The lecture refers to a general approach of **modelling complex systems in life sciences**. In particular, it focuses on application on the field of social dynamics.

The main ingredients of the lecture are:

- Phenomenological analysis of the system under consideration and **development of the related mathematical framework**.
- **Derivation of specific models** according to the mathematical framework.
- Statement of mathematical problems generated by the application of models. The related **qualitative analysis** is mainly focused on the well-posedness of the mathematical problem and the asymptotic behavior in time.
- **Simulations** and a quantitative analysis to complete the qualitative analysis.
Guidelines

It is worth to point out that models are analyzed with special attention to their exploratory and predictive ability. This means that the investigation is addressed to show emerging phenomena and to analyze how certain microscopic interactions and different values of the parameters may generate different types of evolution.
A mathematics for complex living systems

- The **living matter** shows substantial **differences** with respect to the behavior of the **inert matter**.

- A **complex living system** is a system of several individuals interacting in a non-linear manner whose collective behavior cannot be described only by the knowledge of the mechanical dynamics of each element: *The dynamics of a few individuals does not lead straightforwardly to the overall collective dynamics of the whole system.*

- Mathematical models of complex living systems can be “roughly” grouped into three main categories, according to the scale of observation at which the phenomena are analyzed: **microscopic scale**, **statistical (kinetic) scale** and **macroscopic scale**.

- Models of Complex Living Systems are intrinsically **multiscale**, and show at the macroscopic level **emerging phenomena** expressing a **self-organizing ability** which is only the output of the interactions between entities at the microscopic level.
Bearing in mind the complexity related to a mathematical modelling of a complex living system, a mathematical approach may provide useful suggestions to understand the global behavior of a living system, catching the essential features of the complex system and the **emergence of behaviors**.
Mathematical framework

The modelling is referred to: the **mathematical kinetic theory for active particles**, see Bellomo (2008), and Bellomo, Bianca, Delitala (2009).

- The system is constituted by a large number of interacting entities, called **active particles**, whose microscopic state includes, in addition to geometrical and mechanical variables, also an additional variable, called **activity**, representing the ability of each entity to express a specific **strategy** or **function**.

- The activity variable can be either continuous or discrete. **Discretization** is motivated, as in traffic flow modelling or opinion formation, by the need of identifying the state of the particles by ranges of values rather than by a continuous variable.

- The expression of the activity is not the same for all interacting entities. The activity variable is **heterogeneously distributed** over the active particles, while the overall state of the system is described by the **probability distribution function** over such microscopic state.
Reducing the complexity

- Entities of living systems, called active particles, may be organized into several interacting populations. To reduce the complexity, with a modular approach, each population or functional subsystem, expresses a well defined function, the activity, that is collectively expressed by groups of active particles.

- Interactions modify the state of the interacting entities (conservative interactions) according to the strategy that each entity develops, based on the space and state distribution of the other interacting entities. Due to the presence of the activity variable, the output of interactions does not obey to the laws of classical mechanics, but to the individual and collective strategy expressed by the active particles.

- The evolution in time and space of the distribution function over the microscopic state of the interacting active particles is obtained by a suitable balance in each elementary volume of the space of the microscopic states, where the inflow and outflow in the elementary volume is determined by interactions among particles.
In recent years, mathematicians are attempting to formalize in a more rigorous way the quantitative modelling of socioeconomic systems in order to better explore and predict the dynamics of our societies.

The main conceptual difficulty consists in including the so-called living component. Individuals cannot be thought as simple mindless or hyper-rational agents, and their behaviour does not obey, in general, to some universal rule or exact formulae. Mental schemes of human beings is (and may remain) unknown. Moreover, it is not guaranteed at all that two individuals, exactly stimulated in the same way, react in identical ways.

The Theory of Black Swan Events was developed by N. N. Taleb (N.N. Taleb, The Black Swan, Penguin, 2007) to qualitatively explain the disproportionate role of high-impact and rare events that are beyond the realm of normal expectations in history, science, finance and technology,
The market is an excellent example of self-organized systems: each agent decides according to his own perception of the events.

In the simplest framework, these events consist in the price fluctuations, the only available information. Each participant’s action will in turn influence the price.

In a true economy there are external driving factors, such as politics, natural disasters, human psychology etc.
Economics needs a scientific revolution


Axioms of classic economics:

- the **rationality of economic agents**: every economic agent, either a person or company, acts to maximize his profits.

- the **“invisible hand”**: agents, in the pursuit of their own profit, are led to do what is best for society as a whole.

- **market efficiency**: market prices faithfully reflect all known information about assets.

Most current economics theories assume that each participant knows what is best for him given that all other participants are equally intelligent in choosing their best actions.
In reality, markets are not efficient, humans tend to be over-focused in the short-term and blind in the long-term, and errors get amplified, ultimately leading to collective irrationality, panic and crashes. Free markets are wild markets.

In the real world, most of the actions of the actual players are based on trial-and-error inductive thinking, rather than the deductive rationale.

The approach traditionally used in economics is not suitable to include irrationality, and there is the need of exploring new approaches and develop completely different tools in which emerging collective phenomena can be better highlighted, including for instance the so called bounded rationality.
**Behavioural economy, Sociophysics and Econophysics**

*Behavioural economy* and complex systems approaches are relatively new trends in economic sciences that try to give answers to these issues, bearing in mind that many qualitative (and sometimes quantitative) features and emerging patterns do not depend on the microscopic details of the processes under examination (statistical regularities emerge in the behaviour of large populations, just as the law of ideal gases emerges from the chaotic motion of individual molecules).

Various methodologies have been proposed, and the words *socio* and *econophysics* have been used in order to characterize new quantitative approaches towards both sociology and economics.

Some key-points of the models: explain how *small perturbations* can lead to wild effects (classical models of physics). Moreover, although a system may have an *optimum state*, it is sometimes so hard to identify that the system never settles there.
In the literature, there are models mainly at two observation scales: microscopic models focussing on the single individual or mesoscopic models with a statistical description of the systems. Few models are developed at the macroscopic scale.

Most of them are microscopic models like cellular automata or Agent Based models. Referring to sociophysics, one of the most popular models is the bounded confidence model introduced by Deffuant et al. (Adv. Complex Sys, 2000), in which repeatedly two peers are randomly selected and operate a compromise in the positions if their opinions differ by less than a deviation threshold.

Tools of statistical mechanics have been recently used for mesoscopic models to have a deeper insight and a rigorous qualitative analysis on the evolution of the model.
Bibliography


Sociophysics

Focusing on sociophysics, the list of the most explored research fields of investigation includes, among others:

- Opinion formation and cultural dynamics for instance:
  - on diffusion of new technologies (like spread of epidemics);
  - critics’ ratings about the new opening movies or restaurants;
  - reputation systems to measure trust about users while doing transactions over the internet such as e-bay;
  - rough categorization of voters in an election;
  - opinion of the employees about the new company during a company fusion.

- Applause dynamics and problems of herding and imitation.

- Language evolution and problems of social learning

- Problems related to group making a collective decision (fair, correct, efficient decision).

- Social networks analysis.
Which data?

Rating of movies and evolution of Obama’s approval index
Some Data

Networks of sexual contacts and distribution of HIV in communities.
Network of friendship
Bibliography


Let us consider one single population constituted by a large number of particles homogeneously distributed in space, with a discrete microscopic state, the physical variable charged to describe the state of each particle:

$$I_u = \{u_1, \ldots, u_h, \ldots, u_n\}$$

**Distribution function for active particles**: normalized density distribution function

$$f_i(t) = f(t, u_i)$$

**Macroscopic gross variables** can be expressed, under suitable integrability properties, by moments weighted by the above distribution function.

$$\rho(t) = \sum_{i=1}^{n} f(t, u_i), \quad Q(t) = \sum_{i=1}^{n} u_i f(t, u_i).$$
Discrete Kinetic Framework: Single Population

Considering only conservative interactions, let assume that the following quantities can be computed:

• The interaction rate:

\[ \eta_{hk} = \eta(u_h, u_k) : I_u \times I_u \rightarrow \mathbb{R}^+ , \]

• The tables of the game rules (n matrices \( n \times n \)):

\[ A^i_{hk} = A(u_h, u_k; u_i) : I_u \times I_u \times I_u \rightarrow \mathbb{R}^+ . \]

\( u_h \) falls into the state \( u_i \) after an interaction with a field particle with state \( u_k \).

• The tables of the game rules are transition probability densities:

\[ \forall h, k : \sum_{i=1}^{n} A^i_{hk} = 1. \]
The Discrete Generalized Kinetic Framework

- The evolution equation is obtained equating the rate of growth of subjects with microscopic state in volume element \([u, u + du]\), to the inflow and outflow of subjects per unit time in the volume due to interactions. If no source term is considered (no proliferation/destruction term), the evolution equations write as follows:

\[
\frac{df_i}{dt} = \sum_{h=1}^{n} \sum_{k=1}^{n} \eta_{hk} A_{hk}^i f_h f_k - f_i \sum_{k=1}^{n} \eta_{ik} f_k.
\]

- The mathematical problem is defined, for \(i = 1, \ldots, n\), linking the initial conditions to the evolution equations:

\[
f_{i0} = f_i(t = 0) \quad \text{where} \quad \sum_{i=1}^{n} f_{i0} = 1.
\]
Well posedness of the discrete kinetic framework

• **Theorem: Existence and Uniqueness.** Assume \( \eta_{hk} \leq M \) for some positive constant \( M < +\infty \).

The solution \( f(t) = (f_1(t), \ldots, f_n(t)) \) of the Cauchy problem exists and is unique \( \forall t \in [0, +\infty) \).

\[
\forall t \geq 0 : \quad f_i(t) \geq 0 \quad \text{for any} \quad i = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} f_i(t) = 1.
\]

• **Regularity of the solutions.** The solution claimed in the Theorem is of class \( C^\infty \): it guarantees the continuous dependence of solutions on the initial conditions.

• **Theorem: Equilibrium solutions.** Assuming

\[
\eta_{hk} = c \quad \forall h, k = 1, \ldots, n
\]

where \( c \) is a positive constant, then there exists at least one positive equilibrium solution of System.
The framework with external actions

Let further assume that the following quantities can be computed:

- \( g_i = g(t, u_i) \), for \( i = 1, \ldots, n \)
  which are \( n \) given functions representing the distribution at time \( t \) of external actions acting on the activity state \( u_i \).

- \( \mu_{hk} = \mu(u_h, u_k) \) \( \forall h, k = 1, \ldots, n \)
  which is the number of contacts per unit time, at time \( t \), between particles with activity \( u_h \) and external actions acting on the state \( u_k \) (table of the external interaction rates)

- \( B_{hk}^i \in \mathbb{R}_+ \), with \( \sum_{i=1}^n B_{hk}^i = 1 \), \( \forall h, k = 1, \ldots, n \)
  which is the probability density for an active particle with activity state \( u_h \) to end up with state \( u_i \) after an interaction with an external action acting on the state \( u_k \) (table of the external transition probability densities).
The framework with external actions

Equating the time derivative of each $f_i$ to the difference between the **gain** and the **loss** term, and taking into account the contribution of both the internal and the external interactions, the equations of the evolution in time of the $f_i$ take the form:

$$\frac{df_i}{dt} = J_i[f, f] + Y_i[f, g], \quad i = 1, \ldots, n,$$

where

$$J_i[f, f] = \sum_{h=1}^{n} \sum_{k=1}^{n} \eta_{hk} A_{hk}^i f_h f_k - f_i \sum_{k=1}^{n} \eta_{ik} f_k, \quad i = 1, \ldots, n,$$

and

$$Y_i[f, g] = \sum_{h=1}^{n} \sum_{k=1}^{n} \mu_{hk} B_{hk}^i f_h g_k - f_i \sum_{k=1}^{n} \mu_{ik} g_k, \quad i = 1, \ldots, n.$$
Opinion formation

We focus on the evolution of a set of opinions within a population with tools and methods of Kinetic Theory for Active Particles. Several approaches have been proposed in the literature at different representation scales. Several contributions are in the field of physics, the so called socio-physics, while few contributions are developed from mathematicians, from a more recent period.

Here we propose an approach to model the changes in the points of view arising as a consequence of the encounters between individuals performing compromises, then we add the influence of some boredom-like phenomena and finally the presence of external actions is modelled, including the effect of the influence exerted by some agents, the persuaders. Bound of confidence is taken into account.

The modelling is developed with an explorative aim to look at emerging phenomena and critical parameters.
Bibliography

A model in the absence of persuaders

Focussing on the opinion formation problem within a population, with KTAP approach, we have that:

- \( I_u = \{u_1, \ldots, u_i, \ldots, u_n\} \) is the “ordered” finite set of the admissible opinions;
- \( f_i = f_i(t) \quad \forall \quad i = 1, \ldots, n \) is the fraction of individuals sharing the opinion \( u_i \);

For the interaction rates, either we suppose that individuals having the same opinion meet each other more frequently than individuals with different opinion or we suppose that all individuals have the same probability to meet each others:

\[
\eta_{hk} = \begin{cases} 
\alpha & \text{if } h = k, \\
\frac{\alpha + \varepsilon}{2} & \text{if } |h - k| = 1, \\
\varepsilon & \text{if } |h - k| = 2, \\
0 & \text{otherwise.}
\end{cases}
\]
A model in the absence of persuaders

Let us consider a parameter $m$, called **closeness threshold**, related to the **open-mindness** of the individuals.

We assume the occurrence of a **compromise-like processes**: after an interaction, two individuals having a suitably small difference in opinions (depending on the closeness threshold parameter $m$), readjust their opinions letting them become eventually closer.
Clustering and comparison with competing models

This particularization can be viewed as a discrete version of an Agent-Based “bounded confidence model” proposed by Deffuant et al., Adv. Complex Systems (2000), and Weisbuch et al., Complexity (2002).

In this case the discrete model is expressed by a two parameters family of systems of $n$ nonlinear ordinary differential equations, where the parameters $n$ and $m$ respectively represent the finite number of admissible opinions about a given issue and the threshold separating close from distant opinions, i.e. the closeness threshold characterizing the occurrence or not of an adjustment of opinions of interacting individuals.

The properties of the equilibrium configurations of the model are obtained by a qualitative analysis and the results are generalized and visualized by simulations. The role of the parameter related to the “open-mindedness” of the individuals, is analyzed.
Properties of the equilibrium configurations: final distribution of opinions in the cases $n = 27$ with $m = 3, 4, 5, 8$ from left to right.
Simulations

- If individuals are sufficiently **open minded**, all individuals asymptotically will share the same opinion, and correspondingly a **single cluster** emerges.

- If individuals are **less open minded**, two or more **clusters emerge**, corresponding to the situations where different groups of individuals show the same opinion, as for instance groups of interests or parties.
Simulations

The above result is consistent with simulations and results of a competing model, the “bounded confidence model” (Deffuant et al., Adv. complex. Sys., 2000). The main advantage of the approach here proposed, is the possibility to develop analytical proofs of certain features of the asymptotic clustering behavior.
Simulations
Simulations

We define a cluster a configuration displaying at most two non-empty opinion classes with empty classes on the left and right.

All performed simulations relative to systems with $n$ general odd number of opinions and different values of closeness parameter $m$ show that:

For $n$ general odd and $2 \leq m \leq n - 1$, the system of evolution equations admits degenerate equilibria.

Each one of these equilibria corresponds to a distribution containing some clusters.

There are at least $m$ empty classes separating any two consecutive clusters.

This result confirms that:

the number of clusters in a stationary opinion distribution decreases as the closeness threshold $m$ increases.

This result is in good agreement with some experimental evidence which show the emergence of consensus over few opinions, depending on the topology of the interactions and on the “open-mindedness” of the individuals.
To develop a qualitative analysis, we focus on low dimensional cases. For a general number of opinions \( n \) and low number of the parameter \( m \), we can state the following:

- **Theorem.**
  The system of evolution equations of the model with general odd \( n \) and \( m = 2 \) admits several families of degenerate equilibria corresponding to a distribution containing some clusters. There are at least two empty classes separating any two consecutive clusters and any cluster consists of at most two opinion classes.

Analogous theorem can be proved in the case of a general odd \( n \) and \( m = 3 \) where there are at least three empty classes separating any two consecutive clusters.

The result is extended in (Bertotti 2010) where it is proved that for a general \( m \) the clusters are surrounded by at least \( m \) empty classes.
Particular case: \( n = 5 \) and \( m = 3 \)

Consider as an explicit example the \( n = 5 \) and \( m = 3 \), the system of the evolution equations of the model takes then the form:

\[
\begin{align*}
\frac{df_1}{dt} &= -f_1 f_3 - f_1 f_4, \\
\frac{df_2}{dt} &= 2f_1 f_3 + f_1 f_4 - f_2 f_4 - f_2 f_5, \\
\frac{df_3}{dt} &= -f_1 f_3 + f_1 f_4 + 2f_2 f_4 + f_2 f_5 - f_3 f_5, \\
\frac{df_4}{dt} &= 2f_3 f_5 + f_2 f_5 - f_1 f_4 - f_2 f_4, \\
\frac{df_5}{dt} &= -f_3 f_5 - f_2 f_5.
\end{align*}
\]

One may check directly that the equilibria of this system are:

- all points of the form \((f_1, f_2, 0, 0, 0)\) with \(f_1 + f_2 = 1\),
- all points of the form \((0, f_2, f_3, 0, 0)\) with \(f_2 + f_3 = 1\),
- all points of the form \((f_1, 0, 0, 0, f_5)\) with \(f_1 + f_5 = 1\),
- all points of the form \((0, 0, f_3, f_4, 0)\) with \(f_3 + f_4 = 1\),
- all points of the form \((0, 0, 0, f_4, f_5)\) with \(f_4 + f_5 = 1\).
**Particular case: \( n = 5 \) and \( m = 3 \)**

In other words, at least in the case with \( n = 5 \) and \( m = 3 \), we are able to classify the initial profiles leading to structurally different final clusters configurations.

Considering the **average opinion** \( Q \), obtained as first order momentum:

\[
Q[f] = \sum_{i=1}^{n} u_i f_i.
\]

The function \( Q \) can be proved to be a **first integral** for system of the evolution equations with \( n = 5, m = 3 \).

Indeed, given an initial opinion distribution, and consequently its corresponding average opinion \( Q \), then we can prove, by direct methods from dynamical systems theory, that if \( Q = \mu \) for a certain \( \mu \in (0, 1) \), that the long time limit of the opinion distribution under consideration will be expressed:

- if \( \mu \in (0, 1/4] \), by the point \((1 - 4\mu, 4\mu, 0, 0, 0)\),
- if \( \mu \in [1/4, 1/2] \), by the point \((0, 2 - 4\mu, -1 + 4\mu, 0, 0)\),
- if \( \mu \in [1/2, 3/4] \), by the point \((0, 0, -1 + 4(1 - \mu), 2 - 4(1 - \mu), 0)\),
- if \( \mu \in [3/4, 1) \), by the point \((0, 0, 0, 4(1 - \mu), 1 - 4\mu)\).
Particular case: $n = 5$ and $m = 3$

Moreover, let us consider the function:

$$H(f) = \sum_{i=1}^{n} u_i^2 f_i,$$

with $n = 5$, corresponding to the “second” moment, which in an analogy with mechanical problems, can be interpreted as a kind of “energy”.

For any $\mu \in (0, 1)$, the equilibrium point different from the origin is proved to be asymptotically stable, using $H$ as Lyapunov function, for any solution of system of the evolution equations evolving from any point in $D(\mu)$ but $(0, 0, 0, 0, 0)$.
Case $n = 5$ and $m = 3$. Final distribution of the opinions on the left panels and evolution of the distribution of opinions on the right panels. Simulations refer to an initial condition with average opinion $Q = 0.4$ (top panels) and $Q = 0.8$ (bottom panels).
In some cases, it is relevant to consider asymptotic scenarios displaying uniform distribution of opinions. From the modelling viewpoint, it is necessary to include, besides a compromise-like processes, an ingredient of “repulsion”. Thus, we include the effect of a **Boredom-like processes**, and we assume that a portion of individuals having opinion $u_h$ tends to change it, adopting with equal probability the opinions $u_{h-1}$ or $u_{h+1}$, when interacting with other individuals, which share the same opinion $u_h$. 

\[
\begin{array}{c}
\varepsilon \\
1 - 2\varepsilon \\
\varepsilon
\end{array}
\]

boredom–like process
Moreover, referring to the framework with external actions, let us assume the presence of external actions as persuaders:

$$g_i = g(t, u_i) \ \forall \ i = 1, \ldots, n$$

are $n$ given functions representing the fraction at time $t$ of persuaders which sustain the opinion $u_i$.

We assume that $\mu_{hk}$, the number of contacts per unit time between individuals with opinion $u_h$ and persuaders sustaining the opinion $u_k$ is the same as the number of encounters per unit time between individuals with opinion $u_h$ and individuals with opinion $u_k$.

We model the term $B_{hk}^i$ to describe the action of quite efficient persuaders, capable of influencing all individuals they encounter.

$$\mu_{hk} = \eta_{hk} \ \text{and} \ B_{hk}^i = A_{hk}^i \ \text{with} \ \alpha = 1 \ \text{and} \ \varepsilon = 0,$$

for $i, h, k = 1, \ldots, n$. 
**Boredom role: absence of persuaders**

Considering **no persuaders** acting on the system, asymptotic scenarios distributed over the whole range of opinions are obtained. Focussing on low dimensional cases, $n = 3$ and $m = 2$, we have:

On the left: evolution in time of the distribution functions (dashed line $f_1$, dotted line $f_2$, continuous line $f_3$) in the case $n = 3$ and $m = 2$ with no persuaders, starting from a random initial configuration.

On the right, alternative visualization showing the asymptotic distributions of $f_i$. 
Analytical study and simulations

These results are confirmed by the qualitative analysis. Choosing $n = 3$ and $m = 2$, the system admits the following equilibrium point:

$$\left(\frac{2 - \sqrt{2}}{2}, \sqrt{2} - 1, \frac{2 - \sqrt{2}}{2}\right).$$

• This is a “dynamic” equilibrium, representing the shape of the asymptotic trend distribution of the population over the admissible opinions. Even in correspondence to the equilibrium solution it can well happen that exchanges of opinion between single individuals take place.

• The shape is symmetric ($f_1 = f_3$) and is such that the fraction of individuals having the opinion $u_2$ is larger than the fractions with the opinion $u_1$ and $u_3$.

**Theorem:** The equilibrium solution is globally asymptotically stable.

The proof is based on a definition of a entropy-like function $\sum f_i \log(f_i)$ which can be used as Liapunov functional as it can be proved to be strictly decreasing.
Persuaders sustaining a single opinion

- Assume for example the existence of persuaders sustaining, constantly in time, a single extreme opinion, say the opinion $u_1$.
- Let $0 < \delta < 1$ be a measure of the persuaders. In other words, assume $g_1 = \delta$, $g_2 = 0$, $g_3 = 0$.

It is proved the existence of precisely one equilibrium, which is attractive for all solutions evolving from points on the bidimensional two-simplex $\Sigma_2$:

$$\Sigma_2 = \{(f_1, f_2, f_3) \in \mathbb{R}^3 : f_i \geq 0 \forall i = 1, 2, 3 \text{ and } \sum_{i=1}^{3} f_i = 1\}.$$

In particular, in the presence of persuaders sustaining the opinion $u_1$, the asymptotic fraction of individuals having the opinion $u_1$ is effectively larger than in the absence of persuaders.
Asymptotic scenario

Several computational simulations relative to different values on \( n \) and \( m = 2 \) display an asymptotic scenario which confirms the qualitative situation established above.

Moreover, the emerging asymptotic scenario shows analogous patterns for different values of \( n \).

Equilibrium configurations, in the presence of persuaders of measure \( \delta = 0.25 \) in the class \( n \) (left) and \( (n + 1)/2 \) (right).
**Persuaders varying in time**

If the external interaction rates $\mu_{hk}$ and the **persuader’s fractions** $g_i$ **depend periodically on time** (this hypothesis being motivated by periodic elections and other similar phenomena), we analytically prove for some specific low dimensional cases, that a **periodic solution exists**, with the same period as the $\mu_{hk}$ and $g_i$, and it coincides with the asymptotic limit of all solutions of the system of evolution equations.

Equilibrium configurations with $n = 3$ and $m = 2$, in the presence of persuaders of measure $\delta = 0.25$ in the class $n = 1$. Zoom on the asymptotic scenario on the right.
Persuaders varying in time

To summarize, simulations confirm the results obtained by the qualitative analysis:

• in **absence of external actions** (i.e. persuaders), it emerges asymptotically a symmetric and bell-shaped distribution of opinions among individuals;

• with an action of persuaders **sustaining a specific opinion**, as expected, the maximum of the equilibrium distribution of opinions is located on the opinion sustained by the persuaders.

• in the case of **external actions varying in time**, it exists a periodically varying in time stable equilibrium configuration as a consequence of a periodic-type external action.
Conclusions

The discretization generates a new class of nonlinear dynamical systems which can be regarded as an useful and versatile framework to model complex systems and opinion formation problems. The models derived from the above mentioned framework allow to predict some interesting emergent behaviors. To summarize, simulations confirm and extend the results obtained by the qualitative analysis:

- In **absence of external actions** (i.e. persuaders), it is shown:
  - Clustering and emergence of consensus either total or partial within a group of individuals.
  - Emergence of a symmetric and bell-shaped distribution of opinions among individuals when boredom phenomena are included.
Conclusions

• Including the effects of persuaders or opinion leaders allows to take into account heterogeneous type of individuals within the population. It is shown that:

  • with persuaders **sustaining a specific opinion**, as expected, the maximum of the equilibrium distribution of opinions is located on the opinion sustained by the persuaders.
  
  • with **external actions varying in time**, it exists a periodically varying in time stable equilibrium configuration as a consequence of a periodic-type external action.