **Discrete velocity variables**

\[ I_v = \{v_1, \ldots, v_k, \ldots, v_K\}, \quad k = 1, \ldots, K. \]

\[ f_i(t, x, v, u) = \sum_{k=1}^{K} f_i^k(t, x) \delta(v - v_k), \quad f_i^k(t, x, u) = f_i(t, x, u; v_k). \]

- **Discrete velocity and space variable**

\[ I_x = \{x_1, \ldots, x_r, \ldots, x_R\}, \quad r = 1, \ldots, R, \]

\[ f_i(t, x, v, u) = \sum_{r=1}^{R} \sum_{k=1}^{K} f_i^{r,k}(t, u) \delta(x - x_r) \delta(v - v_k), \]

where \( f_i^{r,k}(t) = f_i(t, u, x_r, v_k). \)
Lecture 2. Reducing Complexity and Representations

- **Discrete Microscopic States**

\[ I_u = \{ u_1, \ldots, x_s, \ldots, u_S \}, \quad s = 1, \ldots, S, \]

\[ f_i(t, x, v, u) = \sum_{s=1}^{S} \sum_{r=1}^{R} \sum_{k=1}^{K} f_{srk}^i(t) \delta(u - u_s) \delta(x - x_r) \delta(v - v_k), \]

where \( f_{srk}^i(t) = f_i(t, x_r, v_k, u_s). \)

- **Moment calculations**

Calculation of macroscopic quantities is simply obtained by sums. For instance, the local density is given by

\[ n_i(t, x) = \sum_{k=1}^{K} \int_{D_u} f_{i}^{k}(t, x, u) \, du, \quad \text{or} \quad n_i(t, x) = \sum_{s=1}^{S} \sum_{k=1}^{K} f_{sk}^i(t, x), \]

and so on.
Example: Vehicular Traffic

- **Micro-state:** The state of the micro-system is defined by position, velocity and activity;

- **Functional subsystems:** The simplest case consists in assuming one only functional subsystem. Otherwise, the decomposition can distinguish: Trucks, slow cars and fast cars.

- **Remark 2.7:** Introducing various functional subsystems reduces the need of an heterogeneous activity variable to model the quality of drivers.

- **Remark 2.8:** Multilanes flow can be treated by inserting the lane in the micro-variable of as a functional subsystem.
Example: Immune Competition

- **Micro-state**: The state of the micro-system is defined by the function (activity) expressed by cells of each functional subsystem;

- **Functional subsystems**: The simplest case consists in assuming two functional subsystems. One for cells with a pathologic state, one for the immune cells.

- **Remark 2.9**: Introducing various functional subsystems reduces the need of an heterogeneous activity variable to model the activity of each functional subsystem.

- **Remark 2.10**: The modeling approach should include the onset of new functional subsystems related to Darwinian mutations.
Example: Social Dynamics

- **Micro-state:** The state of the micro-system is defined by the function (activity) expressed by individuals of each functional subsystem. It depends very much on the type of system under consideration.

- **Functional subsystems:** The simplest case consists in assuming one isolated functional subsystems. A general case consists in modeling several functional subsystems interacting in a network.

- **Remark 2.11:** For systems in a network the localization can be used to identify different functional subsystems.

- **Remark 2.12:** The modeling approach should include the onset of new functional subsystems related to aggregation and/or fragmentation events. New subsystems can also be generated by migration phenomena.
Lecture 1. - Common Features of Complex Living Systems

Lecture 2. - Reducing Complexity and Representations

Lecture 3. - The Kinetic Theory of Active Particles with Linear Interactions

*Derivation of Mathematical Tools to deal with Complex Systems.*

Lecture 4. - Nonlinear Interactions and Learning
Lecture 3. The Kinetic Theory for Active Particles with Linear Interactions

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- N. Bellomo, C. Bianca, and M. Delitala, Complexity analysis and mathematical tools towards the modeling of living systems, Physics of Life Reviews, 6 (2009) 144–175.
Classification of Models of the Kinetic Theory

- **Models of classical kinetic theory**  Classical interactions, localized and mean field, with conservation of mass momentum and energy: $f = f(t, x, v)$.

- **Generalized kinetic theory**  Non-classical interactions, localized and mean field, without conservation of mass momentum and energy with the same rules for all particles (typically dissipative interactions): $f = f(t, x, v)$

- **Kinetic theory for active particles**  Interactions by stochastic games with randomly distributed rules: $f = f(t, x, v, u)$.

**Kinetic theory for active particles**  Interactions by stochastic games with rules that are the same for all particles, namely homogeneous activity distribution: $f = f(t, x, v) \delta(u - u_0)$. 
Derivation of Mathematical Structures of the KTAP’s Theory

The strategy to derive these equations follows the guidelines of the classical kinetic theory. Namely, by a balance equation for net flow of particles in the elementary volume of the space of the microscopic state by transport and interactions. The following active particles are involved in the interactions:

- **Test** particles with microscopic state, at the time \( t \), defined by the variable \((x, v, u)\), whose distribution function is \( f = f(t, x, v, u)\).

- **Field** particles with microscopic state, at the time \( t \), defined by the variable \((x^*, v^*, u^*)\), whose distribution function is \( f^* = f(t, x^*, v^*, u^*)\).

- **Candidate** particles with microscopic state, at the time \( t \), defined by the variable \((x_*, v_*, u_*)\), whose distribution function is \( f_* = f(t, x_*, v_*, u_*)\).
Derivation of Mathematical Structures of the KTAP’s Theory

• **Rule:** The *candidate* particles with microscopic state, at the time $t$, defined by the variable $(x_*, v_*, u_*)$, interacts with *field* particles with microscopic state, at the time $t$, defined by the variable $(x^*, v^*, u^*)$ and acquires, in probability the state of the *test* particles with microscopic state, at the time $t$, defined by the variable $(x, v, u)$. *Test* particles interacts with field particles and lose their state.

• **Conservative interactions** modify the microscopic state of particles;

• **Non conservative interactions** generate proliferation or destruction of particles in their microscopic state.

• **Onset of new functional subsystems** Both type of interactions may generate a particle in a new functional subsystem.
Derivation of Mathematical Structures

The mathematical framework refers to the evolution in time and space of the test particle. The derivation related to the distribution functions $f_i$ is based on the following balance of equations in the elementary volume of the phase space:

$$\frac{df_i}{dt} dx dv = \left( G_i[f] - L_i[f] + S_i[f] \right) dx dv,$$

where interactions of candidate and test particles refers to the field particles and $f = \{f_i\}_{i=1}^{n}$. Moreover, for the i-th functional subsystem:

- $G_i[f]$ denotes the \textit{gain} of candidate particles into the state $x,v,u$ of the test particle;
- $L_i[f]$ models the \textit{loss} of test particles;
- $S_i[f]$ models \textit{proliferation/destruction} of test particles in their microscopic state.
Derivation of Mathematical Structures

Let us consider the interactions between candidate or test particles of the $h^{th}$ functional subsystem and the field particles of the $k^{th}$ functional subsystem.

**H.3.1.** The candidate and test particles in $\mathbf{x}$, with state $\mathbf{v}_*, \mathbf{u}_*$ and $\mathbf{v}, \mathbf{u}$, respectively, interact with the field particles in $\mathbf{x}^*$, with state $\mathbf{v}^*, \mathbf{u}^*$ located in its interaction domain $\Omega$, $\mathbf{x}^* \in \Omega$.

**H.3.2.** Interactions are weighted by a suitable term $\eta_{hk}[\rho](\mathbf{x}^*)$, that can be interpreted as an *interaction rate*, which depends on the local density in the position of the field particles.

**H.3.3.** The distance and topological distribution of the intensity of the interactions is weighted by a function $p_{hk}(\mathbf{x}, \mathbf{x}^*)$ such that:

$$\int_{\Omega} p_{hk}(\mathbf{x}, \mathbf{x}^*) d\mathbf{x}^* = 1.$$
Derivation of Mathematical Structures

**H.3.4.** The candidate particle modifies its state according to the probability density $A$ defined as follows: $A_{hk}(v_*, u_* \rightarrow v, u | v_*, v^*, u_*, u^*)$, where $A_{hk}$ denotes the probability density that a candidate particle of the $h$-subsystems with state $v_*, u_*$ reaches the state $v, u$ after an interaction with the field particles $k$-subsystems with state $v^*, u^*$, while the test particle loses its state $v$ and $u$ after interactions with field particles with velocity $v^*$ and activity $u^*$.

**H.3.5.** The test particle, in $x$, can proliferate, due to encounters with field particles in $x^*$, with rate $\mu^i_{hk}(x, x^*)$, which denotes the proliferation rate into the functional subsystem $i$, due the encounter of particles belonging the functional subsystems $h$ and $k$. Destructive events can occur only within the same functional subsystem with rate $\mu^i_{ik}(x, x^*)$. 
Interactions within the action domain

Figure 1: – Active particles interact with other particles in their action domain
Interactions with modification of activity and transition

Figure 2: Active particles during proliferation move from one functional subsystem to the other.
Derivation of Mathematical Structures

**Conjecture 3.1.** Interactions modify the activity variable according to topological stochastic games, however independently on the distribution of the velocity variable, while modification of the velocity of the interacting particles depends also on the activity variable.

\[
\mathcal{A}_{hk}(\cdot) = \mathcal{B}_{hk}(u_* \rightarrow u, |u_*, u^*|) \times \mathcal{C}_{hk}(v_* \rightarrow v |v_*, v^*, u_*, u^*) ,
\]

where \(\mathcal{A}, \mathcal{B}, \text{and } \mathcal{C}\) are, for positive defined \(f\), probability densities:

\[
\int_{D_v \times D_u} \mathcal{A}_{hk}(v_* \rightarrow v, u_* \rightarrow u |v_*, v^*, u_*, u^*) \, dv \, du = 1 , \quad \forall v_*, v^*, u_*, u^* ,
\]

and
Derivation of Mathematical Structures

\[ \int_{D_u} \mathcal{B}_{hk}(u \rightarrow u, |u, u^*|) \, du = 1, \quad \forall \, u, u^*, \]
\[ \int_{D_v} \mathcal{C}_{hk}(v \rightarrow v | v, v^*, u, u^*) \, dv = 1, \quad \forall \, v, v^* u, u^*. \]

**Conjecture 3.2.** The encounter rate, namely, the frequency of interactions depends on velocity and activity of the interacting pairs:

\[ \eta_{ij}(v, v^*, u, u^*), \quad \eta_{hk}(v, v^*, u, u^*) \]

**Conjecture 3.3.** The weight function, which models the intensity of interactions depends on the localization of the interacting pairs:

\[ p_{ij}(x, x^*), \quad p_{hk}(x, x^*) \]
(\partial_t + \mathbf{v} \cdot \partial_\mathbf{x}) f_i(t, \mathbf{x}, \mathbf{v}, u) = \left[ \sum_{j=1}^n \left( G_{ij} [\mathbf{f}] - L_{ij} [\mathbf{f}] \right) \right] + \left[ \sum_{h=1}^n \sum_{k=1}^n S_{hk}^i [\mathbf{f}] \right] (t, \mathbf{x}, \mathbf{v}, u),

G_{ij} = \int_{\Omega \times D_u^2 \times D_v^2} \eta_{ij}(\mathbf{v}_*, \mathbf{v}^*, u_*, u^*) p_{ij}(\mathbf{x}, \mathbf{x}^*) B_{ij}(u_* \to u|u_*, u^*) \times C_{hk}(\mathbf{v}_* \to \mathbf{v}|\mathbf{v}_*, \mathbf{v}^*, u_*, u^*) \times f_i(t, \mathbf{x}, \mathbf{v}_*, u_*) f_j(t, \mathbf{x}^*, \mathbf{v}^*, u^*) \, d\mathbf{v}_* \, d\mathbf{v}^* \, du_* \, du^* \, d\mathbf{x}^*,

L_{ij} = f_i(t, \mathbf{x}, \mathbf{v}) \int_{\Omega \times D_u \times D_v} \eta_{ij}(\mathbf{v}_*, \mathbf{v}^*, u_*, u^*) p_{ij}(\mathbf{x}, \mathbf{x}^*) f_j(t, \mathbf{x}^*, \mathbf{v}^*, u^*) \, d\mathbf{v}^* \, du^* \, d\mathbf{x}^*.

S_{hk}^i = \int_{\Omega \times D_u^2 \times D_v} \eta_{hk}(\mathbf{v}_*, \mathbf{v}^*, u_*, u^*) p_{hk}(\mathbf{x}, \mathbf{x}^*) \mu_{hk}^i(u_*, u^*) \times f_h(t, \mathbf{x}, \mathbf{v}, u_*) f_k(t, \mathbf{x}^*, \mathbf{v}^*, u^*) \, d\mathbf{v}^* \, du_* \, du^* \, d\mathbf{x}^*. 
Open Systems - Macro-actions

Modeling \textit{macroscopic actions applied by the outer environment} essentially means the identification of the term \( K_i = K_i(t, \mathbf{x}, \mathbf{u}) \) supposed to be a known function of its arguments. The action \( K_i \) acts over the variable \( \mathbf{u} \) for each functional subsystem. The resulting equation, for \( i = 1, \ldots, n \) is as follows:

\[
(\partial_t + \mathbf{v} \cdot \partial_x) f_i(t, \mathbf{x}, \mathbf{v}, \mathbf{u}) + \partial_u (K_i(t, \mathbf{x}, \mathbf{u}) f_i(t, \mathbf{x}, \mathbf{v}, \mathbf{u})) = J_i[f].
\]

Modeling \textit{external actions at the microscopic scale} means modeling the action of functional subsystems generated by the outer system delivered by the distribution functions:

\[
g_r(t, \mathbf{x}, w), \quad r = 1, \ldots, m, \quad w \in D_w = D_u,
\]

which depends on time, space and on a variable \( w \) modeling the activity of the outer functional subsystem.
Open Systems - Micro-actions

\[
(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}}) \; f_i(t, \mathbf{x}, \mathbf{v}, u) = J_i[f](t, \mathbf{x}, \mathbf{v}, u) + Q_i[f, g](t, \mathbf{x}, \mathbf{v}, u),
\]

\[
Q_i[f, g] = \sum_{r=1}^{m} C^e_{ir}[f, g](t, \mathbf{x}, \mathbf{v}, u) + \sum_{h=1}^{n} \sum_{r=1}^{m} S^e_{hr}(i)[f, g](t, \mathbf{x}, \mathbf{v}, u),
\]

\[
C^e_{ir} = \int_{\Omega \times D_u^2 \times D_v} \eta^e_{ir}(\mathbf{v}^*, \mathbf{v}^*, u^*, u^*) p^e_{ir}(\mathbf{x}, \mathbf{x}^*) B_{ir}(u^* \rightarrow u|u^*, u^*) \times C_{hk}(\mathbf{v}^* \rightarrow \mathbf{v}|\mathbf{v}^*, \mathbf{v}^*, u^*, u^*) \times f_i(t, \mathbf{x}, \mathbf{v}^*, u^*) g_r(t, \mathbf{x}, \mathbf{v}^*, \mathbf{v}^*, \mathbf{w}^*) \; d\mathbf{v}^* \; d\mathbf{v}^* \; du^* \; dw^* \; d\mathbf{x}^*,
\]

\[
S^e_{hr}(i) = \int_{\Omega \times D_u^2 \times D_v} \eta^e_{hr}(\mathbf{v}^*, \mathbf{v}^*, u^*, \mathbf{w}^*) p_{hr}(\mathbf{x}, \mathbf{x}^*) \mu^e_{hr}(i)(u^*, \mathbf{w}^*) \times f_h(t, \mathbf{x}, \mathbf{v}, u^*) g_r(t, \mathbf{x}, \mathbf{v}^*, \mathbf{w}^*) \; d\mathbf{v}^* \; du^* \; dw^* \; d\mathbf{x}^*,
\]
Derivation of Mathematical Structures - Open Systems

$\eta_{hk}^e$ models the encounter rates between the $k^{th}$ external action with state $w^*$ and the $h^{th}$ candidate particle with state $u_*$. 

$B_{ij}^e(u_* \rightarrow u|u_*, w^*)$ denotes the probability density that the candidate particle the $i^{th}$ functional subsystem with state $u_*$, $h$ falls into the state $u$ of the same functional subsystem due to interactions with the $j^{th}$ action with state $w^*$. 

$C_{ij}^e(v_* \rightarrow v|v_*, v^*, u_*, w^*)$ models the velocity dynamics, conditioned by the activity of the interacting pairs of the $i^{th}$ and $j^{th}$ functional subsystems respectively. 

$\mu_{hk}^e(i)(u_*, w^*; u)$ models the net proliferation into the $i^{th}$ population, due to interactions, which occur with rate $\eta_{hk}$, of the candidate particle of the population $h^{th}$ with state $u_*$ with the $k^{th}$ action with state $w^*$. 

Homogeneous activity Systems with Linear Interactions

The generalisation to a system of several interacting functional subsystems is immediate. It simply needs including interactions of active particles for a large system of $p$ functional subsystems labelled by the index $i = 1, \ldots, n$:

$$(\partial_t + \mathbf{v} \cdot \partial_x) f_i(t, \mathbf{x}, \mathbf{v}) = \sum_{j=1}^{n} \left( G_{ij}[f] - L_{ij}[F] \right) (t, \mathbf{x}, \mathbf{v}),$$

where

$$G_{ij}[f] = \int_{\mathcal{K}} \eta_{ij}(\mathbf{v}_*, \mathbf{v}^*, \mathbf{u}_*, \mathbf{u}^*) p_{ij}(\mathbf{x}, \mathbf{x}^*) A_{ij}(\mathbf{v}_* \rightarrow \mathbf{v} | \mathbf{v}, \mathbf{v}^*, f) \times f_i(t, \mathbf{x}, \mathbf{v}_*) f_j(t, \mathbf{x}^*, \mathbf{v}^*) \, d\mathbf{v}_* \, d\mathbf{v}^* \, d\mathbf{x}^*,$$

and

$$L_{ij}[f] = f_i(t, \mathbf{x}, \mathbf{v}) \int_{\mathcal{H}} \eta_{ij}(\mathbf{v}_*, \mathbf{v}^*, \mathbf{u}_*, \mathbf{u}^*) p_{ij}(\mathbf{x}, \mathbf{x}^*) f_j(t, \mathbf{x}^*, \mathbf{v}^*) \, d\mathbf{v}^* \, d\mathbf{x}^*.$$ 

where $\mathcal{K} = \Omega \times D_{\mathbf{v}} \times D_{\mathbf{v}}$ and $\mathcal{H} = \Omega \times D_{\mathbf{v}}$. 
\[
\begin{align*}
\partial_t f_i(t, u) + \mathcal{F}_i(t) \partial_u f_i(t, u) &= J_i[f](t, u) + Q_i[f](t, u) \\
&= \sum_{j=1}^{n} \int_{D_u \times D_u} \eta_{ij}(u_*, u^*) \mathcal{B}_{ij}(u_* \rightarrow u|u_*, u^*) f_i(t, u_*) f_j(t, u^*) \, du_* du^* \\
&- f_i(t, u) \sum_{j=1}^{n} \int_{D_u} \eta_{ij}(u, u^*) f_j(t, u^*) \, du^* \\
&+ \sum_{h=1}^{n} \sum_{k=1}^{n} \int_{D_u \times D_u} \eta_{hk}(u_*, u^*) \mu_{hk}^i(u, u^*) f_h(t, u_*) f_k(t, u^*) \, du_* du^* \\
&+ \sum_{r=1}^{m} \int_{D_u \times D_v} \eta_{ij}^e(u_*, v^*) \mathcal{C}_{ij}(u_* \rightarrow u|u_*, v^*) f_i(t, u_*) g_r(t, u^*) \, du_* dv^* \\
&- f_i(t, u) \sum_{r=1}^{m} \int_{D_v} \eta^e(u, v^*) g_r(t, v^*) \, dv^* \\
&+ \sum_{h=1}^{r} \sum_{r=1}^{m} \int_{D_u} \int_{D_u} \eta_{hk}^e(u_*, v^*) \mu_{hk}^e(i)(u_*, v^*; u) f_h(t, u_*) g_r(t, v^*) \, du_* dv^*.
\end{align*}
\]
Spatially homogeneous case

The modeling of multicellular systems and immune competition motivates to study the particular cases in which active particles have a velocity distribution $P = P(v)$ that is constant in time and that is not modified by interactions.

$$\frac{\partial}{\partial t} f(t, u) = \eta^0 \int_{D_u \times D_u} B(u_* \rightarrow u | u_*, u^*) f(t, u_*) f(t, u^*) \, du_* \, du^*$$
$$- \eta^0 f(t, u) \int_{D_u} [1 - \mu(u, u^*)] f(t, u^*) \, du^* ,$$

where

$$\eta^0 = \int \eta[\rho_0] w(x, x^*) P(v_*) P(v^*) \, dv_* \, dv^* \, dv_* \, dv^* \, dx^* .$$

where

$$\mu = \int_{\Omega} \mu(x, x^*, u, u^*) \, dx^* .$$
Lecture 4. - Nonlinear Interactions and Learning

Lecture 1. - Common Features of Complex Living Systems

Lecture 2. - Reducing Complexity and Representations

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Lecture 4. - Nonlinear Interactions and Learning

Derivation of Mathematical Tools to deal with Complex Systems: Nonlinear Interactions and Learning Dynamics
Sources of Nonlinearity

- Encounter rate related to *hiding and learning dynamics*;

- *Density distribution* conditioning the encounter rate;

- *Density distribution* conditioning the table of games;

- Nonlinearity induced by *topological distribution of the interacting entities*. 
Hiding Learning Dynamics - A Reference Mathematical Structure

\[
\partial_t f_i(t, u) = J_i[f](t, u) = \sum_{j=1}^{2} J_{ij}[f](t, u) \\
= \sum_{j=1}^{2} \int_{D_u \times D_u} \eta_{ij}(u_*, u^*) \mathcal{B}_{ij}(u_* \rightarrow u | u_*, u^*) \\
\times f_i(t, u_*) f_j(t, u^*) du_* du^* \\
- f_i(t, u) \sum_{j=1}^{2} \int_{D_u} \eta_{ij}(u, u^*) [1 - \mu_{ij}(u, u^*)] f_j(t, u^*) du^*.
\]

where: \( \eta_{ij}(u, u^*) \) and \( \mu_{ij}(u, u^*) \) the encounter rate and the proliferation rate related to the candidate active particle, with state \( u_* \), of the \( i \)-th functional subsystem and the field active particle, with state \( u^* \), of the \( j \)-th functional subsystem.
**Lecture 4. - Nonlinear Interactions and Learning**

**Hiding Learning Dynamics - A Reference Mathematical Structure**

$B_{ij}$ is the probability density that a candidate particle, with state $u_*$, of the $i$-th functional subsystem ends up into the state $u$ of the test particle of the same functional subsystem after the interaction with the field particle, with state $u^*$, of the $j$-th functional subsystem. $B_{ij}$ satisfies for all $i, j \in \{1, 2\}$, the following condition:

$$\int_{D_u} B_{ij}(u_* \rightarrow u|u_*, u^*) \, du = 1, \quad \forall u_*, u^* \in D_u.$$

Interactions, modeled by the terms $B_{ij}$, have been called *stochastic games* since the microscopic state of the active particles is known in probability and the output is identified by a probability density.

This class of equations is derived by assuming linearly additive interactions.
Modeling the H-L Dynamics

The mathematical structure can be specialized into a mathematical model when the interaction terms $\eta_{ij}$ and $B_{ij}$ are properly modeled according to a dynamics of hiding and learning actions. Specifically, let us consider the encounter rate in the following general form

$$\eta_{ij} = \eta_{ij}^0 e^{-c \beta_{ij}^2(t|f)},$$

where $c$ is a positive constant and $\beta_{ij}^2(t|f)$ models the distance between the two interacting particles. In details, the following cases can be considered:

- **I** $\beta_{ij}$ is equal to zero;
- **II** $\beta_{ij}$ is the distance between the microscopic states of the two interacting pairs;
- **III** $\beta_{ij}$ is the distance between the shapes of the distribution of the two interacting entities.
Modeling the H-L Dynamics

- **I** $\beta_{ij}$ is equal to zero:
  This corresponds to simplest approach consists in assuming that it is identified only by the interacting pairs, so that the encounter rate is a constant $\eta_{ij}^0$ for each pair of interacting functional subsystems.

- **II** $\beta_{ij}$ is the distance between the microscopic states of the two interacting pairs:
  This corresponds to suppose that it is given by the difference between the microscopic state of the interacting particles, and $\beta_{ij}$ can be related to the interaction rate as follows:

  \[
  \beta_{ij}(u_*, u^*) = \sqrt{(u_* - u^*)^2},
  \]

  Therefore

  \[
  \eta_{ij} = \eta_{ij}^0 e^{-c(u_* - u^*)^2}.
  \]
Modeling the H-L Dynamics

- **III** $\beta_{ij}$ is the distance between the shapes of the distribution of the two interacting entities:

This corresponds to suppose that it is given by the difference between the shapes of the interacting particles. More precisely distance $\beta_{ij}$ between the $i$-th and the $j$-th functional subsystem can be defined as follows:

$$\beta_{ij}^2(t|\mathbf{f}) = \int_{D_u} (f_i - f_j)^2(t, u) du,$$

Therefore

$$\eta_{ij} = \eta_{ij}^0 e^{-c\beta_{ij}^2(t|\mathbf{f})} = \eta_{ij}^0 e^{-c \int_{D_u} (f_i - f_j)^2(t, u) du}.$$

More in general, the distance $\beta_{ij}$ can be given by the norm of the difference between $f_i$ and $f_j$ in the appropriate metric space $\beta_{ij} = ||f_i - f_j||$. 
Hiding Learning Dynamics

A simple approach of modeling the terms $B_{ij}$ can be obtained by supposing that the output of the interaction is defined by the most probable value $m_{ij}$ and the variance $\sigma_{ij}$. In various cases it can be assumed that encounters within the same functional subsystem do not modify the state of the interacting pair:

- The candidate particle with activity $u_*$ does not shows any modification of its state related when it encounters a field particle of the same subsystem: $m_{11} = m_{22} = u_*$.

**Hiding:**
- The candidate particle of the first subsystem, with activity $u_*$, shows a trend to increase the distance when it encounters the candidate particle of the second subsystem:

$$m_{12} = \begin{cases} 
  u_* - \varepsilon_1 (u^* - u_*), & \text{if } u_* > u^* \\
  u_* + \varepsilon_1 (u^* - u_*), & \text{if } u_* \leq u^*
\end{cases}$$
Hiding Learning Dynamics - *Learning*

- The candidate particle, with activity $u_*$, of the second subsystem shows a trend to reduce the distance with respect to the state of the candidate particle of the first subsystem:

$$m_{21} = \begin{cases} 
    u_* - \varepsilon_2 (u^* - u_*), & \text{if } u_* < u^* \\
    u_* + \varepsilon_2 (u^* - u_*), & \text{if } u_* \geq u^* 
\end{cases}$$

The dynamics is given by the following equation:

$$\partial_t f_i(t, u) = J_i[f](t, u) = \sum_{j=1}^{2} J_{ij}[f](t, u)$$

$$= \sum_{j=1}^{2} \int_{D_u \times D_u} \eta_{ij}^0 e^{-c \beta_{ij}^2(t|\mathbf{f})} B_{ij}(u_* \rightarrow u|u_*, u^*) f_i(t, u_*) f_j(t, u^*) du_* du^*$$

$$- f_i(t, u) \sum_{j=1}^{2} \int_{D_u} \eta_{ij}^0 e^{-c \beta_{ij}^2(t|\mathbf{f})} [1 - \mu_{ij}(u, u^*)] f_j(t, u^*) du^*.$$
Hiding Learning Dynamics - Evolution of the distance

The time evolution of the distance between the two system is obtained by solving the initial value problem with suitable initial conditions and subsequently by computing the time evolution of the distance $\beta_{ij}$. Technical calculations, under suitable differentiability assumptions, yield:

$$
\partial_t \beta_{ij}(t, u) = Q_{ij}[f](t, u) = \frac{\int_{D_u} (J_i[f] - J_j[f])^2(t, u) \, du}{\sqrt{\int_{D_u} (f_i - f_j)^2(t, u) \, du}}.
$$
Hiding Learning Dynamics with Variable Number of Particles

The mathematical models we have seen until now are limited to the case of systems with constant number of particles. However, a consequence of the H-L dynamics is that proliferative and/or destructive events may be generated by the competition between the functions subsystems.

\[
\partial_t f_i(t, u) = N_i[f](t, u) = \sum_{j=1}^{2} N_{ij}[f](t, u)
\]

\[
= \sum_{j=1}^{2} \int_{D_u \times D_u} \eta_{ij}^0 e^{-c \beta_{ij}^2(t|f)} B_{ij}(u_* \rightarrow u|u_*, u^*)
\]

\[
\times f_i(t, u_*) f_j(t, u^*) \, du_* du^*
\]

\[
- f_i(t, u) \sum_{j=1}^{2} \int_{D_u} \eta_{ij}^0 e^{-c \beta_{ij}^2(t|f)} [1 - \mu_{ij}(u, u^*)] f_j(t, u^*) \, du^*.
\]

A conceivable modeling of the terms \(\mu_{ij}\) should preliminarily consider their sign. For instance \(\mu_{11} = \mu_{22}\) in absence of recruitment, while \(\mu_{12} \leq 0\) due to the output of chasing; and \(\mu_{21} = 0\).
Hiding Learning Dynamics with Generation of New Functional Subsystems
Consider now the case of generation new functional subsystems with characteristics that reduce the learning ability:

\[
\partial_t f_i(t, u) = G_i[f](t, u) = \sum_{h=1}^{p} \sum_{k=1}^{p} G_{hk}[f](t, u)
\]

\[
= \sum_{h=1}^{p} \sum_{k=1}^{p} \int_{D_u \times D_u} \eta_{hk}^0 e^{-c \beta_{hk}^2(t|f)} B_{hk}^i (u_* \rightarrow u|u_*, u^*) \times f_h(t, u_*) f_k(t, u^*) \, du_* \, du^* 
\]

\[
- \sum_{h=1}^{p} \sum_{k=1}^{p} \int_{D_u} \eta_{hk}^0 e^{-c \beta_{hk}^2(t|f)} [1 - \mu_{hk}^i (u_*, u^*)] f_h(t, u_*) f_k(t, u^*) \, du_* \, du^*,
\]

\(B_{hk}^i\) is the probability density that a \(h\)-th candidate particle, with state \(u_*\) ends up into the state \(u\) of the \(i\)-th f.s. after the interaction with the \(k\)-th field particle, with state \(u^*\). Moreover, \(\mu_{hk}^i\) is the proliferative/destructive rate in the \(i\)-th f.s. of the test particle, due to the encounter between the \(h\)-th candidate particle, with state \(u_*\), of the \(k\)-th particle (field), with state \(u^*\).
Nonlinear Interactions

- Nonlinear interactions means that $B_{ij}$ depend on the distribution function of the interaction particles. For instance by moments of such distribution.

\[
\partial_t f_i(t, u) = J_i[f](t, u) = \sum_{j=1}^{n} J_{ij}[f_i, f_j](t, u)
\]

\[
= \sum_{j=1}^{n} \int_{D_u \times D_u} \eta_{ij}(t|f_i, f_j) B_{ij}(u_* \rightarrow u|u_*, u^*, \mathbb{E}^P[f_i], \mathbb{E}^P[f_j])
\times f_i(t, u_*) f_j(t, u^*) \, du_* \, du^*
\]

\[-f_i(t, u) \sum_{j=1}^{n} \int_{D_u} \eta_{ij}(t|f_i, f_j) f_j(t, u^*) \, du^*,
\]

The modeling of the terms $B_{ij}$, for $i, j \in \{1, 2, \ldots, n\}$, is based on the idea that the candidate particle interacts with the field particles within its space interaction domain and feels an action identified by the low order moments of the field active particles. The action can be also related, whether consistent with the specific system under consideration, to their most probable value.
Nonlinear Interactions

In general the interaction domain of the candidate particle with state $u_*$ may not be the whole domain $D_u$ but a subset $\Omega_{u_*} \subseteq D_u$, which contains the field particles $u^* \in \Omega_{u_*}$ that are able to interact with candidate particle. Thus interactions only occur if the distance, in the space of microscopic states of the interacting particles, are sufficiently small. Therefore a positive function $\omega(u_*, u^*)$, normalized with respect to integration over $u^*$, is introduced to take into account such dynamics. This function, which weight the interactions among the active particles, is assumed to have a compact support in the domain of influence $\Omega_{u_*} \subseteq D_u$ of the interactions. Moreover:

$$
\int_{D_u} \omega(u_*, u^*) \, du^* = \int_{\Omega_{u_*}} \omega(u_*, u^*) \, du^* = 1.
$$

Accordingly we define the $p$th order weighted moment as follows:

$$
E^p_w[f_i](t, u_*) = \int_{D_u} (u^*)^p \omega(u_*, u^*) \, f_i(t, u^*) \, du^* = \int_{\Omega_{u_*}} (u^*)^p \omega(u_*, u^*) \, f_i(t, u^*) \, du^*.
$$
Nonlinear Interactions

The mathematical structure is as follow:

\[ \partial_t f_i(t, u) = J_i[f](t, u) = \sum_{j=1}^{n} J_{ij}[f_i, f_j](t, u) \]

\[ = \sum_{j=1}^{n} \int_{D_u \times D_u} \eta_{ij}(t|f_i, f_j) B_{ij}(u_* \rightarrow u|u_*, u^*, \mathbb{E}_w^p[f_i], \mathbb{E}_w^p[f_j]) \]

\[ \times f_i(t, u_*) f_j(t, u^*) \ du_* \ du^* \]

\[ - f_i(t, u) \sum_{j=1}^{n} \int_{D_u} \eta_{ij}(t|f_i, f_j) f_j(t, u^*) \ du^*. \]

The modeling of the terms \( B_{ij} \), for \( i, j \in \{1, 2, \ldots, n\} \), is based on the idea that the candidate particle interacts with the field particles within its interaction domain, defined by \( \Omega_{u^*} \), of the space of the activity variables and feels an action identified by the low order moments of the field active particles.
Nonlinear Interactions

An immediate technical generalization consists in inserting the modeling of proliferative and/or destructive events as well as transition from one functional subsystem to the other. The following mathematical framework is obtained:

\[
\partial_t f_i(t, u) = Q_i[f](t, u) = \sum_{h=1}^{n} \sum_{k=1}^{n} Q_{hk}[f_h, f_k](t, u)
\]

\[
= \sum_{h=1}^{n} \sum_{k=1}^{n} \int_{D_u \times D_u} \eta_{hk}^0 e^{-c \frac{\alpha_{hk}^2}{\lambda_{hk}}(t|f_h, f_k)} B_{hk}(u_* \rightarrow u | u_*, u^*, \mathbb{E}_w[f_h], \mathbb{E}_w[f_k])
\]

\[
\times f_h(t, u_*) f_k(t, u^*) \, du_* \, du^*
\]

\[
- \sum_{h=1}^{n} \sum_{k=1}^{n} \int_{D_u} \eta_{hk}^0 e^{-c \frac{\alpha_{hk}^2}{\lambda_{hk}}(t|f_h, f_k)} [1 - \mu_{hk}(u_*, u^*)] f_h(t, u_*) f_k(t, u^*) \, du_* \, du^*.
\]
where the number of particles is not any more constant in time due to the term that models proliferative and/or destructive events:

- $\mu_{h,k}^i$ models the proliferative/destructive rate of particles of the $hth$ functional subsystem, with state $u_*$, into the state $u$ of the $ith$ functional subsystem due to the encounter with the particle (field) of the $kth$ functional subsystem, with state $u^*$. In particular, destructive events occur only within the functional subsystem of the field particles.

Moreover, conservative interactions include transition from one functional subsystem to the other. This dynamics is modeled by the following term:

- $B_{h,k}^i$ models the probability density that a candidate particle of the $hth$ functional subsystem, and with state $u_*$, ends up into the state $u$ of the $ith$ functional subsystem after the interaction with the field particle, with state $u^*$, of the $kth$ functional subsystem.
Nonlinear Interaction - Open Problems

- Functional subsystems on a network;

- Including space phenomena;

- Modeling flocking problems;

- Including learning-hiding phenomena in specific models until now derived by assuming linear interactions.
Ten Lectures on Mathematical Modelling of Complex Living Systems - Part II - Real Systems

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Bibliography

Review Papers and Books Traffic


**FIVE KEY CHARACTERISTICS OF VEHICULAR DYNAMICS AS A LIVING SYSTEM**

**I NONLINEAR INTERACTIONS:** Vehicles on roads should be regarded as a complex living system, which interacts in a nonlinear manner. Interactions follow specific strategies generated both by the ability to communicate with the other entities and to organize the dynamics according to their own strategy and interpretation of that of the others. Nonlinearity means that the action over the driver-vehicle (or pedestrian) from the surrounding ones is not the addition of all individual actions. Nonlinearly depends on their state and localization.

**II HETEROGENEOUS EXPRESSION OF STRATEGIC ABILITY:** The individual behavior of the driver-vehicle system, regarded as a micro-system, is heterogeneously distributed. The shape of the distribution has an influence over the strategy developed in the interactions, which can modify this distribution.
III Granular dynamics: The dynamics shows the behavior of granular matter with aggregation and vacuum phenomena. Namely, the continuity assumption of the distribution function in kinetic theory cannot be straightforwardly taken.

IV Influence of the environmental conditions: The dynamics is remarkably affected by the quality of environment, including weather conditions and quality of the road or of the domain of pedestrians. Therefore, the models should include parameters to be tuned with respect to the variability of these conditions.

V Parameters: The modeling approach needs to include parameters suitable to describe some essential characteristics of the system. These parameters should be related to specific observable phenomena and their identification should be pursued either by using existing experimental data or by experiments to be properly designed.
Scaling - Vehicular Traffic

- **Microscopic modeling:** Dynamics of each single vehicle under the action of the surrounding vehicles. The solution of a large system of ordinary differential equations can provide the description of vehicular flow conditions.

- **Macroscopic description:** Evolution equations for the mass density and linear momentum regarded as macroscopic observable of the flow assumed to be continuous. Mathematical models are stated in terms of nonlinear PDEs derived on the basis of conservation equations closed by phenomenological models.

- **Statistical description:** Evolution equation for the statistical distribution function on the position and velocity of the vehicle along the road. This type of modeling is based on vehicle interactions modeled at the microscopic scale. The *kinetic theory for active particles* can be used.
Daganzo’s Critical Analysis - Vehicular Traffic


– *Shock waves and particle flows in fluid dynamics refer to thousands of particles, while only a few vehicles are involved by traffic jams*;

– *A vehicle is not a particle but a system linking driver and mechanics, so that one has to take into account the reaction of the driver, who may be aggressive, timid, prompt etc. This criticism also applies to kinetic type models*;

– *Increasing the complexity of the model increases the number of parameters to be identified.*
**Derivation of Mathematical Structures**

The derivation of the evolution equation for the $f_i$s is obtained by a balance for net flow of active particles in the elementary volume of the space of the microscopic state by transport and interactions. The following particles are involved in the interactions:

- **Test** particles with microscopic state $(x, v, u)$, at the time $t$, and distribution function is $f = f(t, x, v, u)$.

- **Field** particles with microscopic state $(x^*, v^*, u^*)$, at the time $t$, and distribution function is $f^* = f(t, x^*, v^*, u^*)$.

- **Candidate** particles with microscopic state $(x_*, v_*, u_*)$, and distribution function is $f_* = f(t, x_*, v_*, u_*)$. 
Interactions

**H.5.1.** The candidate and test particles in $\mathbf{x}$, with state $\mathbf{v}_*, \mathbf{u}_*$ and $\mathbf{v}, \mathbf{u}$, respectively, interact with the field particles in $\mathbf{x}^*$, with state $\mathbf{v}^*, \mathbf{u}^*$ located in its interaction domain $\Omega$, $\mathbf{x}^* \in \Omega$.

**H.5.2.** Interactions are weighted by a suitable term $\eta_{h,k}[\rho](\mathbf{x}^*)$, that can be interpreted as an *interaction rate*, which depends on the local density in the position of the field particles.
**H.5.3.** The distance and topological distribution of the intensity of the interactions is weighted by a function $p_{hk}(x, x^*)$ such that: $\int_{\Omega} p_{hk}(x, x^*) \, dx^* = 1$.

**H.5.4.** The candidate particle modifies its state according to the probability density $A$ defined as follows:

$$A_{hk}(v_*, v; u_*, u|v_*, v^*, u_*, u^*)$$

where $A$ denotes the probability density that a candidate particles with state $v_*, u_*$ reaches the state $v, u$ after an interaction with the filed particles with state $v^*, u^*$, while the test particle looses its state $v$ and $u$ after interactions with field particles with velocity $v^*$ and activity $u^*$. 
Lecture 5 - Vehicular Traffic Dynamics

The following factorization:

\[ A_{hk}(\cdot) = B_{hk}(u_\star \to u, |u_\star, u_\star^*) \times C_{hk}(v_\star \to v | v_\star, v_\star^*, u_\star, u_\star^*) , \]

can be used in a variety of applications.

\( A, B, \) and \( C \) are, for positive defined \( f \), probability densities:

\[
\int_{D_v \times D_u} A_{hk}(v_\star \to v, u_\star \to u | v_\star, v_\star^*, u_\star, u_\star^*) \, dv \, du = 1, \quad \forall \, v_\star, v_\star^*, u_\star, u_\star^* .
\]

and

\[
\int_{D_u} B_{hk}(u_\star \to u, |u_\star, u_\star^*) \, du = \int_{D_v} C_{hk}(v_\star \to v | v_\star, v_\star^*, u_\star, u_\star^*) \, dv = 1 .
\]

The above structures characterize linearly additive interactions. Nonlinear interactions occur when the argument include some dependance on the distribution function. For instance by moments.
\( \left( \partial_t + \mathbf{v} \cdot \partial_x \right) f_i(t, x, \mathbf{v}, u) \)

\[
= \sum_{j=1}^{n} \left( G_{ij}[\mathbf{f}] - L_{ij}[\mathbf{f}] \right)(t, x, \mathbf{v}, u)
\]

\[
= \int_{\Lambda} \eta_{ij}[\rho_j](t, \mathbf{x}^*) p_{ij}(x, \mathbf{x}^*) B_{ij}(u_* \rightarrow u|u_*, u^*) C_{hk}(\mathbf{v}_* \rightarrow \mathbf{v}|\mathbf{v}_*, \mathbf{v}^*, u_*, u^*)
\times f_i(t, x, \mathbf{v}_*, u_*) f_j(t, \mathbf{x}^*, \mathbf{v}_*, u^*) d\mathbf{v}_* d\mathbf{v}^* du_* du^* dx^*,
\]

\[- f_i(t, x, \mathbf{v}) \int_{\Gamma} \eta_{ij}[\rho_j](t, \mathbf{x}^*) p_{ij}(x, \mathbf{x}^*) f_j(t, \mathbf{x}^*, \mathbf{v}_*, u^*) d\mathbf{v}_* du^* dx^*, \]

where \( \Lambda = \Omega \times D_v^2 \times D_u^2 \), and \( \Gamma = \Omega \times D_v \times D_u \).
Reference Physical Quantities - Vehicular Traffic

- \( n_M \) is the maximum density of vehicles corresponding to bumper-to-bumper traffic jam;

- \( V_M \) is the maximum admissible mean velocity which can be reached, in average, by vehicles running in free flow conditions, while a fast isolated vehicle can reach velocities larger that \( V_M \). Specifically, a limit velocity can be defined as follows: \( V_\ell = (1 + \mu) V_M \), \( \mu > 0 \), such that no vehicle can reach, simply by mechanical reasons, a velocity larger than \( V_\ell \). Moreover, it is convenient, identifying the critical time \( T_c \) as the ratio between \( \ell \) and \( V_M \).

- \( t \) is the dimensionless time variable obtained referring the real time \( t_r \) to a suitable critical time \( T_c = \ell / V_M \).

- \( x \) is the dimensionless space variable obtained dividing the real space \( x_r \) by the length \( \ell \) of the lane.
**Microscopic Representation**

The state of the whole system is defined by dimensionless position and velocity of the vehicles regarded as points:

\[ x_i = x_i(t), \quad v_i = v_i(t), \quad i = 1, \ldots, N, \]

where the subscript refers to each vehicle, while \( x_i \in [0, 1] \) and \( v_i \in [0, 1 + \mu] \) are dimensionless variables being referred to \( \ell \) and \( V_M \) respectively.

Suitable averaging processes provide gross quantities such as density and mass velocity. Averaging can be either at fixed time over a certain space domain or at fixed space over a certain time range. For instance the number density is given by the number of vehicles \( n(t) \), which at the time \( t \) are found in the tract \([x - \Delta, x + \Delta]\):

\[ \rho(t; x) \approx \frac{1}{2\Delta} \frac{n(t; x)}{n_M}, \]
Kinetic Theory Representation

The state of the whole system is defined by the statistical distribution of position and velocity of the vehicles.

\[ f = f(t, x, v) : \mathbb{R}_+ \times [0, 1] \times [0, 1 + \mu] \rightarrow \mathbb{R}_+ , \]

where \( f(t, x, v)dx dv \) is normalized with respect to \( n_M \) and gives the number of vehicles which, at the time \( t \), are in the phase space domain \( [x, x + dx] \times [v, v + dv] \).

Macroscopic observable quantities can be obtained, under suitable integrability assumptions, by moments of the above distribution function. In particular, the \textbf{dimensionless local density} and the \textbf{total number of vehicles} are given by

\[ \rho(t, x) = \int_0^{1+\mu} f(t, x, v) dv , \quad N(t) = \int_0^1 \int_0^{1+\mu} f(t, x, v) dv dx . \]

\[ q(t, x) = \int_0^{1+\mu} v f(t, x, v) dv . \]
Critical Analysis

- The system is with finite degrees of freedom. However, microscopic models induce considerable errors in the computation of macroscopic quantities.

- The flow is not continuous, hence hydrodynamic models should not be used. Moreover, it is difficult to evaluate the entity of the approximation induced by macro-models.

- The number of individual entities is not large enough to allow the use of continuous distribution functions within the framework of the mathematical kinetic theory. Moreover, interactions are not localized, considering that drivers adapt the dynamics of the vehicle to the flow conditions in the visibility zone.

- Individual entities cannot be regarded as classical particles, but as active particles due to their ability to modify their dynamics according to specific strategies.

- Increasing the complexity of the model increases the number of parameters to be identified.
**Empirical data:** Empirical data can be, or ought to be, used to validate mathematical models. On the other hand, the difficulty to obtain these data, due to the great variation of the environment and of the individuals belonging to the system, reduces the amount of available data useful to validate theoretical models. Three types of empirical data can be classified:

*Quantitative data:* which provide accurate data of macroscopic quantities such as number density, mean velocity, and/or flow. These data should be used to verify if the model has the ability to reproduce quantitatively them.

*Qualitative data on emerging behaviors:* corresponding to specific conditions, for instance in vehicular traffic interactions of clusters of fast and slow vehicles, bottlenecks, panic conditions. One should verify if the model has the ability to reproduce qualitatively them.
The averaging process which leads to macroscopic quantities introduces unavoidable fluctuations due not only to measurement errors, but also to the stochastic and granular essence of the flow, where deceleration and acceleration of vehicles are observed even in almost steady flow *stop and go dynamics*;

- Generally, experimental results refer to *steady state conditions*, while rarely traffic reaches a steady state;

- Empirical data are *results very sensitive to the quality of environmental conditions*. Therefore, it is impossible identifying a unique deterministic representation.

- Kerner remarks that additional transitions can be observed when the flow is congested, for instance related to a considerable dispersion of data.

Empirical data provide macroscopic quantities, while the dynamics is ruled at the microscopic scale;
Discrete kinetic theory representation

Let us consider **discrete velocity models**, where the velocity variable belongs to the following set:

\[ I_v = \{ v_1 = 0, \ldots, v_i, \ldots, v_n = 1 \} . \]

The corresponding discrete representation is obtained by linking the discrete distribution functions to each \( v_i \):

\[ f_i = f_i(t, x) : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+ , \]

for \( i = 1, \ldots, n \).

**Discretization of the velocity domain is not a mere mathematical procedure, but it represents a possible manner to take into account the strongly granular nature of traffic.**
Macroscopic Quantities

The following gross quantities are obtained by weighted sums:

\[ \rho(t, x) = \sum_{i=1}^{n} f_i(t, x), \]

\[ q(t, x) = \sum_{i=1}^{n} v_i f_i(t, x), \quad q(t, x) = \xi(t, x) \rho(t, x) \]

\[ \sigma(t, x) = \frac{1}{u(t, x)} \sum_{i=1}^{n} (v_i - \xi(t, x))^2 f_i(t, x), \]

\[ H(t, x) = \sum_{i=1}^{n} f_i(t, x) \log f_i(t, x). \]
The structure to generate specific models is as follows:

$$\partial_t f_i + v_i \partial_x f_i = \sum_{h,k=1}^{n} \int_{x}^{x+\xi} \eta[\rho](t, y) A_{hk}[\rho; \alpha] f_h(t, x) f_k(t, y) w(x, y) \, dy$$

$$- f_i(t, x) \sum_{h=1}^{n} \int_{x}^{x+\xi} \eta[\rho](t, y) f_h(t, y) w(x, y) \, dy,$$

where:

$$A_{hk}[\rho; \alpha] \geq 0, \quad \sum_{i=1}^{n} A_{hk}[\rho] = 1, \quad \forall h, k = 1, \ldots, n, \quad \rho(t, x) \in [0, 1).$$

$$w(x, y) \geq 0, \quad \int_{x}^{x+\xi} w(x, y) \, dy = 1,$$

while $A_{hk}$ denotes the probability density that a candidate particle with velocity $v_h$ falls into the state $v_i$ after an encounter with a field particle with velocity $v_k$. 

(1) A parameter $\alpha \in [0, 1]$ is related to the quality of the road:

$$\alpha = \frac{\rho_c}{\rho_{cM}}, \quad \alpha \in [0, 1],$$

where $\rho_{cM}$ is the maximal admissible value corresponding to the best highway in optimal environmental conditions.

(2) The velocity is discretized by coupling of a steady grid when the density $\rho$ is less than the critical one $\rho_c$ and with an adaptive grid when $\rho$ is greater than $\rho_c$:

$$v_i = \begin{cases} 2 \frac{i - 1}{n - 1} & \text{if } 0 \leq \rho \leq \rho_c \\ (1 + \alpha) (1 - \rho^3)^2 \frac{i - 1}{n - 1} & \text{if } \rho_c < \rho \leq 1 \end{cases} \quad \text{for } i \in \{1, \ldots, n\}$$

(3) The encounter rate $n_{hk}$, for $h, k \in \{1, \ldots, n\}$, is assumed as follows:

$$\eta_{hk} = |v_h - v_k|.$$
Assumptions on The Table of the Games $A_{hk}^i$

$\epsilon_{hk} = \alpha \frac{(1-\rho)}{|h-k|^2}$, it is assumed:

- The trend of the candidate particle to change its velocity decreases in probability when the “distance” $|v_h - v_k|$ between the velocity of the candidate and field vehicle decreases.

- The candidate vehicle $h$ can attain only a velocity $v_i \in \{v_h, v_{h-1}, \ldots, v_k\}$ if it interacts with a field vehicle $k$ such that $v_h > v_k$ while if the candidate vehicle interacts with a field vehicle such that $v_h < v_k$ it can only attain a velocity $v_i \in \{v_h, v_{h+1}, \ldots, v_k\}$.

- If the candidate and field vehicles have the same velocity, the interaction does not imply a change in velocity.

- When a candidate vehicle, with velocity $v_h$, encounters a (faster field vehicle), $v_k > v_h$, the candidate vehicle has a trend to increase its velocity from $h$ to $k$.

- When a candidate vehicle, with velocity $v_h$, encounters a (slower field vehicle), $v_k < v_h$, the candidate vehicle has a trend to decrease its velocity from $h$ to $k$. 
### The Table of Games

<table>
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<tr>
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</table>
Figure 1: Fundamental diagrams for the average velocity (left panel) and the macroscopic flux (right panel) versus the macroscopic density, for $n = 7$, $\rho_{cM} = 0.2$, $\alpha = 1$. 
Figure 2: Fundamental diagram for the average velocity versus and the macroscopic flux for $n = 7$, $\rho_{cM} = 0.2$, $\alpha = 0.6$ (left panel). The jump of the velocity versus $\alpha$ for $n = 7$, $\rho_{cM} = 0.2$ (right panel).
Lecture 5. - Application: Vehicular Traffic

Lecture 6. - Application: Crowd Dynamics and Swarms

This Lecture develops an analogous approach to modeling crowd dynamics looking at the beautiful shapes of swarms

Lecture 7. - Looking Ahead for a Mathematical Theory
Some characteristics of crowd dynamics are similar to that of vehicular traffic. For instance:

- The system is with finite degrees of freedom. However, microscopic models induce considerable errors in the computation of macroscopic quantities.

- The flow is not continuous, hence hydrodynamic models should not be used. Moreover, it is difficult to evaluate the entity of the approximation induced by macro-models.

- The number of individual entities is not large enough to allow the use of continuous distribution functions within the framework of the mathematical kinetic theory. Moreover, interactions are not localized, considering that drivers adapt the dynamics of the vehicle to the flow conditions in the visibility zone.

- Individual entities cannot be regarded as classical particles, but as active particles due to their ability to modify their dynamics according to specific strategies.
However additional technical differences can be indicated:

**Pedestrians are non isotropic particles**
Pedestrians have a target
Dynamics in the presence of obstacles
Dynamics in the presence of obstacles - The Jamarat Bridge
Validation of models by empirical data presents technical difficulties analogous to that of traffic:

**Empirical data**: Empirical data can be, or ought to be, used to validate mathematical models. On the other hand, the difficulty to obtain these data, due to the great variation of the environment and of the individuals belonging to the system, reduces the amount of available data useful to validate theoretical models. Three types of empirical data can be classified:

- **Quantitative data**: which provide accurate data of macroscopic quantities such as number density, mean velocity, and/or flow. These data should be used to verify if the model has the ability to reproduce quantitatively them.

- **Qualitative data on emerging behaviors**: corresponding to specific conditions, for instance in vehicular traffic interactions of clusters of fast and slow vehicles, bottlenecks, panic conditions. One should verify if the model has the ability to reproduce qualitatively them.
Mathematical Structures

\[ f = f(t, x, v) \delta(u - u_0) : \mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}_+ , \quad p \geq 2. \]

\[ \partial_t f(t, x, v) + v \cdot \partial_x f(t, x, v) = J[f](t, x, v) \]

\[ = \int \eta[\rho](t, z) w(x, z) A(v_\ast \rightarrow v|v_\ast, v^\ast, \rho) f(t, x_\ast, v_\ast) f(t, z, v^\ast) dv_\ast dv^\ast dz \]

\[ -f(t, x, v) \int_{\Gamma \times D_u} \eta[\rho](t, z) w(x, z) f(t, z, v^\ast) dv^\ast dz , \]
Particularizations and Generalizations

- \( f = f(t, x, v, \theta, u) : \mathbb{R}_+ \times \mathbb{R}^p \times [1 + \mu] \times [0, 2\pi] \times [0, 1] \rightarrow \mathbb{R}_+ \), General case with velocity in plane polar coordinates

- \( f = f(t, x, v, \theta) \delta(u - u_0) : \mathbb{R}_+ \times \mathbb{R}^p \times [1 + \mu] \times [0, 2\pi] \rightarrow \mathbb{R}_+ \), Velocity in plane polar coordinates

- \( f = f(t, x, \theta) \delta(v - v_0) \delta(u - u_0) : \mathbb{R}_+ \times \mathbb{R}^p \times [0, 2\pi] \rightarrow \mathbb{R}_+ \), Velocity in plane polar coordinates with one velocity modulus

- \( f = f(t, x, \theta) \delta(u - u_0) \sum_{i=1}^{n} \delta(u - u_0) : \mathbb{R}_+ \times \mathbb{R}^p \times [0, 2\pi] \rightarrow \mathbb{R}_+ \), Velocity in plane polar coordinates with discrete velocity modules
Table of games to change direction

- **Interaction with a upper direction and target:** It is assumed that both interactions contribute to an anticlockwise rotation:

\[
B^j_{hk} [\rho_i] = \alpha (1 - \rho_i) (\varepsilon_1 + \varepsilon_2) \quad \text{if} \quad \theta_k > \theta_h; \quad \hat{\theta}_i > \theta_h; \quad j = h + 1,
\]

\[
B^j_{hk} [\rho_i] = \alpha (1 - \rho_i) (1 - \varepsilon_1 - \varepsilon_2) \quad \text{if} \quad \theta_k > \theta_h; \quad \hat{\theta}_i > \theta_h; \quad j = h,
\]

\[
B^j_{hk} [\rho_i] = 0 \quad \text{otherwise}.
\]

- **Interaction with a upper direction and a lower target:** It is assumed that the two interactions contribute, respectively to an anticlockwise and a clockwise rotation:

\[
B^j_{hk} [\rho_i] = \alpha (1 - \rho_i) \varepsilon_1 \quad \text{if} \quad \theta_k > \theta_h; \quad \hat{\theta}_i < \theta_h; \quad j = h + 1,
\]

\[
B^j_{hk} [\rho_i] = \alpha (1 - \rho_i) (1 - \varepsilon_1 - \varepsilon_2) \quad \text{if} \quad \theta_k > \theta_h; \quad \hat{\theta}_i < \theta_h; \quad j = h,
\]

\[
B^j_{hk} [\rho_i] = 1 - \alpha (1 - \rho_i) \varepsilon_2 \quad \text{if} \quad \theta_k > \theta_h; \quad \hat{\theta}_i < \theta_h; \quad j = h - 1,
\]

\[
B^j_{hk} [\rho_i] = 0 \quad \text{otherwise}.
\]
• Interaction with a lower direction and an upper target: It is assumed that the two interactions contribute, respectively, to a clockwise and an anticlockwise rotation:

\[ B^j_{hk}[\rho_i] = \alpha (1 - \rho_i) \varepsilon_2 \quad \text{if} \quad \theta_k < \theta_h; \quad \hat{\theta}_i > \theta_h; \quad j = h + 1, \]

\[ B^j_{hk}[\rho_i] = \alpha (1 - \rho_i) (1 - \varepsilon_1 - \varepsilon_2) \quad \text{if} \quad \theta_k < \theta_h; \quad \hat{\theta}_i < \theta_h; \quad j = h, \]

\[ B^j_{hk}[\rho_i] = 1 - \alpha (1 - \rho_i) \varepsilon_1 \quad \text{if} \quad \theta_k < \theta_h; \quad \hat{\theta}_i > \theta_h; \quad j = h - 1, \]

\[ B^j_{hk}[\rho_i] = 0 \quad \text{otherwise}. \]

• Interaction with a upper direction and target: It is assumed that both interactions contribute to a clockwise rotation:

\[ B^j_{hk}[\rho_i] = \alpha (1 - \rho_i) (\varepsilon_1 + \varepsilon_2) \quad \text{if} \quad \theta_k < \theta_h; \quad \hat{\theta}_i < \theta_h; \quad j = h - 1, \]

\[ B^j_{hk}[\rho_i] = \alpha (1 - \rho_i) (1 - \varepsilon_1 - \varepsilon_2) \quad \text{if} \quad \theta_k < \theta_h; \quad \hat{\theta}_i < \theta_h; \quad j = h, \]

\[ B^j_{hk}[\rho_i] = 0 \quad \text{otherwise}. \]

where \( B^j_{hk}[\rho_i] \) stands for \( B(\theta_h \rightarrow \theta_j|\theta_h, \theta_k, \rho_i(t)) \) and \( \alpha \) is a parameter that models the quality of the environment.
Lecture 6 - Crowds and Swarms

Density vs. space - Jamarat Bridge

[Graph showing density distribution along a bridge]

Ten Lectures on Mathematical Modelling of Complex Living Systems - Part II - Real Systems – p. 41/60
Looking at the beautiful shapes of swarms

6.1. Interactions between active particles of a swarm are in three space coordinates, while those of particles of a crowd are defined over two-space coordinates.

6.2. Mathematical problems are stated in unbounded domains with initial conditions with compact support. The solution of problems should provide the evolution in time of the domain of the initial conditions.

6.3. The swarm has the ability to express a common strategy, which is a nonlinear elaboration of all individual contributions, generated by each individuals based on the microscopic state of all other individuals. In general a swarm has the ability to express a collective intelligence that is generated by a cooperative strategy.
Time evolution of the swarm’s domain

$D_{t_0}$

$D_{t_1}$

$D_{t_2}$
Looking at the beautiful shapes of swarms

- 6.4. The insight on emerging strategies needs to be specifically referred to the type of individuals composing the swarm, and the specific applications under consideration. Heterogeneity of individual behaviors plays an important role in the dynamics of swarms of cells in biology, where several complex events, such as proliferative/destructive events or mutations, may arise in short time intervals. In the animal world heterogeneity means, in several cases, hierarchy.

- 6.5. The strategy includes a clustering ability (flocking) that prevents the fragmentation of the moving domain of the swarm. Moreover, when a fragmentation of occurs, the clustering ability attempts to induce an aggregation.
• **6.6.** The dynamics of interactions differs in the various zones of the swarm. For instance, from the border to the center of the swarm’s domain. Stochastic behaviors are an essential characteristics of the dynamics.

• **6.7.** Recent studies conjecture, on the basis of empirical data, that some systems of animal world develop a common strategy based on interactions depending on topological rather than metric distances. This definitely is a valuable suggestion to be used towards modeling.

• **6.8.** The concept of swarm can be extended to various types of micro organisms and ultimately to cells in a multicellular system. In this case the strategy expressed by the interacting entities depends on the biological functions that characterize the population. Moreover, the modeling approach should include proliferative and/or destructive events.
Lecture 5. - Application: Vehicular Traffic

Lecture 6. - Application: Crowd Dynamics and Swarms

Lecture 7. - Looking Ahead for a Mathematical Theory

A critical analysis - questions and answers looking at research perspectives
The mathematical structures that have been derived for linearly additive interactions and also for some cases of nonlinear interactions offer the framework for the derivation of specific models. These frameworks may possibly interpreted as a mathematical theory.

The derivation of models is obtained by modeling interactions at the microscopic level by a phenomenological interpretation of empirical data. On the other hand, when these models are obtained by a robust theory (for instance in biology) then we may talk about a biological-mathematical theory.

The present state-of-the-art does not offer such challenging result. However, some preliminary analysis can be developed by looking at a well defined field of applied or natural sciences. The reasonings in the following refer specifically to biological sciences.
Focusing on biological sciences, the derivation of models for multicellular systems is obtained when a mathematical description of cell interactions can be derived, by phenomenological interpretation, from empirical data. On the other hand, only when the above interactions are delivered by a theoretical interpretation delivered within the framework of biological sciences, then we may talk about a **biological-mathematical theory**.

The various theoretical approaches known in the literature postulate probabilistic models of gene expression, while gene interactions among genes and with the outer environments should be taken into account. Considering that a robust theory is not yet available, a conjecture is here proposed developing at the molecular scale some ideas already exploited at the cellular scale. This conjecture is proposed in what follows at a preliminary stage still waiting to be properly developed.
• Therefore the approach to a mathematical biological theory means derivation of the cellular dynamics form the underlying dynamics at the molecular scale.

• The process needs to be continued at the higher scale of tissues, where the macroscopic models of tissues depend on the dynamics at the lower scale of cells. Therefore, ultimately, at the molecular scale.

• It follows that the structure of models of biological phenomena evolve in time (models that change of type).
Let us now consider the coupling with the lower scale, where the overall state is defined by the distribution function of gene expression:

$$ \varphi = \varphi(t, v) : [0, T] \times D_v \to \mathbb{R}_+ , $$

over the microscopic state $v \in D_v$ of the interacting entities regarded as active particles.

The system at the lower scale interacts with the outer environment, that has the ability of modifying the gene expression by an action of the type

$$ \psi = \psi(t, v) : [0, T] \times D \to \mathbb{R}_+ , \quad \int_0^T \int_D \psi(t, v) \, dv \, dt \leq M , $$

for some constant $M$. 

Lecture 7. - Looking Ahead for a Mathematical Theory

The interaction scheme from the lower to the higher scale can be represented as follows:

\[
\mathcal{L} \varphi = \mathcal{N}[\varphi, \varphi] + \mathcal{M}[\varphi, \psi] \quad \rightarrow \quad \mathcal{L} f = J[f] + Q[f, \varphi],
\]

that corresponds to the following dynamics:

- The evolution of the system at the lower scale is determined by the interaction between active particles within the population, and with particles of the outer environment.
- The evolution of the system at the higher scale is determined by the interaction between active particles, of both populations among themselves, and, for each of them, with particles of the lower system.
- A simplified approach consists in modelling the parameters of the equation at the cellular scale using the distance \(d(\varphi, \varphi_0)\) of the distribution \(\varphi\) form the initial distribution \(\varphi_0\).
Let us consider a stochastic perturbation in velocity of the whole system,

\[
(\partial_t + \mathbf{v} \cdot \nabla_x) f_i = \nu_i \mathcal{L}_i[f_i] + J_i[f] + Q_i[f, g],
\]

where \( J_i[f] \) and \( Q_i[f, g] \) are the inner and outer operators.

- \( \nu_i \) is the turning rate or turning frequency, hence \( \tau_i = \frac{1}{\nu_i} \) is the mean run time.
- The linear transport term describes the dynamics of biological organisms modelled by a velocity-jump process,

\[
\mathcal{L}_i[f_i] = \int_{D_v} \left( T_i(v^* \rightarrow v)f_i(t, x, v^*, u) - T_i(v \rightarrow v^*)f_i(t, x, v, u) \right) dv^*,
\]

where \( T_i(v^* \rightarrow v) \) is, for the \( i^{th} \) subsystem, the probability kernel for the new velocity \( v \in D_v \) assuming that the previous velocity was \( v^* \).
The hypotheses on the turning operators $L_i$ are as follows:

**H.1.** Each turning operator $L_i$ satisfies the following solvability conditions:

$$
\int_{D_v} L_i[f] \, dv = \int_{D_v} v L_i[f] \, dv = 0.
$$

**H.2.** There exists a unique function $M_{\rho, U}^i \in L^1(D_v, (1 + |v|) \, dv)$, for all $\rho \geq 0$ and $U \in D_v$, verifying

$$
L_i(M_{\rho, U}^i) = 0, \quad \int_{D_v} M_{\rho, U}^i(v) \, dv = \rho, \quad \int_{D_v} v M_{\rho, U}^i(v) \, dv = \rho U.
$$

Here, variables $t$, $x$ and $u$ act as parameters. These hypotheses allow to derive macroscopic scale hyperbolic systems.
Let us consider a hyperbolic scaling formally corresponding to the following choice of scale:

\[ t \rightarrow \varepsilon t, \quad x \rightarrow \varepsilon x \quad \Rightarrow \quad t \nu = \frac{1}{\varepsilon}, \]

which produces the following non-dimensional model:

\[ \varepsilon (\partial_t + \mathbf{v} \cdot \nabla_x) f_i^\varepsilon = L_i[f_i^\varepsilon] + \varepsilon q_i J_i^\varepsilon[f^\varepsilon] + \varepsilon Q_i^\varepsilon[f^\varepsilon, g], \quad i = 1, 2, 3, 4, \]

The closed system interaction operator is scaled as follows

\[ J_i^\varepsilon[f^\varepsilon] = \sum_{j=1}^{4} \left( G_{ij}^\varepsilon[f^\varepsilon] - L_{ij}^\varepsilon[f^\varepsilon] \right)(t, \mathbf{x}, \mathbf{v}, \mathbf{u}) + \varepsilon \delta_i \sum_{j=1}^{4} \sum_{k=1}^{4} S_{jk}^i[f^\varepsilon], \]

where we have retained the same notation for the non-dimensional gain \( G_{ij}, \) lost \( L_{ij} \)
and proliferative/destructive \( S_{jk}^i \) term.
The hyperbolic macroscopic behaviour is deduced from the limit $\varepsilon \to 0$. First, taking $\varepsilon = 0$ we formally obtain $L_i[f_i^0] = 0$, so each $f_i^0$ verifies the conditions of hypothesis H.2. Then, we have four limiting distributions of the form $f_i^0 = M_{\rho_i^0}^0, U_i^0$ corresponding to our four subsystems, and we have to study the equations satisfied by the equilibrium variables $\rho_i^0$ and $U_i^0$. To do that, integration over $\mathbf{v}$ yields:

$$
\partial_t \rho_i^\varepsilon + \text{div}(\rho_i^\varepsilon U_i^\varepsilon) = \varepsilon^{q_i-1} \sum_{j=1}^{4} \int_{D_{\mathbf{v}}} \left( G_{ij}[f^\varepsilon] - L_{ij}[f^\varepsilon] \right) d\mathbf{v} + \varepsilon^{q_i+\delta_i-1} \sum_{j=1}^{4} \sum_{k=1}^{4} \int_{D_{\mathbf{v}}} S_{jk}^i[f^\varepsilon] d\mathbf{v}
$$

$$
+ \varepsilon^{q_i-1} \sum_{j=1}^{m} \int_{D_{\mathbf{v}}} \left( G_{ij}^c[f^\varepsilon] - L_{ij}^c[f^\varepsilon] \right) d\mathbf{v} + \varepsilon^{q_i+\delta_i-1} \sum_{j=1}^{m} \int_{D_{\mathbf{v}}} S_{ij}^c[f^\varepsilon, g] d\mathbf{v}.
$$
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Analogous calculations for the momentum equation yield:

\[ \partial_t (\rho_i^\varepsilon U_i^\varepsilon) + \text{Div} \left( \int_{D\mathbf{v}} \mathbf{v} \otimes \mathbf{v} f_i^\varepsilon \, d\mathbf{v} \right) \]

\[ = \varepsilon^{q_i-1} \sum_{j=1}^{4} \int_{D\mathbf{v}} \mathbf{v} \left( G_{ij}^\varepsilon[f^\varepsilon] - L_{ij}^\varepsilon[f^\varepsilon] \right) \, d\mathbf{v} + \varepsilon^{q_i+\delta_i-1} \sum_{h=1}^{4} \sum_{k=1}^{4} \int_{D\mathbf{v}} \mathbf{v} S_{hk}^i[f^\varepsilon] \, d\mathbf{v} \]

\[ + \varepsilon^{q_i-1} \sum_{j=1}^{m} \int_{D\mathbf{v}} \mathbf{v} \left( G_{ij}^e[f^\varepsilon] - L_{ij}^e[f^\varepsilon] \right) \, d\mathbf{v} + \varepsilon^{q_i+\delta_i-1} \sum_{j=1}^{m} \int_{D\mathbf{v}} \mathbf{v} S_{ij}^e[f^\varepsilon, g] \, d\mathbf{v}. \]

Moreover we use the pressure tensor \( P_i^0 \) as a measure of the statistical variation in velocity around the expected mean velocity \( U_i^0 \),

\[ P_i^0(t, x, u) = \int_{D\mathbf{v}} (\mathbf{v} - U_i^0) \otimes (\mathbf{v} - U_i^0) \, f_i^0 \, d\mathbf{v}. \]

\[ \int_{D\mathbf{v}} \mathbf{v} \otimes \mathbf{v} M_{\rho_i^0, U_i^0} \, d\mathbf{v} = P_i^0 + \rho_i^0 (U_i^0 \otimes U_i^0). \]
Perturbation of the equilibrium $f_i = M_i^0, u_i^0 + \varepsilon h_i$, with $M = \{M_i^0, u_i^0\}_{i=1}^4$:

$$
\partial_t \rho_i^0 + \text{div}(\rho_i^0 u_i^0) = O(\varepsilon^q_i)
$$

$$
+ \varepsilon^{q_i-1} \sum_{j=1}^4 \int_{Dv} \left( G_{ij}[M] - L_{ij}[M] \right) dv + \varepsilon^{q_i+\delta_i-1} \sum_{j=1}^4 \sum_{k=1}^4 \int_{Dv} S_{jk}^i[M] dv
$$

$$
+ \varepsilon^{q_i-1} \sum_{j=1}^m \int_{Dv} \left( G_{ij}^e[M, g] - L_{ij}^e[M, g] \right) dv + \varepsilon^{q_i+\delta_i-1} \sum_{j=1}^m \int_{Dv} S_{ij}^e[M, g] dv,
$$

$$
\partial_t (\rho_i^0 u_i^0) + \text{Div} \left( \int_{Dv} v \otimes v M^0_i, u^0_i dv \right) = O(\varepsilon^q_i)
$$

$$
+ \varepsilon^{q_i-1} \sum_{j=1}^4 \int_{Dv} v \left( G_{ij}[M] - L_{ij}[M] \right) dv + \varepsilon^{q_i+\delta_i-1} \sum_{j=1}^4 \sum_{k=1}^4 \int_{Dv} v S_{hk}^i[M] dv
$$

$$
+ \varepsilon^{q_i-1} \sum_{j=1}^m \int_{Dv} v \left( G_{ij}^e[M, g] - L_{ij}^e[M, g] \right) dv + \varepsilon^{q_i+\delta_i-1} \sum_{j=1}^m \int_{Dv} v S_{ij}^e[M, g] dv.
$$
Lecture 7. - Looking Ahead for a Mathematical Theory

Case 1. $\delta_i \geq 0$ and $q_i > 1$: This is the simple conservative hyperbolic system:

$$\begin{align*}
\partial_t \rho_i^0 + \text{div}(\rho_i^0 \mathbf{U}_i^0) &= 0, \\
\partial_t (\rho_i^0 \mathbf{U}_i^0) + \text{Div}(\rho_i^0 (\mathbf{U}_i^0 \otimes \mathbf{U}_i^0) + P_i^0) &= 0.
\end{align*}$$

Case 2. $\delta_i > 0$ and $q_i = 1$: In this case we preserve a source term related to conservative actions, and therapy actions into the closed system:

$$\begin{align*}
\partial_t \rho_i^0 + \text{div}(\rho_i^0 \mathbf{U}_i^0) &= \sum_{j=1}^{4} \int_{D_{\mathbf{v}}} \left( G_{ij}^{e} [\mathbf{M}] - L_{ij}^{e} [\mathbf{M}] \right) d\mathbf{v} \\
&\quad + \sum_{j=1}^{m} \int_{D_{\mathbf{v}}} \left( G_{ij}^{e} [\mathbf{M}, \mathbf{g}] - L_{ij}^{e} [\mathbf{M}, \mathbf{g}] \right) d\mathbf{v}, \\
\partial_t (\rho_i^0 \mathbf{U}_i^0) + \text{Div}(\rho_i^0 (\mathbf{U}_i^0 \otimes \mathbf{U}_i^0) + P_i^0) &= \sum_{j=1}^{4} \int_{D_{\mathbf{v}}} \mathbf{v} \left( G_{ij}^{e} [\mathbf{M}] - L_{ij}^{e} [\mathbf{M}] \right) d\mathbf{v} \\
&\quad + \sum_{j=1}^{m} \int_{D_{\mathbf{v}}} \mathbf{v} \left( G_{ij}^{e} [\mathbf{M}, \mathbf{g}] - L_{ij}^{e} [\mathbf{M}, \mathbf{g}] \right) d\mathbf{v}.
\end{align*}$$
Case 3. \( \delta_i = 0 \) and \( q_i = 1 \): In this last case we preserve all the macroscopic information about the closed system, including proliferative, destructive interactions, and therapy actions:

\[
\begin{cases}
    \partial_t \rho^0_i + \text{div}(\rho^0_i U^0_i) = \sum_{j=1}^{4} \int_{D_v} \left( G_{ij}[M] - L_{ij}[M] + \sum_{k=1}^{4} S^i_{jk}[M] \right) dv \\
    + \sum_{j=1}^{m} \int_{D_v} \left( G^e_{ij}[M, g] - L^e_{ij}[M, g] + S^e_{ij}[M, g] \right) dv,
\end{cases}
\]

\[
\begin{cases}
    \partial_t (\rho^0_i U^0_i) + \text{Div} \left( \rho^0_i (U^0_i \otimes U^0_i) + P^0 \right) \\
    = \sum_{j=1}^{4} \int_{D_v} v \left( G_{ij}[M, g] - L_{ij}[M, g] + \sum_{k=1}^{4} S^i_{jk}[M] \right) dv \\
    + \sum_{j=1}^{m} \int_{D_v} v \left( G^e_{ij}[M, g] - L^e_{ij}[M, g] + S^e_{ij}[M, g] \right) dv.
\end{cases}
\]
Theorem Let $f^\varepsilon$ verify

$$
\|f^\varepsilon\|_{C(0,\infty;L^p(D_x \times D_v \times D_u))}^4 \leq C < \infty
$$

for some $p > 2$, and such that each $f^\varepsilon_i$ converges pointwise. We also assume that the microscopic state space has finite measure and that the probability densities $B_{jk}, C_{jk}, B^e_{jk}$ and $B^e_{jk}$ are bounded functions while the interactions rates $\eta_{ij}$ and $\eta^e_{ij}$, intensity rates $p_{ij}$ and $p^e_{ij}$ and proliferation/destruction rates $\mu^1_{ij}$ and $\mu^2_{ij}$ are all square integrable with respect to their variables. Finally, we assume that $\mu^1_{i,j}$ and $\mu^2_{i,j}$ are continuous. Then, the pointwise limit of $f^\varepsilon$ is the vector valued function $M = \{M^i_0, U^0_i\}^4_{i=1}$ given by hypothesis H.2. with

$$
\rho^0_i = \lim_{\varepsilon \to 0} \rho[f^\varepsilon_i], \quad U[f^\varepsilon_i] = \lim_{\varepsilon \to 0} U^\varepsilon_i,
$$

i.e., the weak and pointwise limits of the local density velocity of $f_\varepsilon$. Moreover, in the three regimes introduced above, the limiting densities $\rho^0_i$ and velocities $U^0_i$ verify the three case considered above.
Dynamical Systems and Measure Theory

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PhD Course on
*Mathematical Methods and Models for Complex Systems in Life Sciences*

Lecture 3 – December 3, 2010
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Introduction

- Abstract Measure Theory and Probability Theory speak, to a large extent, of the same things but often with different terminology.
- Each terminology is rooted in the history of the corresponding theory, and nowadays it would be impossible to unify them de jure.
- In this lecture we try to present the main ingredients common to both theories, making constantly a comparison between their specific languages.
- Our main interest will be in probabilistic concepts, also in view of the application to crowd dynamics.
Sets and Events

Language of Set Theory

- $\Omega =$ abstract set, i.e., a set with no special structure
- $A \subseteq \Omega =$ subset of $\Omega$
- Set operations (given two subsets $A, B \subseteq \Omega$):
  - complement $A^c = \{ \omega \in \Omega : \omega \notin A \}$
  - intersection $A \cap B = \{ \omega \in \Omega : \omega \in A \text{ and } \omega \in B \}$
  - union $A \cup B = \{ \omega \in \Omega : \omega \in A \text{ or } \omega \in B \}$
- $\emptyset =$ the empty set
- $\Omega =$ the universe set
- If $A \cap B = \emptyset$ then the sets $A, B$ are said to be disjoint

Language of Probability

- $\Omega =$ state space, i.e., the set of all possible outcomes of a random experiment
- $A \subseteq \Omega =$ event, i.e., a property which can be observed to hold or not to hold after the random experiment is performed
- Event operations (given two events $A, B \subseteq \Omega$):
  - event contrary to $A$: $A^c$
  - event $A$ and $B$: $A \cap B$
  - event $A$ or $B$: $A \cup B$
- $\emptyset =$ the impossible event
- $\Omega =$ the sure event
- If $A \cap B = \emptyset$ then the events $A, B$ are said to be incompatible

Example: Random experiment of tossing two coins

The state space is $\Omega = \{hh, ht, th, tt\}$. The event “The output of the first toss is head” is the subset $A = \{hh, ht\}$. The event “The output of the second toss is tail” is $B = \{ht, tt\}$. The event “The first toss gives head and the second one tail” is $A \cap B = \{ht\}$. 
\(\sigma\)-Algebras

- We consider now families of subsets of \(\Omega\), or, in the Probability terminology, families of events. We denote by \(\mathcal{A}\) any of such families.
- The interesting families are those stable with respect to the logical operations discussed before.

**Definition (\(\sigma\)-algebra)**

A family \(\mathcal{A}\) of subsets of \(\Omega\) is said to be a \(\sigma\)-algebra if the following properties hold true:

1. \(\emptyset, \Omega \in \mathcal{A}\)
2. if \(A \in \mathcal{A}\) then also \(A^c \in \mathcal{A}\)
3. \(\mathcal{A}\) is closed under countable unions and intersections: if \((A_i)_{i \in \mathbb{N}} \subseteq \mathcal{A}\) then \(\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}\), \(\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}\).

**Examples:**

- \(\mathcal{A} = \{\emptyset, \Omega\}\) is the so-called trivial \(\sigma\)-algebra.
- If \(A \subseteq \Omega\) then \(\sigma(A) = \{\emptyset, A, A^c, \Omega\}\) is the \(\sigma\)-algebra generated by \(A\). It is the smallest \(\sigma\)-algebra containing \(A\) (i.e., if \(\mathcal{A}\) is any other \(\sigma\)-algebra containing \(A\) then \(\sigma(A) \subseteq \mathcal{A}\)).
- If \(\Omega\) is a topological space, we denote by \(\mathcal{B}(\Omega)\) the \(\sigma\)-algebra generated by the open sets. It is called the Borel \(\sigma\)-algebra.

**Definition (Measurable space)**

Each element \(A \in \mathcal{A}\) is called a measurable set. The pair \((\Omega, \mathcal{A})\) is called a measurable space.
Measures and Probabilities

Definition (Measure)

A (real-valued) measure on the measurable space \((\Omega, A)\) is a mapping \(\mu : A \to \mathbb{R}\) satisfying the following property (called \(\sigma\)-additivity):

- For every countable sequence \((A_i)_{i \in \mathbb{N}}\) of elements of \(A\) pairwise disjoint, i.e., such that \(A_i \cap A_j = \emptyset\) for all \(i \neq j\), it results

\[
\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i).
\]

Remark: This definition is consistent, in fact \(A_i \in A\) all \(i\) implies \(\bigcup_{i=1}^{\infty} A_i \in A\). Therefore this union still belongs to the domain of the measure \(\mu\).

Definition (Probability Measure)

A probability on the measurable space \((\Omega, A)\) is a real-valued measure \(\mu\) defined on \(A\), such that

- \(0 \leq \mu(A) \leq 1\) for all \(A \in A\)
- \(\mu(\Omega) = 1\).

The triple \((\Omega, A, \mu)\) is called a measure space, or, if \(\mu\) is a probability, a probability space.
Now we go from \((\Omega, \mathcal{A}, \mu)\) to the measurable space \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\), \(d \geq 1\), with functions.

**Language of Real Analysis**

A function \(f : \Omega \to \mathbb{R}^d\) is said to be **measurable** if

\[
f^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^d).
\]

**Language of Probability**

A mapping \(X : \Omega \to \mathbb{R}^d\) is said to be a **random variable** if

\[
X^{-1}(B) \in \mathcal{A}, \quad \forall B \in \mathcal{B}(\mathbb{R}^d).
\]

- In practice, a random variable is nothing but a measurable function. In particular, it is not a “variable” in the classical analytical sense!

**Theorem (Measurable functions transport measures)**

*Given a measurable function \(X : \Omega \to \mathbb{R}^d\), define the mapping \(\nu : \mathcal{B}(\mathbb{R}^d) \to \mathbb{R}\) by letting

\[
\nu(B) := \mu(X^{-1}(B)), \quad \forall B \in \mathcal{B}(\mathbb{R}^d).
\]

Then \(\nu\) is a measure on the measurable space \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\). If \(\mu\) is a probability, so is \(\nu\).*

- **Remark:** \(X^{-1}(B) \in \mathcal{A}\) because \(X\) is measurable, hence \(\mu(X^{-1}(B))\) is well defined.

- The measure \(\nu\) is called the **image of \(\mu\)** under \(X\), or, in the Probability terminology, the **law of \(X\)**. It is often indicated by the notation \(\nu = X \# \mu\), termed **push forward**.
Integration w.r.t. an Abstract Measure

- A real-valued simple function (or simple random variable) is a measurable function $s : \Omega \to \mathbb{R}$ taking only a finite number, say $n$, of values. It can be represented as

$$s(\omega) = \sum_{i=1}^{n} \alpha_i 1_{A_i}(\omega)$$

where:

- $\alpha_i \in \mathbb{R}$ is the $i$-th value taken by $s$
- $A_i = \{\omega \in \Omega : s(\omega) = \alpha_i\}$ is the subset of $\Omega$ where $s$ takes the value $\alpha_i$. Notice that $A_i = s^{-1}(\{\alpha_i\})$, therefore $A_i \in \mathcal{A}$ because $s$ is measurable
- $1_{A_i}$ is the indicator function of $A_i$, such that $1_{A_i}(\omega) = 1$ if $\omega \in A_i$, $1_{A_i}(\omega) = 0$ if $\omega \in A^c$

**Definition (Integral of a simple function)**

The integral of $s$ on $\Omega$ with respect to the measure $\mu$ is

$$\int_{\Omega} s \, d\mu := \sum_{i=1}^{n} \alpha_i \mu(A_i).$$

Notice that $\int_{\Omega} s \, d\mu \in \mathbb{R}$.

- In practice, $E[s] = \int_{\Omega} s \, d\mu$ using a probability measure $\mu$.

**Definition (Expectation of a simple random variable)**

The expectation (or expected value) of $s$ on $\Omega$ with respect to the probability measure $\mu$ is

$$E[s] := \sum_{i=1}^{n} \alpha_i \mu(A_i).$$

Notice that $E[s] \in \mathbb{R}$. 
Integration w.r.t. an Abstract Measure

- We consider now a positive measurable function \( f : \Omega \to \mathbb{R} \), or a positive random variable \( X : \Omega \to \mathbb{R} \).

**Definition (Integral of a positive measurable function)**

The integral of \( f \geq 0 \) on \( \Omega \) with respect to the measure \( \mu \) is

\[
\int_{\Omega} f \, d\mu := \sup \left\{ \int_{\Omega} s \, d\mu : s \text{ simple, } 0 \leq s \leq f \right\}.
\]

**Definition (Expectation of a positive random variable)**

The expectation (or expected value) of \( X \geq 0 \) on \( \Omega \) with respect to the probability measure \( \mu \) is

\[
E[X] := \sup \{ E[s] : s \text{ simple, } 0 \leq s \leq X \}.
\]

- For \( f \) arbitrary measurable function we define its positive part \( f^+ := \max\{f, 0\} \) and its negative part \( f^- := \max\{-f, 0\} \). Notice: \( f^+, f^- \geq 0, f = f^+ - f^- \). We do the same for an arbitrary random variable \( X \).

**Definition (Integral of an arbitrary measurable function)**

The integral of \( f \) on \( \Omega \) with respect to the measure \( \mu \) is

\[
\int_{\Omega} f \, d\mu := \int_{\Omega} f^+ \, d\mu - \int_{\Omega} f^- \, d\mu
\]

provided either \( \int_{\Omega} f^+ \, d\mu < +\infty \) or \( \int_{\Omega} f^- \, d\mu < +\infty \).

**Definition (Expectation of an arbitrary random variable)**

The expectation (or expected value) of \( X \geq 0 \) on \( \Omega \) with respect to the probability measure \( \mu \) is

\[
E[X] := E[X^+] - E[X^-]
\]

provided either \( E[X^+] < +\infty \) or \( E[X^-] < +\infty \).
Theorem (Integrals depend only on the image measure)

Let $X : \Omega \to \mathbb{R}$ be a measurable function from $(\Omega, \mathcal{A}, \mu)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$, where $\nu = X \# \mu$. Let $h : \mathbb{R} \to \mathbb{R}$ be another measurable function on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then

$$\int\limits_\Omega h(X) \, d\mu = \int\limits_{\mathbb{R}} h \, d\nu$$

provided integrals exist (finite or infinite).

- Choosing in particular $h(x) = x$ yields

$$\int\limits_\Omega X \, d\mu = \int\limits_{\mathbb{R}} x \, d\nu.$$  

- In terms of expectation, this takes the form

$$\mathbb{E}[X] = \int\limits_{\mathbb{R}} x \, d\nu.$$  

- In practice, if the law of $X$ is known then the expectation $\mathbb{E}[X]$, as well as any other expectation $\mathbb{E}[h(X)]$, can be computed ignoring both the state space $\Omega$ and the probability $\mu$. 

We consider a certain number of agents, say $N$, distributed in the space $\mathbb{R}^d$, $d \geq 1$.

The number $N$ may be as large as desired but finite.

Each agent is identified by its position in the physical space $\mathbb{R}^d$. Hence, the state of the $i$-th agent is the vector $(x^i_1, x^i_2, \ldots, x^i_d) \in \mathbb{R}^d$, $x^i_k \in \mathbb{R}$ being the $k$-th coordinate of its position. The positions of the agents change in time in consequence of certain dynamics of the system.

We now move to a probabilistic description of the distribution of the agents in $\mathbb{R}^d$.

Let $X^i_t : \Omega \to \mathbb{R}^d$ be a random variable, from an abstract probability space $(\Omega, \mathcal{A}, P)$ into the “physical” probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu_t)$, $\mu_t = X^i_t \# P$, representing the position of the $i$-th agent at time $t \geq 0$.

The probability space $(\Omega, \mathcal{A}, P)$ is fixed, i.e., time-independent.

The probability $\mu_t$ does not depend on $i$, i.e., the random variables $X^i_t$, $i = 1, \ldots, N$, have all the same law. This amounts to assuming that agents are indistinguishable.

$\mu_t$ gives the distribution of the agents in $\mathbb{R}^d$ at time $t \geq 0$: for all $B \in \mathcal{B}(\mathbb{R}^d)$ it results

$$\mu_t(B) = P\{X^i_t \in B\} = \text{probability to find an agent in the set } B \text{ at time } t.$$  

Remark: the knowledge of $\mu_t$ makes it unnecessary to detail the space $(\Omega, \mathcal{A}, P)$, which thus remains only for formal theoretical purposes.
In order to count the number of agents contained in a certain set $B \in \mathcal{B}(\mathbb{R}^d)$ at time $t$, we introduce the random variable $Y_{t,B} : \Omega \to \mathbb{N}$ defined as

$$Y_{t,B}(\omega) = \sum_{i=1}^{n} \mathbb{1}_{\{\omega \in \Omega : X_t^i(\omega) \in B\}}(\omega)$$

and we take its expectation on $\Omega$:

$$\mathbb{E}[Y_{t,B}] = \sum_{i=1}^{n} P(\{\omega \in \Omega : X_t^i(\omega) \in B\}) = \sum_{i=1}^{n} P((X_t^i)^{-1}(B)) = N \mu_t(B).$$

$\mathbb{E}[Y_{t,\bullet}]$, thought of as a mapping on $\mathcal{B}(\mathbb{R}^d)$, is a measure, say $m_t$, proportional to $\mu_t$.

Since $m_t(B) := \mathbb{E}[Y_{t,B}]$ is the (average) number of agents in $B$ at time $t$, the measure $m_t : \mathcal{B}(\mathbb{R}^d) \to [0, N]$ can be identified with the mass of the system at time $t$.

Notice that $m_t(\mathbb{R}^d) = N$ for all $t \geq 0$, therefore the total mass of the system is constant in time.
Evolution of the System in Space and Time

- Since the total mass of the system is constant in time, we can postulate that the evolution takes place under the principle of the **conservation of the mass**.
- Correspondingly, we assume that $m_t$ satisfies the **continuity equation**:

$$\frac{\partial m_t}{\partial t} + \nabla \cdot (m_t v_t) = 0. \quad (1)$$

- $v_t(x)$ is the velocity field, responsible for mass transportation, at time $t$ in the point $x \in \mathbb{R}^d$.
- If the agents of the system **interact** among one another, $v_t$ depends also on their distribution $\mu_t$, hence $v_t(x) = v[\mu_t](x)$.
- Using the proportionality $m_t = N\mu_t$, we convert (1) into an equation for the probability $\mu_t$:

$$\frac{\partial \mu_t}{\partial t} + \nabla \cdot (\mu_t v[\mu_t]) = 0, \quad \mu_0 = \bar{\mu} \ (\text{given initial distribution of the agents}) \quad (2)$$

where the field $v[\mu_t]$ has to be prescribed $\rightarrow$ **modeling** (we will see this in the next lecture).
- The precise meaning of Eq. (2) is recovered by understanding derivatives of measures in the sense of distributions:

$$\int_{\mathbb{R}^d} \phi \, d\mu_t = \int_{\mathbb{R}^d} \phi \, d\bar{\mu} + \int_0^t \int_{\mathbb{R}^d} v[\mu_\tau] \cdot \nabla \phi \, d\mu_\tau \, d\tau, \quad \forall \phi \in C^\infty_c(\mathbb{R}^d), \ \forall t \geq 0. \quad (3)$$

- If $\mu_t$ solves Eq. (2) in the sense (3) then $m_t$ solves Eq. (1), again in the sense (3), with $v_t = v[m_t/N]$ and initial condition $m_0 = N\bar{\mu}$. 
Exercises

**Note:** Unless otherwise stated, in the following exercises we assume that an abstract state space $\Omega$ is given, possibly together with a $\sigma$-algebra $\mathcal{A}$ and a measure $\mu : \mathcal{A} \rightarrow \mathbb{R}$.

1. Let $2^\Omega$ denote the family of all subsets of $\Omega$. Prove that $2^\Omega$ is a $\sigma$-algebra.

2. Let $\Omega$ be a finite set, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$, and let $p_1, p_2, \ldots, p_n$ be given numbers in $[0, 1]$ such that $\sum_{i=1}^n p_i = 1$. Set
   \[ \mu(\{\omega_i\}) := p_i, \quad i = 1, \ldots, n. \]
   Prove that this uniquely defines a probability measure $\mu$ on $2^\Omega$.

3. For any two $A, B \in \mathcal{A}$, show that $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

4. Let $\mu$ be positive, i.e., $\mu(A) \geq 0$ for all $A \in \mathcal{A}$. Given $A, B \in \mathcal{A}$ with $A \subseteq B$, show that $\mu(A) \leq \mu(B)$.

5. Let $\mu$ be a probability and let $A, B \in \mathcal{A}$ be such that $\mu(A) = p$, $\mu(B) = q$ with $p, q \in [0, 1]$. Show that $\max\{0, p + q - 1\} \leq \mu(A \cap B) \leq \min\{p, q\}$.

6. Let $f : (\Omega, \mathcal{A}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be measurable. Prove that $\nu = f \# \mu$ is a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and that it is a probability if so is $\mu$.

7. If $\Omega$ is a topological space, we define the **support** of $\mu$, denoted $\text{supp} \mu$, to be the smallest closed subset of $\Omega$ such that $\mu(A) = 0$ for all $A \in \mathcal{A}$ with $A \cap \text{supp} \mu = \emptyset$. Let $\mu_t$ be any solution to Eq. (2) in the sense (3) and assume that a bounded set $U \subset \mathbb{R}^d$ exists such that $\text{supp} \mu_t \subseteq U$ for all $t \geq 0$. Prove that $\mu_t(U) = \bar{\mu}(U) = 1$ for all $t > 0$. 
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Crowd Dynamics I

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PhD Course on
Mathematical Methods and Models for Complex Systems in Life Sciences

Lecture 4 – December 10, 2010
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   - Phenomenology
   - Literature Review

2 Modeling
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4 References
The Physics of Crowds

- Pedestrians are not **inert** matter, in the sense that when moving they are not passively dragged by external forces.
- Pedestrians are **active** particles with **decision-based** dynamics.
- Pedestrians develop **individual strategies** based on a few elementary **behavioral rules**:
  - They have **preferential directions** for reaching their destinations
  - They want to **avoid collisions** and excessive aggregation with other pedestrians.
- These rules are valid in **normal** conditions. Under particular circumstances, *e.g.*, **panic** conditions, entirely different behaviors may emerge.
- Pedestrians usually **do not cooperate**, *i.e.*, they do not have a goal as a group.

- Crowds are systems of **interacting active** particles.
- Interactions are:
  - **Nonlocal**: pedestrians can see even far group mates
  - **Anisotropic**: pedestrians have limited visual fields oriented mainly ahead.
- Complex behaviors emerge, such as **self-organization**, *i.e.*, spontaneous group organization in specific patterns.
Mathematical Literature on Crowd Dynamics

- Models for the description of the spatiotemporal patterns of the motion of crowds have been proposed in the literature at least since the Seventies.
- First proposals were poorly organized in well-defined research lines. Rather, they were conceived as by-products of the more developed theory of vehicular traffic.
- Nowadays research on crowd dynamics is definitely stand-alone. It has proved to be strategic for a wide range of scientific areas such as Sociology, Psychology, Physics, and of course Applied Mathematics.
- Models in the literature can be divided in two main categories:
  - Microscopic (or discrete) models, based on the individual representation of pedestrians. They can be formalized by, e.g., ordinary differential equations, if the point of view is the dynamical one of the spatial trajectories of pedestrians, or cellular automata, which focus instead on an algorithmic description of the behavioral rules followed by pedestrians. Examples of models in this category are Helbing et al. 1995 – 2010, Hoogendoorn et al. 2003 – 2005, Maury and Venel 2007, 2008.
  - Macroscopic (or continuous or hydrodynamical) models, based on the idea that the equations describing the flow of fluids can also describe the flow of pedestrians, provided a large-scale point of view is adopted. Pedestrians are regarded as small particles, and an overall description of the crowd is provided in terms of their density. These models are formalized by partial differential equations of conservation/balance law type. Examples include Henderson 1974, Hughes 2000 – 2003, Colombo and Rosini 2005, 2009, Bellomo and Dogbé 2008, Coscia and Canavesio 2008, Piccoli and Tosin 2009, 2010, Maury et al. 2010.
- Models for collective behaviors in groups of interacting particles are available also for animals (swarms) and other kind of agents, possibly inanimate such as robots, see e.g., the works by Carrillo et al. 2010 and Cristiani et al. 2010.
Recall that we describe the distribution of pedestrians in the physical space $\mathbb{R}^d$, $d \geq 1$, by means of the **probability distribution** $\mu_t$ of their positions (random variables) $X^i_t$, $i = 1, \ldots, N \geq 2$.

Pedestrians are **indistinguishable**, therefore $\mu_t$ does not depend on $i$.

Under the principle of the conservation of mass, the evolution in time and space of the probability $\mu_t$ is ruled by the **continuity equation**:

\[
\frac{\partial \mu_t}{\partial t} + \nabla \cdot (\mu_t v[\mu_t]) = 0
\]

along with a prescribed initial condition $\mu_0 = \bar{\mu}$, where $\bar{\mu}$ is a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Equation (1) has a precise mathematical meaning in the sense of distributions. Heuristically, it says that the number of pedestrians contained in any region $B \in \mathcal{B}(\mathbb{R}^d)$ may change in time only because of pedestrians crossing the boundary $\partial B$. 
Modeling the Interactions Among Pedestrians

- The velocity \( v[\mu_t] \) originates from the **individual strategy** developed by pedestrians:
  - **Non-cooperative** interactions
  - **Nonlocal** interactions
  - **Anisotropic** interactions.

- We distinguish two main contributions to the total velocity:
  \[
  v[\mu_t](x) = v_d(x) + \nu[\mu_t](x)
  \]
  
  desired velocity interaction velocity

- The **desired velocity** is a given field, depending only on \( x \in \mathbb{R}^d \), which expresses the preferential direction of motion of pedestrians, *i.e.*, the one they would follow if they were alone.

- The **interaction velocity** accounts for the tendency of pedestrians to avoid crowding. It is a correction to \( v_d \) decided on the basis of the distribution of pedestrians in a neighborhood \( S^\alpha_R(x) \):
  \[
  \nu[\mu_t](x) = N \int_{S^\alpha_R(x)} f(|y - x|) r(y - x) \, d\mu_t(y)
  \]
More on the Interaction Velocity

\[ \nu[\mu_t](x) = N \int_{S^\alpha_R(x)} f(|y - x|) r(y - x) \, d\mu_t(y) \]  

(2)

- Borrowing the terminology from the *kinetic theory for active particles*, we call:
  - **test pedestrian** the pedestrian located in \( x \in \mathbb{R}^d \), whose velocity is affected by the interactions
  - **field pedestrian** the pedestrian located in \( y \in S^\alpha_R(x) \), who induces the test pedestrian to correct her desired velocity.

- \( \nu[\mu_t](x) \) results from **binary** interactions that the test pedestrian in \( x \) experiences with all of the field pedestrians located in the neighborhood \( S^\alpha_R(x) \).
  - In Eq. (2) integration is performed w.r.t. the mass \( N \mu_t \) of the field pedestrians.

- \( f : [0, R] \rightarrow \mathbb{R} \) is the strength of the interaction, depending on the distance between the test and the field pedestrian:
  - \( f < 0 \sim \text{repulsion} \), usually decreasing with the distance
    Example: \( f(|y - x|) \propto -\frac{1}{|y - x|} \) for \( |y - x| \geq \epsilon > 0 \)
  - \( f > 0 \sim \text{attraction/cohesion} \), usually increasing with the distance
    Example: \( f(|y - x|) \propto |y - x| \)

- \( r : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is the direction of the interaction, depending on the relative position of the test and the field pedestrian and such that \( |r| \leq 1 \).
Microscopic Models

Let us consider an atomic initial condition:

$$\bar{\mu} = \frac{1}{N} \sum_{k=1}^{N} \delta_{\bar{x}_k} \quad (\bar{x}_k \in \mathbb{R}^d). \quad (3)$$

We can find a solution to Eq. (1), with initial condition $\bar{\mu}$, by the ansatz:

$$\mu_t = \frac{1}{N} \sum_{i=k}^{N} \delta_{x_k(t)},$$

where $x_k : (0, T] \rightarrow \mathbb{R}^d$ are mappings to be determined.

It turns out that the $x_k$'s must solve the following Cauchy problem:

$$\begin{cases}
\dot{x}_k = v_d(x_k) + \sum_{j=1, \ldots, N} f(|x_j - x_k|)r(x_j - x_k) \\
x_k(0) = \bar{x}_k
\end{cases}$$

$k = 1, \ldots, N.$

This is a microscopic model in which the $x_k : (0, T] \rightarrow \mathbb{R}^d$ are the trajectories of pedestrians.

In probabilistic terms, given the initial distribution (3) we know that the position at time $t$ of any of our pedestrians is certainly one of the $x_k$'s, with furthermore

$$P\left(\{X^i_t = x_k\}\right) = \mu_t(\{x_k\}) = \frac{1}{N}, \quad \forall k = 1, \ldots, N.$$
Let us consider an initial condition \textit{absolutely continuous} w.r.t. Lebesgue:

\[ d\bar{\mu} = \bar{\rho} \, dx, \]

where \( \bar{\rho} : \mathbb{R}^d \rightarrow [0, +\infty) \) is such that

\[ \int_{\mathbb{R}^d} \bar{\rho}(x) \, dx = 1 \quad \text{and} \quad \bar{\mu}(B) = \int_B \bar{\rho}(x) \, dx, \quad \forall \ B \in \mathcal{B}(\mathbb{R}^d). \quad (4) \]

We can find a solution to Eq. (1), with initial condition \( \bar{\mu} \), by the ansatz: \( d\mu_t = \rho_t \, dx \),

where for each \( t \) the function \( \rho_t : \mathbb{R}^d \rightarrow [0, +\infty) \) satisfies (4) with \( \bar{\rho}, \bar{\mu} \) replaced by \( \rho_t, \mu_t \).

It turns out that \( \rho_t \) must solve, at least in a suitable weak sense,

\[ \frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t v[\rho_t]) = 0, \quad \rho_0 = \bar{\rho} \]

with

\[ v[\rho_t](x) = v_d(x) + N \int_{S_R^\alpha(x)} f(|y - x|) r(y - x) \rho_t(y) \, dy. \]

This is a \textit{macroscopic} model, in which pedestrian trajectories are blurred into the \textit{probability density} \( \rho_t \) of the distribution of pedestrian positions.

\begin{itemize}
  \item The function \( N\rho_t \) can be interpreted as the \textit{mass density} of pedestrians.
  \item This model is expected to be a good approximation of the microscopic one for \textit{large} \( N \).
  \item For \textit{small} \( N \) this model may represent a condition of high uncertainty in the determination of the positions of pedestrians.
\end{itemize}
Let us prescribe now an initial condition of the form:

\[ d\bar{\mu}(x) = \theta \frac{1}{N} \sum_{k=1}^{N} \delta_{\bar{x}_k}(x) + (1 - \theta)\bar{\rho}(x) \, dx, \quad \theta \in [0, 1], \]

_i.e., an interpolation_ between an atomic and an absolutely continuous measure.

A solution to Eq. (1), with initial condition \( \bar{\mu} \), can be found by the ansatz:

\[ d\mu_t(x) = \theta \frac{1}{N} \sum_{k=1}^{N} \delta_{x_k(t)}(x) + (1 - \theta)\rho_t(x) \, dx. \]

It turns out that the \( x_k \)'s and \( \rho_t \) must solve

\[
\begin{align*}
\dot{x}_k &= v[\mu_t](x_k) \\
x_k(0) &= \bar{x}_k, \\
\partial_t \rho_t + \nabla \cdot (\rho_t v[\mu_t]) &= 0 \\
\rho_0 &= \bar{\rho}
\end{align*}
\]

where

\[ v[\mu_t](x) = v_d(x) + \theta \sum_{j=1, \ldots, N}^{N} f(|x_j - x|)r(x_j - x) + N(1 - \theta) \int_{S^\alpha_R(x)} f(|y - x|)r(y - x)\rho_t(y) \, dy. \]

This is a _multiscale_ model in which pedestrian trajectories and density affect each other. It allows one to study intermediate situations between a fully continuous (\( \theta = 0 \)) and a fully discrete (\( \theta = 1 \)) mass of pedestrians.
Case Study: Pedestrian Flow Through a Narrow Passage

- Two crowds, each with $N = 30$ pedestrians, walk in opposite directions and share a narrow passage.

- $\theta = 0$
  - Fully continuous mass
  - Clogging of the bottleneck

- $\theta = 0.3$
  - Coupled multiscale model
  - Oscillations of the passing direction

- $\theta = 1$
  - Fully discrete mass
  - Alternate lanes
A Very Partial List of References

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Opinion Formation and Social Systems

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The lecture refers to a general approach of **modelling complex systems in life sciences**. In particular, it focuses on application on the field of social dynamics and behavioral economy.

The main ingredients of the lecture are:

- Phenomenological analysis of the system under consideration and **development of the related mathematical framework**.
- **Derivation of specific models** according to the mathematical framework.
- Statement of mathematical problems generated by the application of models. The related **qualitative analysis** is mainly focused on the well-posedness of the mathematical problem and the asymptotic behavior in time.
- **Simulations** and a quantitative analysis to complete the qualitative analysis.
Guidelines

It is worth to point out that models are analyzed with special attention to their exploratory and predictive ability. This means that the investigation is addressed to show emerging phenomena and to analyze how certain microscopic interactions and different values of the parameters may generate different types of evolution.
The living matter shows substantial differences with respect to the behavior of the inert matter.

A complex living system is a system of several individuals interacting in a non-linear manner whose collective behavior cannot be described only by the knowledge of the mechanical dynamics of each element:

The dynamics of a few individuals does not lead straightforwardly to the overall collective dynamics of the whole system.

Mathematical models of complex living systems can be “roughly” grouped into three main categories, according to the scale of observation at which the phenomena are analyzed: microscopic scale, statistical (kinetic) scale and macroscopic scale.

Models of Complex Living Systems are intrinsically multiscale, and show at the macroscopic level emerging phenomena expressing a self-organizing ability which is only the output of the interactions between entities at the microscopic level.
Multiscale modelling

Bearing in mind the complexity related to a mathematical modelling of a complex living system, a mathematical approach may provide useful suggestions to understand the global behavior of a living system, catching the essential features of the complex system and the emergence of behaviors.
Mathematical framework

The modelling is referred to: the **mathematical kinetic theory for active particles**, see Bellomo (2008), and Bellomo, Bianca, Delitala (2009).

- The system is constituted by a large number of interacting entities, called **active particles**, whose microscopic state includes, in addition to geometrical and mechanical variables, also an additional variable, called **activity**, representing the ability of each entity to express a specific **strategy** or **function**.

- The activity variable can be either continuous or discrete. **Discretization** is motivated, as in traffic flow modelling or opinion formation, by the need of identifying the state of the particles by ranges of values rather then by a continuous variable.

- The expression of the activity is not the same for all interacting entities. The activity variable is **heterogeneously distributed** over the active particles, while the overall state of the system is described by the **probability distribution function** over such microscopic state.
Reducing the complexity

- Entities of living systems, called **active particles**, may be organized into several interacting populations. To reduce the complexity, with a modular approach, each population or **functional subsystem**, expresses a well defined function, the activity, that is collectively expressed by groups of active particles.

- **Interactions** modify the state of the interacting entities (**conservative interactions**) according to the strategy that each entity develops, based on the space and state distribution of the other interacting entities. Due to the presence of the activity variable, the output of interactions does not obey to the laws of classical mechanics, but to the individual and collective strategy expressed by the active particles. Moreover, in some cases (not in our problems of opinion formation), interactions may generate **proliferative and/or destructive** events.
Reducing the complexity

- Methods of the mathematical kinetic theory can be used to derive the evolution equations. The evolution in time and space of the distribution function over the microscopic state of the interacting active particles is obtained by a suitable balance in each elementary volume of the space of the microscopic states, where the inflow and outflow in the elementary volume is determined by interactions among particles.
Bibliography


Let us consider one single population constituted by a large number of particles homogeneously distributed in space, with a discrete microscopic state, the physical variable charged to describe the state of each particle:

\[ I_u = \{ u_1, \ldots, u_n \} \]

- Distribution function for active particles: normalized density distribution function

\[ f_i(t) = f(t, u_i) \]

- Macroscopic gross variables can be expressed, under suitable integrability properties, by moments weighted by the above distribution function.

\[ \rho(t) = \sum_{i=1}^{n} f(t, u_i), \quad Q(t) = \sum_{i=1}^{n} u_i f(t, u_i). \]
Considering only conservative interactions, let assume that the following quantities can be computed:

- **The interaction rate**:
  
  \[ \eta_{hk} = \eta(u_h, u_k) : I_u \times I_u \rightarrow \mathbb{R}^+ , \]

- **The tables of the game rules** (n matrices \( n \times n \)):
  
  \[ A^i_{hk} = A(u_h, u_k; u_i) : I_u \times I_u \times I_u \rightarrow \mathbb{R}^+ . \]

  \( u_h \) falls into the state \( u_i \) after an interaction with a field particle with state \( u_k \).

- **The tables of the game rules are transition probability densities**:
  
  \[ \forall h, k : \sum_{i=1}^{n} A^i_{hk} = 1. \]
The Discrete Generalized Kinetic Framework

- The **evolution equation** is obtained equating the rate of growth of subjects with microscopic state in volume element \([u, u + du]\), to the inflow and outflow of subjects per unit time in the volume due to interactions. If no source term is considered (no proliferation/destruction term), the evolution equations write as follows:

\[
\frac{df_i}{dt} = \sum_{h=1}^{n} \sum_{k=1}^{n} \eta_{hk} A_{hk}^i f_h f_k - f_i \sum_{k=1}^{n} \eta_{ik} f_k.
\]

- The **mathematical problem** is defined, for \(i = 1, \ldots, n\), linking the initial conditions to the evolution equations:

\[
f_{i0} = f_i(t = 0) \quad \text{where} \quad \sum_{i=1}^{n} f_{i0} = 1.
\]
Well posedness of the discrete kinetic framework

- **Theorem: Existence and Uniqueness.** Assume $\eta_{hk} \leq M$ for some positive constant $M < +\infty$.

  The solution $f(t) = (f_1(t), \ldots, f_n(t))$ of the Cauchy problem exists and is unique $\forall t \in [0, +\infty)$.

  $\forall t \geq 0 : f_i(t) \geq 0$ for any $i = 1, \ldots, n$ and $\sum_{i=1}^{n} f_i(t) = 1$.

- **Regularity of the solutions.** The solution claimed in the Theorem is of class $C^\infty$: it guarantees the continuous dependence of solutions on the initial conditions.

- **Theorem: Equilibrium solutions.** Assuming

  $\eta_{hk} = c \quad \forall h, k = 1, \ldots, n$

  where $c$ is a positive constant, then there exists at least one positive equilibrium solution of System.
The framework with external actions

Let further assume that the following quantities can be computed:

- \( g_i = g(t, u_i) \), for \( i = 1, \ldots, n \)
  which are \( n \) given functions representing the distribution at time \( t \) of external actions acting on the activity state \( u_i \).

- \( \mu_{hk} = \mu(u_h, u_k) \), \( \forall h, k = 1, \ldots, n \),
  which is the number of contacts per unit time, at time \( t \), between particles with activity \( u_h \) and external actions acting on the state \( u_k \) (table of the external interaction rates).

- \( B^i_{hk} \in \mathbb{R}_+ \), with \( \sum_{i=1}^{n} B^i_{hk} = 1 \), \( \forall h, k = 1, \ldots, n \),
  which is the probability density for an active particle with activity state \( u_h \) to end up with state \( u_i \) after an interaction with an external action acting on the state \( u_k \) (table of the external transition probability densities).
The framework with external actions

Equating the time derivative of each $f_i$ to the difference between the gain and the loss term, and taking into account the contribution of both the internal and the external interactions, the equations of the evolution in time of the $f_i$ take the form:

$$\frac{df_i}{dt} = J_i[f, f] + Y_i[f, g], \quad i = 1, \ldots, n,$$

where

$$J_i[f, f] = \sum_{h=1}^{n} \sum_{k=1}^{n} \eta_{hk} A_{hk}^i f_h f_k - f_i \sum_{k=1}^{n} \eta_{ik} f_k, \quad i = 1, \ldots, n,$$

and

$$Y_i[f, g] = \sum_{h=1}^{n} \sum_{k=1}^{n} \mu_{hk} B_{hk}^i f_h g_k - f_i \sum_{k=1}^{n} \mu_{ik} g_k, \quad i = 1, \ldots, n.$$

Constructing a model amounts to assign specific values to the tables of the interaction rates $\eta_{hk}$, the transition probability densities $A_{hk}^i$, the external interaction rates $\mu_{hk}$, the external transition probability densities $B_{hk}^i$ and the $g_i$ as well.
In recent years, mathematicians are attempting to formalize in a more rigorous way the quantitative modelling of socioeconomic systems in order to better explore and predict the dynamics of our societies. The main conceptual difficulty consists in including the so-called living component. Individuals cannot be thought as simple mindless or hyper-rational agents, and their behaviour does not obey, in general, to some universal rule or exact formulae. Mental schemes of human beings is (and may remain) unknown. Moreover, it is not guaranteed at all that two individuals, exactly stimulated in the same way, react in identical ways.

The **Theory of Black Swan Events** was developed by N. N. Taleb (*N.N. Taleb, The Black Swan, Penguin, 2007*) to explain the disproportionate role of high-impact and rare events that are beyond the realm of normal expectations in history, science, finance and technology.

**Behavioural economy** and complex systems approaches are relatively new trends in economic sciences that try to give answers to these issues, bearing in mind that many qualitative (and sometimes quantitative) features and emerging patterns do not depend on the microscopic details of the processes under examination.
Various methodologies have been proposed, and the words socio and econophysics have been used in order to characterize new quantitative approaches towards both sociology and economics. In the literature, there are models mainly at two observation scales: microscopic models focussing on the single individual or mesoscopic models with a statistical description of the systems. Few models are developed at the macroscopic scale.

Most of them are microscopic models like cellular automata or Agent Based models; one of the most popular models is the bounded confidence model introduced by Deffuant et al. (Adv. Complex Sys, 2000), in which repeatedly two peers are randomly selected and operate a compromise in the positions if their opinions differ by less than a deviation threshold.

Tools of statistical mechanics have been recently used for mesoscopic models to have a deeper insight and a rigorous qualitative analysis on the evolution of the model.


Examples

The list of the most explored research fields of investigation includes, among others:

- Opinion formation and cultural dynamics for instance:
  - on diffusion of new technologies (like spread of epidemics);
  - critics’ ratings about the new opening movies or restaurants;
  - reputation systems to measure trust about users while doing transactions over the internet such as e-bay;
  - rough categorization of voters in an election;
  - opinion of the employees about the new company during a company fusion.

- Applause dynamics and problems of herding and imitation.
- Language evolution and problems of social learning
- Problems related to group making a collective decision (fair, correct, efficient decision).
- Social networks analysis.
Which data?

Rating of movies and evolution of Obama’s approval index
Networks of sexual contacts and distribution of HIV in communities.
Network of friendship
Opinion formation

We focus on the evolution of a set of opinions within a population with tools and methods of Kinetic Theory for Active Particles. Several approaches have been proposed in the literature at different representation scales. Several contributions are in the field of physics, the so called socio-physics, while few contributions are developed from mathematicians, from a more recent period.

Here we propose an approach to model the changes in the points of view arising as a consequence of the encounters between individuals performing compromises, then we add the influence of some boredom-like phenomena and finally the presence of external actions is modelled, including the effect of the influence exerted by some agents, the persuaders.

The modelling is developed with an explorative aim to look at emerging phenomena and critical parameters.


A model in the absence of persuaders

Focussing on the opinion formation problem within a population, with KTAP approach, we have that:

- \( I_u = \{ u_1, \ldots, u_i, \ldots, u_n \} \) is the “ordered” finite set of the admissible opinions;
- \( f_i = f_i(t) \ \forall \ i = 1, ..., n \) is the fraction of individuals sharing the opinion \( u_i \);

For the interaction rates, either we suppose that individuals having the same opinion meet each other more frequently than individuals with different opinion or we suppose that all individuals have the same probability to meet each others:

\[
\eta_{hk} = \begin{cases} 
\alpha & \text{if } h = k , \\
\frac{\alpha + \varepsilon}{2} & \text{if } |h - k| = 1 , \\
\varepsilon & \text{if } |h - k| = 2 , \\
0 & \text{otherwise.}
\end{cases}
\]
A model in the absence of persuaders

Let us consider a parameter $m$, called **closeness threshold**, related to the **open-mindness** of the individuals.

We assume the occurrence of a **compromise-like processes**: after an interaction, two individuals having a suitably small difference in opinions (depending on the closeness threshold parameter $m$), readjust their opinions letting them become eventually closer.

![Diagram](attachment:image.png)
Clustering and comparison with competing models

This particularization can be viewed as a discrete version of an Agent-Based “bounded confidence model” proposed by Deffuant et al., Adv. Complex Systems (2000), and Weisbuch et al., Complexity (2002).

In this case the discrete model is expressed by a two parameters family of systems of \( n \) nonlinear ordinary differential equations, where the parameters \( n \) and \( m \) respectively represent the finite number of admissible opinions about a given issue and the threshold separating close from distant opinions, i.e. the closeness threshold characterizing the occurrence or not of an adjustment of opinions of interacting individuals.

The properties of the equilibrium configurations of the model are obtained by a qualitative analysis and the results are generalized and visualized by simulations. The role of the parameter related to the “open-mindedness” of the individuals, is analyzed.
Properties of the equilibrium configurations: final distribution of opinions in the cases $n = 27$ with $m = 3, 4, 5, 8$ from left to right.
Simulations

- If individuals are sufficiently **open minded**, all individuals asymptotically will share the same opinion, and correspondingly a **single cluster** emerges.

- If individuals are **less open minded**, two or **more clusters emerge**, corresponding to the situations where different groups of individuals show the same opinion, as for instance groups of interests or parties.
Simulations

The above result is consistent with simulations and results of a competing model, the “bounded confidence model” (Deffuant et al., Adv. complex. Sys., 2000). The main advantage of the approach here proposed, is the possibility to develop analytical proofs of certain features of the asymptotic clustering behavior.
Simulations

We define a cluster a configuration displaying at most two non-empty opinion classes with empty classes on the left and right. All performed simulations relative to systems with \( n \) general odd number of opinions and different values of closeness parameter \( m \) show that:

For \( n \) general odd and \( 2 \leq m \leq n - 1 \), the system of evolution equations admits degenerate equilibria. Each one of these equilibria corresponds to a distribution containing some clusters. There are at least \( m \) empty classes separating any two consecutive clusters.

This result confirms that:

the number of clusters in a stationary opinion distribution decreases as the closeness threshold \( m \) increases.

This result is in good agreement with some experimental evidence which show the emergence of consensus over few opinions, depending on the topology of the interactions and on the “open-mindness” of the individuals.
To develop a qualitative analysis, we focus on low dimensional cases. For a general number of opinions $n$ and low number of the parameter $m$, we can state the following:

- **Theorem.**
  The system of evolution equations of the model with general odd $n$ and $m = 2$ admits several families of degenerate equilibria corresponding to a distribution containing some clusters. There are at least two empty classes separating any two consecutive clusters and any cluster consists of at most two opinion classes.

Analogous theorem can be proved in the case of a general odd $n$ and $m = 3$ where there are at least three empty classes separating any two consecutive clusters.

The result is extended in (Bertotti 2010) where it is proved that for a general $m$ the clusters are surrounded by at least $m$ empty classes.
**Particular case: \( n = 5 \) and \( m = 3 \)**

Consider as an explicit example the \( n = 5 \) and \( m = 3 \), the system of the evolution equations of the model takes then the form:

\[
\begin{align*}
\frac{df_1}{dt} &= -f_1 f_3 - f_1 f_4 , \\
\frac{df_2}{dt} &= 2f_1 f_3 + f_1 f_4 - f_2 f_4 - f_2 f_5 , \\
\frac{df_3}{dt} &= -f_1 f_3 + f_1 f_4 + 2f_2 f_4 + f_2 f_5 - f_3 f_5 , \\
\frac{df_4}{dt} &= 2f_3 f_5 + f_2 f_5 - f_1 f_4 - f_2 f_4 , \\
\frac{df_5}{dt} &= -f_3 f_5 - f_2 f_5 .
\end{align*}
\]

One may check directly that the equilibria of this system are:

- all points of the form \((f_1, f_2, 0, 0, 0)\) with \(f_1 + f_2 = 1\),
- all points of the form \((0, f_2, f_3, 0, 0)\) with \(f_2 + f_3 = 1\),
- all points of the form \((f_1, 0, 0, 0, f_5)\) with \(f_1 + f_5 = 1\),
- all points of the form \((0, 0, f_3, f_4, 0)\) with \(f_3 + f_4 = 1\),
- all points of the form \((0, 0, 0, f_4, f_5)\) with \(f_4 + f_5 = 1\).
**Particular case: \( n = 5 \) and \( m = 3 \)**

In other words, at least in the case with \( n = 5 \) and \( m = 3 \), we are able to classify the initial profiles leading to structurally different final clusters configurations.

Considering the **average opinion** \( Q \), obtained as first order momentum:

\[
Q[f] = \sum_{i=1}^{n} u_i f_i.
\]

The function \( Q \) can be proved to be a **first integral** for system of the evolution equations with \( n = 5, m = 3 \).

Indeed, given an initial opinion distribution, and consequently its corresponding average opinion \( Q \), then we can prove, by direct methods from dynamical systems theory, that if \( Q = \mu \) for a certain \( \mu \in (0, 1) \), that the long time limit of the opinion distribution under consideration will be expressed:

- if \( \mu \in (0, 1/4] \), by the point \((1 - 4\mu, 4\mu, 0, 0, 0)\),
- if \( \mu \in [1/4, 1/2] \), by the point \((0, 2 - 4\mu, -1 + 4\mu, 0, 0)\),
- if \( \mu \in [1/2, 3/4] \), by the point \((0, 0, -1 + 4(1 - \mu), 2 - 4(1 - \mu), 0)\),
- if \( \mu \in [3/4, 1) \), by the point \((0, 0, 0, 4(1 - \mu), 1 - 4\mu)\).
**Particular case: \( n = 5 \) and \( m = 3 \)**

Moreover, let us consider the function:

\[
H(f) = \sum_{i=1}^{n} u_i^2 f_i ,
\]

with \( n = 5 \), corresponding to the “second” moment, which in an analogy with mechanical problems, can be interpreted as a kind of “energy”.

For any \( \mu \in (0, 1) \), the equilibrium point different from the origin is proved to be **asymptotically stable**, using \( H \) as Lyapunov function, for any solution of system of the evolution equations evolving from any point in \( D(\mu) \) but \( (0, 0, 0, 0, 0) \).
Case $n = 5$ and $m = 3$. Final distribution of the opinions on the left panels and evolution of the distribution of opinions on the right panels. Simulations refer to an initial condition with average opinion $Q = 0.4$ (top panels) and $Q = 0.8$ (bottom panels).
In some cases, it is relevant to consider asymptotic scenarios displaying uniform distribution of opinions. From the modelling viewpoint, it is necessary to include, besides a compromise-like processes, an ingredient of “repulsion”.

Thus, we include the effect of a **Boredom-like processes**, and we assume that a portion of individuals having opinion $u_h$ tends to change it, adopting with equal probability the opinions $u_{h-1}$ or $u_{h+1}$, when interacting with other individuals, which share the same opinion $u_h$. 

\[
\begin{array}{c}
\varepsilon \\
1 - 2\varepsilon \\
\varepsilon
\end{array}
\]

*: boredom–like process*
External persuaders

Moreover, referring to the framework with external actions, let us assume the presence of external actions as persuaders:

\[ g_i = g(t, u_i) \quad \forall \quad i = 1, \ldots, n \]

are \( n \) given functions representing the fraction at time \( t \) of persuaders which sustain the opinion \( u_i \).

We assume that \( \mu_{hk} \), the **number of contacts per unit time** between individuals with opinion \( u_h \) and persuaders sustaining the opinion \( u_k \) is the same as the number of encounters per unit time between individuals with opinion \( u_h \) and individuals with opinion \( u_k \).

We model the term \( B_{hk}^i \) to describe the action of quite **efficient persuaders**, capable of influencing all individuals they encounter.

\[ \mu_{hk} = \eta_{hk} \quad \text{and} \quad B_{hk}^i = A_{hk}^i \quad \text{with} \quad \alpha = 1 \quad \text{and} \quad \varepsilon = 0 , \]

for \( i, h, k = 1, \ldots, n \).
Boredom role: absence of persuaders

Considering **no persuaders** acting on the system, asymptotic scenarios distributed over the whole range of opinions are obtained. Focussing on low dimensional cases, $n = 3$ and $m = 2$, we have:

On the left: evolution in time of the distribution functions (dashed line $f_1$, dotted line $f_2$, continuous line $f_3$) in the case $n = 3$ and $m = 2$ with no persuaders, starting from a random initial configuration.

On the right, alternative visualization showing the asymptotic distributions of $f_i$. 

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Analytical study and simulations

These results are confirmed by the qualitative analysis. Choosing $n = 3$ and $m = 2$, the system admits the following equilibrium point:

$$
\left( \frac{2 - \sqrt{2}}{2}, \sqrt{2} - 1, \frac{2 - \sqrt{2}}{2} \right).
$$

- This is a “dynamic” equilibrium, representing the shape of the asymptotic trend distribution of the population over the admissible opinions. Even in correspondence to the equilibrium solution it can well happen that exchanges of opinion between single individuals take place.
- The shape is symmetric ($f_1 = f_3$) and is such that the fraction of individuals having the opinion $u_2$ is larger than the fractions with the opinion $u_1$ and $u_3$.

**Theorem:** The equilibrium solution is globally asymptotically stable.

The proof is based on a definition of a entropy-like function $\sum f_i \log(f_i)$ which can be used as Liapunov functional as it can be proved to be strictly decreasing.
Persuaders sustaining a single opinion

- Assume for example the existence of persuaders sustaining, constantly in time, a single extreme opinion, say the opinion $u_1$.
- Let $0 < \delta < 1$ be a measure of the persuaders. In other words, assume $g_1 = \delta$, $g_2 = 0$, $g_3 = 0$.

It is proved the existence of precisely one equilibrium, which is attractive for all solutions evolving from points on the bidimensional two-simplex $\Sigma_2$:

$$
\Sigma_2 = \{(f_1, f_2, f_3) \in \mathbb{R}^3 : f_i \geq 0 \forall i = 1, 2, 3 \text{ and } \sum_{i=1}^{3} f_i = 1\}.
$$

In particular, in the presence of persuaders sustaining the opinion $u_1$, the asymptotic fraction of individuals having the opinion $u_1$ is effectively larger than in the absence of persuaders.
Asymptotic scenario

Several computational simulations relative to different values on \( n \) and \( m = 2 \) display an asymptotic scenario which confirms the qualitative situation established above.

Moreover, the emerging asymptotic scenario shows analogous patterns for different values of \( n \).

Equilibrium configurations, in the presence of persuaders of measure \( \delta = 0.25 \) in the class \( n \) (left) and \( (n + 1)/2 \) (right).
Persuaders varying in time

If the external interaction rates $\mu_{hk}$ and the persuader’s fractions $g_i$ depend periodically on time (this hypothesis being motivated by periodic elections and other similar phenomena), we analytically prove for some specific low dimensional cases, that a periodic solution exists, with the same period as the $\mu_{hk}$ and $g_i$, and it coincides with the asymptotic limit of all solutions of the system of evolution equations.

Equilibrium configurations with $n = 3$ and $m = 2$, in the presence of persuaders of measure $\delta = 0.25$ in the class $n = 1$. Zoom on the asymptotic scenario on the right.
Persuaders varying in time

To summarize, simulations confirm the results obtained by the qualitative analysis:

- **in absence of external actions** (i.e. persuaders), it emerges asymptotically a symmetric and bell-shaped distribution of opinions among individuals;
- with an action of persuaders **sustaining a specific opinion**, as expected, the maximum of the equilibrium distribution of opinions is located on the opinion sustained by the persuaders.
- in the case of **external actions varying in time**, it exists a periodically varying in time stable equilibrium configuration as a consequence of a periodic-type external action.
Conclusions

The discretization generates a new class of nonlinear dynamical systems which can be regarded as an useful and versatile framework to model complex systems and opinion formation problems. The models derived from the above mentioned framework allow to predict some interesting emergent behaviors. To summarize, simulations confirm and extend the results obtained by the qualitative analysis:

- In **absence of external actions** (i.e. persuaders), it is shown:
  - Clustering and emergence of consensus either total or partial within a group of individuals.
  - Emergence of a symmetric and bell-shaped distribution of opinions among individuals when boredom phenomena are included.
Conclusions

• Including the effects of persuaders or opinion leaders allows to take into account heterogeneous type of individuals within the population. It is shown that:

• with persuaders sustaining a specific opinion, as expected, the maximum of the equilibrium distribution of opinions is located on the opinion sustained by the persuaders.

• with external actions varying in time, it exists a periodically varying in time stable equilibrium configuration as a consequence of a periodic-type external action.